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Agricultural Research Institute
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PROCEEDINGS

OF THE

ROYAL SOCIETY OF LONDON

SERIES A

CONTAINING PAPERS OF A MATHEMATICAL AND
PHYSICAL CHARACTER.

VOL XCIV

LONDON
PRINTED FOR THE ROYAL SOCIETY AND SOLD BY
HARRISON AND SONS, ST MARTIN'S LANE,
PRINTERS IN ORDINARY TO HIS MAJESTY

AUGUST, 1918

LONDON
HARRISON AND SONS, PRINTERS IN ORDINARY TO HIS MAJESTY,
ST MARTIN'S LANE

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PROCEEDINGS OF

THE ROYAL SOCIETY.

Section A.—Mathematical and Physical Sciences.

The Fourth Colourless Sensation in the Three-Sensation Spectrum Curves when Measured on the Centre of the Retina.

By Sir W de W Abney, K C B, F R S, *With a Note by* Prof. S W J Smith, F.R.S.

(Received March 19, 1917.)

When the image of a small disc of a spectrum colour is received on the retina of a person possessing normal sense of colour, he observes changes in the hue varying with the portion of the retina on which it falls. On the centre the colour is that of the projected beam, and, as it travels away from the centre, the hue becomes paler as the angular distance becomes greater, and, when a certain angular distance is reached (varying with the intensity of the beam and with its wave-length), the colour vanishes, and the disc appears as white of fair brightness. On this principle the boundaries of colour fields can be mapped.

It is evident that, as the disc of light travels across the retina, the composition of the colour changes, and that an increased proportion of "white" is felt as the white boundary of the colour field is approached, whilst the colour itself gradually diminishes.

The intensity of the white at the boundary of the colour field can be measured with some certainty, but it is more difficult to obtain a measure of the white which exists in the colour when it falls on the centre of the retina.

In two papers* communicated to the Royal Society, the writer has shown that the spectrum colours can be formed by admixture of three colour sensations, but no place was found here for an admixture of the "white" sensation, as the colours were received on the centre of the retina. Had they been received at a place outside the centre (*i.e.* on part of the periphery), it is probable that the three colour sensations would have required a percentage of "white" to make a complete match The white sensation we are considering is not the same as is found in the spectrum colours. It is white which cannot be analysed by refraction or diffraction, but exists as an entity by itself It thus differs from the white which exists in the ordinary spectrum colours, which is found to be due to the admixture of the three sensations of colour in fixed proportions, the proportions depending on the quality of the light forming the spectrum. The "white" we are considering appears always to be of the same degree of "colourlessness" whatever may be the light used to form the spectrum

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* 'Phil. Trans.,' A, vol. 193, p. 259 (1900), and 'Phil. Trans.,' A, vol. 205, p. 333 (1906).

For the sake of avoiding any ambiguity, it is proposed to define this "white" as a fourth sensation—or the fundamental sensation of light, which is a definition that the author has employed in more than one communication which he has sent to the Royal Society

In a recent paper by Dr W. Watson and the author, it has been pointed out that, in the work by eminent foreign workers, it has been supposed that in the sensations of colour on the retina the cones are the principal agents which are active, whilst for the sensation of the "white" the rods are the sensation agents In that paper a slight discussion is made on these proposals. It may be remarked that, whilst there are both in the rods and cones distinctive features, there are some processes which it would be difficult to class as possessing the features of either cones or rods.

In the same paper retinas are classified under two headings· one class, those in which at the fovea cones alone existed, as, when extinguishing a small disc of a spectrum colour, the light and colour both "went out' at the same instant, in the other class, the retinas on which the same coloured disc fell, when received on the fovea, first lost the sensation of colour, leaving a "white" which was extinguished afterwards by a still further reduction of the intensity of the colour disc It is too early from our experience to say that in this last class both cones and rods were present in the fovea. It may be that the processes existing there were hermaphrodite, being neither pure cones nor pure rods

This is mentioned here because, in the subjoined account of experiments undertaken to find what very small amount of the fourth sensation—if any—was present when the colour sensations of the spectrum of ordinary intensity were measured at the centre of the retina, the observers both had retinas of the second class.

In an hermaphrodite cone and rod it may be that some species are most sensitive to colour, whilst in others the greater sensitiveness may be for the "white"

At the beginning of 1898 I carried out an investigation as to the form of the luminosity curves of the spectrum of very feeble arc electric light The measures made by myself and my assistant, Mr Bradfield, were put on one side owing to press of other work. It seems that these curves would show the point I wanted to obtain, as stated above, by the amount of action on the cones when the light was a disc falling on the centre of the retina

The method adopted was as follows, the colour-patch apparatus being that which I ordinarily use :—

The various spectrum colours came through a movable slit at the focus, and a patch of the colour was formed by a lens placed in the coloured beam

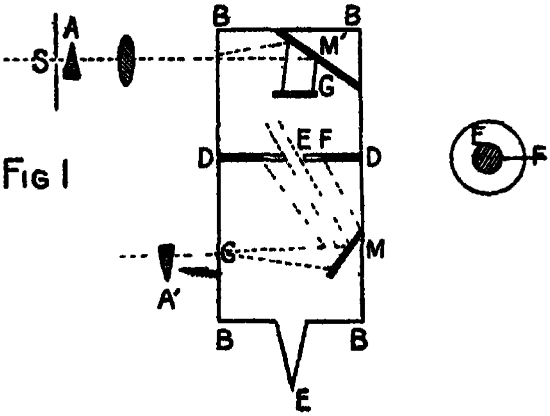


FIG I

The patch of colour was caused to fall on a plate of glass (finely ground on both surfaces) by reflection from M′, which occupied a position shown in the box BBBB—a partition DD was pierced by a hole some ½ inch in diameter In front of the hole discs could be placed, in which clean-cut holes of ⅓, ¼, and ⅛ inch diameter were bored An aperture C in the side of the box was made, which allowed the white light reflected from the front surface of the first prism of the apparatus to fall on M. That portion of the light reflected from M which passed through the centre of the disc placed at DD passed through to the far end of the blackened box. An eye placed at an eye-hole E at the end of the box saw the white rim of the disc illuminated, and in its centre a small disc of coloured light. In the spectrum ray an annulus A was placed, the centre of rotation being accurately in the continuation of the slit through which the coloured ray passed Every 25° of the annulus diminished to half the initial intensity when the D light was passing through the slit and the annulus was at its zero point. The brightness of yellow beam was found to be 1-1 of a Heffner lamp at a foot distance.

To diminish the intensity of the white, a second annulus A′ was placed at the point where the reflected beam crossed and formed a line. With this annulus every 30° of angle cut off half the white. The white light passing through the annulus at its zero point was measured, and found to equal 0 2 Heffner lamp-light at 1 foot distance from a screen, so that, when the light passed through annulus at 60°, the intensity of the white light was reduced to 0 05 of the Heffner lamp-light. For reasons which need not be given, it was determined to compare the luminosity of the colours with the white when, for each series of readings, the white light was kept constant

It was determined to take the following intensities of white light The annulus was set —

At 60°	and the white transmitted		was	0 05 H,
120°	„	„	„	0 0125 H,
180°	„	„	„	0 0031 H,
240°	„	„	„	0 0008 H,
300°	„	„	„	0 0002 H

Three sets of measures were made by my assistant and myself, and a mean value for $\frac{1}{4}$-inch disc of colour obtained by using the annulus in the coloured beam to secure equality of luminosity.

These luminosities were readily found The white surround to the colour disc seemed to aid the equalisation of luminosities, the small oscillation of the angle of the annulus in the colour beam showing very accordant readings.

The following Table gives the mean readings of the annulus taken by my assistant throughout the spectrum and at the four intensities of white light 240°, 180°, 120°, and 60° (The readings at 300° were practically the same as those for 240°, so have been omitted.) The first column is the empiric scale of the spectrum, and the second is that scale reduced to standard scale, which has been used throughout all previous communications

Table I gives the original readings. The values of these intensities or luminosities are calculated from a Table which shows each 25° of the annulus as diminishing the light by $\frac{1}{2}$. For the sake of simplicity, the values shown in Table II are derived by deducting the values obtained *from* the highest reading given in the several columns, making the luminosity curve to show a maximum of 100 Table III shows the intensities when the whole numbers of the S.S.N. are employed. The areas of these curves are taken when the scale of S.S.N. is applied

Table I.—(A) Annulus Readings in Degrees

Temporary scale of spectrum	Standard scale (S S N)	A′ Annulus in fixed positions			
		240°	180°	120°	60°
	°				
72	60 2	80	50	28	—
74	58	123	95	70	30
76	55 8	166	127	100	55
78	53 6	199	155	123	82
80	51 ·4	225	178	143	108
82	49 2	250	208	160	115
84	47	275	228	172	117
86	44 8	296	244	185	115
88	42 6	312	257	190	105
90	40 4	318	262	190	94
92	38 2	314	252	182	80
94	36 0	305	245	167	65
96	33 8	292	231	252	45
98	31 6	275	218	141	28
100	29 4	260	197	118	5
110	27 2	250	182	100	—

Table V shows the intensity values of each of the columns when the areas of the curves are made equal to that of the spectrum of normal intensity.

When plotting Table V, it appeared that it would be more complete if the curve for 90° could be added The original curves of annulus readings were brought into requisition, and the closely approximate annulus curve for 90° deduced, and from it intensity curve with a maximum of 100 calculated as shown in Table V This curve was reduced so as to be of the same approximate area as those shown for the 0° to 240° curves, and is also shown in Table V

The results of the tabulated "equal area" curves are shown graphically in fig 2

The diagram shows that all the curves of equal area very approximately intersect at S S N 44 (and no doubt if the measures had been more accurate the intersections would have been still more closely at the same point) There must be some reason for such a coincidence in intersection.*

---

* In the note which follows on at the end of the paper, the referee gives a mathematical reason for this intersection on a certain assumption The author had arrived at a similar result by a different plan, but hesitated to give it as it seems that in the "A" curve some small corrections have to be made when the intensity is altered, and also that when the radiation is reduced as it was in the experiments the white comparison is not equally reduced with the radiation This difference between radiation and luminosity is not known exactly, but the method given in the referee's note makes it practical to find the difference in the reduction in the white with reduction of radiation.—*May* 23, 1917.

Table II

Original scale	S S N	240°	180°.	120°	60°
72	60 2	0·1	0 8	0 9	—
74	58	0 6	1 0	3 5	9 0
76	55 8	1 3	2 4	8·0	20 5
78	53 6	2 2	5 0	15 0	38 0
80	51 4	8 0	7 0	26	68 5
82	49 2	15 5	19 5	41	96 0
84	47	31	39	58	100 0
86	44 8	58	61 5	68	96 0
88	42 6	85	87	95	72
90	40 4	100	100	95	58
92	38 2	90	76	76	36
94	36	70	62	50 5	28 5
96	33 8	50	42 5	33 5	14
98	31 6	31	26	24	8 5
100	29 4	20	17	13	4 4
110	27 2	15 5	11	4·7	—

Table III

S S N	240°	180°	120°	60°.	Normal spectrum.
60	—	—	0	3	2
58	—	1	2	14	15
56	2	4	8 5	39 1	32
54	4	8	15 5	57	61
52	6 1	12	25	74	75
50	13 3	19	35	90	91
48	23	32	47	100	100
46	42	55	61	98	94
44	73	75	80	88	75
42	92	96	98	74	54
40	99 5	98	96	58	37
38	89	80	75	39	25
36	70	60	52	27	16
34	54	44	35	18	9
32	39	32	24	6	4
30	28	24	16	2	—
28	19	17	10	1	
26	13	12	7		
	640	680	700	783	665

Table IV

Deduced luminosity curve for 90°	
60	2
58	8
56	16
54	26
52	42
50	64
48	92
46	96
44	91
42	86
40	76
38	60
36	42
34	29
32	19
30	13
28	8
26	5

Table V—Curves of Equal Areas Calculated from Table III and Table IV.

S S N	240°	180°	120°	90°	60°	0°
60	—	—	—	1 7	2	2
58	—	1	2	7 1	13	15
56	2 0	4	7 5	13 5	29	32
54	4 0	8	14 5	22	40	61
52	6 1	12	24	35 5	64	75
50	13 3	15	34	54	74	90
48	28 4	35	45 5	77	87	100
46	42 5	55	59	79	85	94
44	74	75	77	76 5	76 5	75
42	94	96	99·5	72 5	69	54
40	102	98	92·5	64	50 5	37
38	90 5	80	72 5	50·8	33	25
36	71	60	50·5	35 5	23 5	16
34	54·7	44	34	24 5	15 5	9
32	39 5	32	23	16	10	4
30	28 5	24	15 5	10 3	5	1
28	19 5	17	9 6	5	1	
26	13 5	12	6 7	2		

The theory that the cones in the retina are sensitive to the coloured rays whilst the rods similarly respond to the colourless rays is a partial answer to these results But there are difficulties in accepting this theory as altogether sufficient In a large portion of the spectrum from near D to the extreme

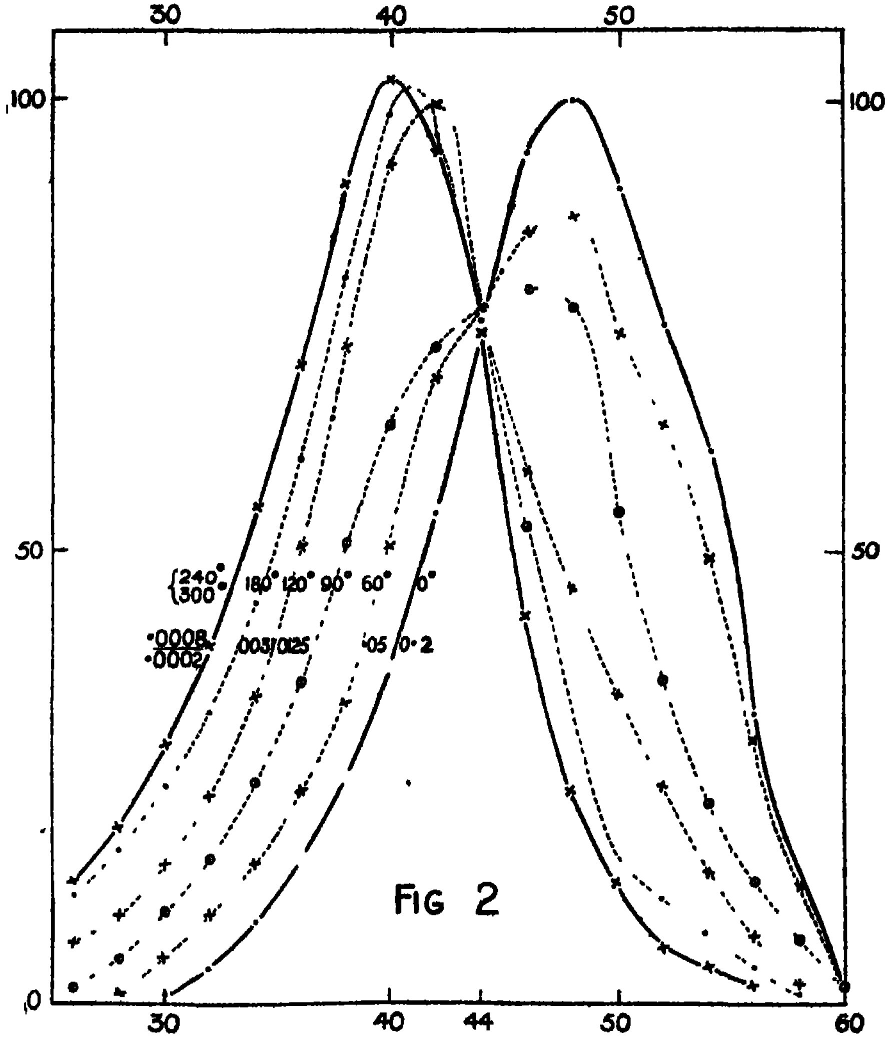


FIG 2

blue the sensation curves, as given in the 'Philosophical Transactions,' show that the overlapping of the red, green, and blue sensations are blended in various degrees, and the spectrum colour is a compound of the one or two pure sensations and white light Thus, at one spectrum green, the colour is produced by a mixture of 60 per cent of white light and the pure green sensation Experiment shows that by adding white to the spectrum colours, impure as they are in this region of the spectrum, the colour may be indistinguishable in the mixture and only white is perceived by the eye If the intensity of the spectrum is diminished, a point, varying for each wave-length, is reached in which the colours disappear, and here we have the cones in the dark-adapted retina showing no colour The fact that the boundaries of colour fields are white, shows that the fourth sensation "white" has the same effect as the mixed white of the three-colour sensation, in drowning a colour (In the intensities of the white comparison light and of the spectrum used in the above results there are only three which showed any signs of colours, the ordinary bright spectrum of 0° through the annulus, that of 60°, and again, partially, that of 90° In the white light which forms the spectrum coming through 180° and 240°,* the whole of the rays appear as white, or, perhaps more accurately, may be described as greyish Yet, at the most refrangible part of the spectrum, the cones must be receiving the same light as when the colour is still visible It thus appears if the cones are sensitive to colour they are also sensitive to the same white rays as the rods, but in a minor degree, whilst the rods are probably unaffected by the colour rays)

In 'Phil. Trans.,' vol. 7, 1897, "On the Sensitiveness of the Retina to Light and Colour," a large section is devoted to colour fields, in which measures of the extinction of colour by white light on the periphery are made. The colours operated with are not feeble, but of a measurable brightness. The loss of colour at a good many degrees from the axis of the retina is very sharply defined as a rule, and it is a matter for consideration whether the boundaries of the colour fields are fixed by extinction of colours by the white light formed by mixtures of the colour sensations or principally the sensitiveness of the rods to what I have called the fourth colour-sensation, which, apparently, as recently remarked, may be equally effective as the white of the mixed colour sensations extinguishes

In any case, there is an enormous difference in the sensitiveness of the retina at the periphery to the extinction of colour to that formed at the central part We may take it that, when the spectrum is formed of white light of 0 0008 candle at 1 foot distance from the screen, the colourless light will be so small that it is negligible in the three-sensation curves.

In another communication it is hoped that the colour field boundaries of white may throw more light on the action of the rods and cones, and of the

---

* When making the observations the intensity at which the colour disappeared from the rays was recorded See also the Author's book on the 'Trichromatic Theory (Longmans, 1914)

S.S N	I	II	III	IV
	Luminosity of spectrum with maximum of 100 formed by white light		Absolute luminosity of spectrum in II compared with I	Percentage of III compared with I
	Of 1 candle at 1 foot distant from screen	Of 0 0008 candle at 1 foot distant from screen		
56	32	2	0·0016	0·00005
52	75	6 1	0 0048	0 000017
48	100	23	0 0184	0·00018
44	75	75	0 0600	0·0008
40	37	99 5	0 0796	0 0021
36	18	70	0 0560	0 003
32	4	39	0·0812	0 008
28	1	19	0·0152	0 015
26	0 6	13	0·0104	0 017

hermaphrodite processes which more or less replace them in the retina. So far, we have from the present enquiry gathered that the fourth sensation is very small in the spectrum colours of an ordinary bright spectrum when falling on the centre of the retina—so small, indeed, that the sensations are 100,000 times more intense than any cone effect, and so do not appreciably affect the sensation curves which are given in the 'Philosophical Transactions.'

This paper is sent in to the Royal Society after some delay Its communication was delayed by the fact that Dr (now Lieut.-Colonel) W. Watson was commandeered as scientific adviser at the Front Up to the time of his leaving the country, the author and Colonel Watson were in the thick of some interesting researches Amongst others was a repetition of the experiments, with slight alterations, which are given above In this repetition, instead of leaving the reflected white light used in the apparatus unaltered and altering the intensity of the colour to match the luminosity, the intensity of the spectrum colours in each set of observations remained unaltered, and the matches of luminosity were made by alteration in the intensity of the white The colour-patch apparatus was used in its usual way, and without any external reflection of beams, as shown in the above observations We hope that, when the war is over, these new results will be published, showing that similar results were obtained to those above, the difference being that the intersection of equal area curves took place slightly nearer the most refrangible part of the spectrum Again, we had made a partial examination of some congenital night-blind cases. The remarks as to the night-blind apply to the central part of the retina.

One point of interest connected with these examinations may be mentioned We found that, without any restriction to the fovea in diminishing the intensity of a spectrum beam, the colour and light were lost in them at the same time, the colour and light disappearing at the same intensity as the colour disappears in the normal eye. This is similar to the effect of receiving the spot of light on the fovea of Class I retina, which is supposed to be coneless

If this be true, then, the night-blind, according to the foreign theory, possess only cones in the whole retina

We propose at some future date to give some account of what happens at the periphery of the retina of these congenital night-blind cases

Appendix

S S N	λ	S S N	λ
60	6728	40	5270
58	6521	38	5172
56	6330	36	5085
54	6152	34	5002
52	5996	32	4924
50	5850	30	4848
48	5720	28	4778
46	5596	26	4707
44	5481	24	4630
42	5372	22	4578

Note by Prof S W J SMITH, F R S.

The important result illustrated by fig 2 that the equal area curves have, approximately, a common ordinate at the scale reading $s = 44$ admits of a simple interpretation, which may be formulated in the following manner.

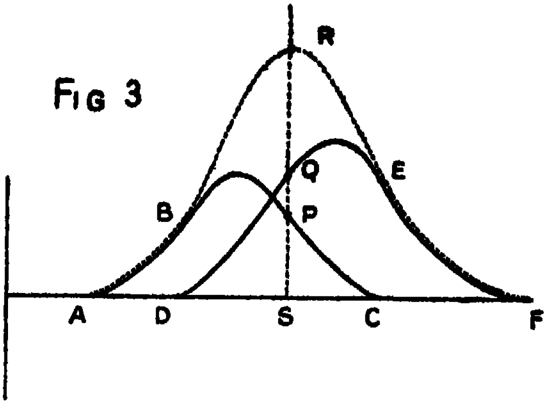
Suppose that the dotted curve ABREF is obtained by plotting the sum of the ordinates (at each abscissa) of two curves ABC and DEF

If these curves (ABC and DEF) overlap it is obvious that, whatever the respective areas included between them and AF may be, an abscissa S can be found which satisfies the condition

$$\frac{\mathrm{SP}}{\text{area ABC}} = \frac{\mathrm{SQ}}{\text{area DEF}} = \frac{\mathrm{SP}+\mathrm{SQ}}{\text{area ABC}+\text{area DEF}} = \frac{\mathrm{SR}}{\text{area ABREF}}$$

Suppose now that a second curve AB′R′E′F is obtained by plotting the sums of the ordinates of two curves AB′C and DE′F Let AB′C be such that each ordinate is α times the corresponding ordinate of ABC, and let DE′F be such

FIG 3



that each ordinate is β times the corresponding ordinate of DEF Then we shall obviously have

$$\frac{SP'}{\text{area } AB'C} = \frac{SP}{\text{area } ABC} \quad \text{and} \quad \frac{SQ'}{\text{area } DE'F} = \frac{SQ}{\text{area } DEF}$$

whence

$$\frac{SP' + SQ'}{\text{area } AB'C + \text{area } DE'F} = \frac{SP + SQ}{\text{area } ABC + \text{area } DEF},$$

or

$$\frac{SR'}{\text{area } AB'R'E'F} = \frac{SR}{\text{area } ABREF}$$

Hence if we plot the curves to such scales that the areas AB'R'E'F and ABREF are equal we shall at the same time make

$$SR' = SR$$

In other words, *the curves will have a common ordinate at* S

This result suggests that the luminosity curves of fig 2 arise from the fact that, at each scale reading s, the sum of two luminosities x_a and x_b is recorded It suggests also that the curves representing the variations of x_a and x_b with s remain always of the same type so long as the measurements are made at constant total illumination.

When this constant illumination is decreased continuously it would appear that one of the luminosities, say x_a, becomes continuously less important than the other, say x_b. Finally, the luminosity curve practically coincides with that of the "B sensation" to which, we may suppose, x_b is due

Conversely, when the illumination is as bright as possible, x_b becomes negligible in comparison with x_a and the luminosity curve exhibits the typical variation, with s, of the luminosity x_a due to the "A sensation"

If this interpretation be correct (and it seems to be what is intended) the B sensation would be what the Author calls the "fourth" (colourless) sensation, while the A sensation would be, presumably, the sum of the other three. Further it should be possible to deduce, from the two extreme curves, the forms of all observable intermediate curves.

Taking the curves of fig 2 marked 0° and 240°, respectively, to represent sufficiently nearly the forms of the curves of the A and B sensations, the dotted curves shown in fig 4 (attached) can be calculated.

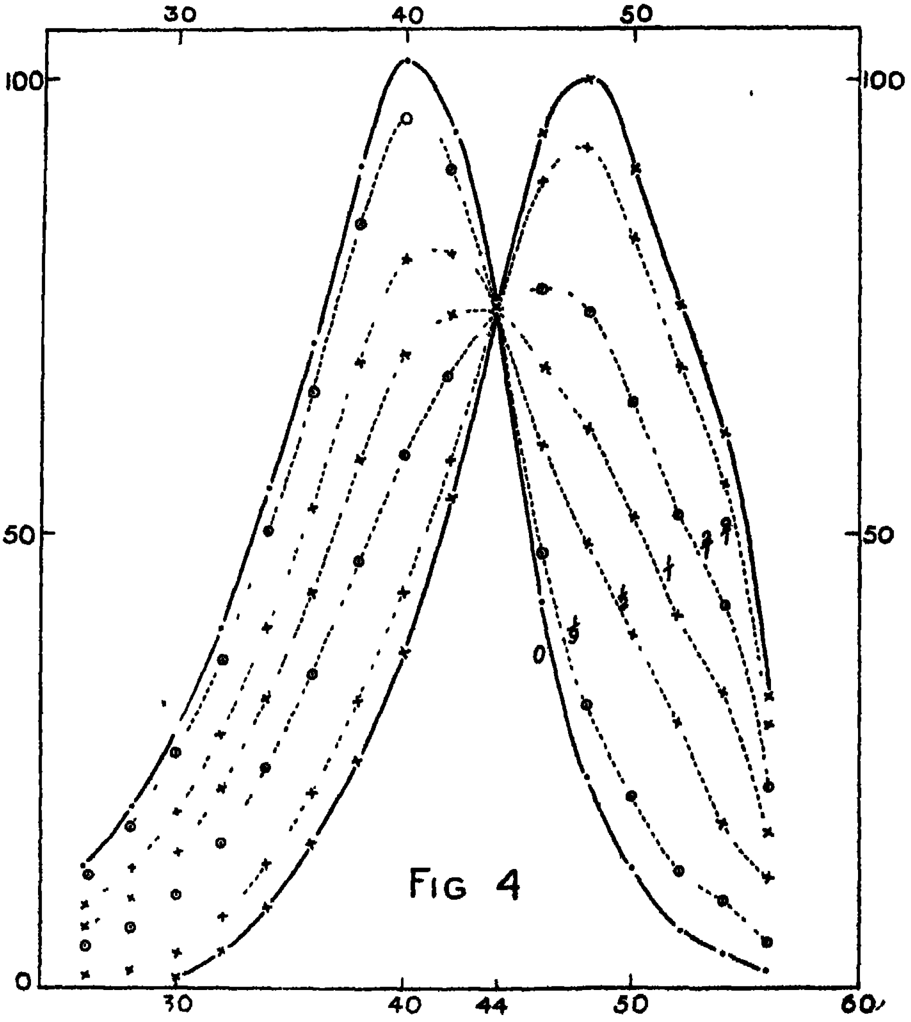


FIG 4

They represent results obtainable by combining the A and B sensations in different relative proportions The assumed ratios of the areas of the A and B curves are, respectively,

1·0, 9 1, 2:1, 1 1, 1 2, 1·9 and 0:1

It will be seen that the calculated results are not very dissimilar from those obtained experimentally and give reasonable support to the interpretation put forward

These curves would serve also to show, more exactly than is shown in the text, the relative insignificance of the fourth-sensation-contribution to luminosity curves of ordinary brightness They form also a more concrete presentment of the argument in favour of the existence of this fourth sensation.

Periodic Disturbance of Level arising from the Load of Neighbouring Oceanic Tides

By K Terazawa

(Communicated by Sir Joseph Larmor, F R.S Received April 17, 1916)

(Abstract)

In Hecker's observations on the lunar deflection of gravity, the force apparently acting on the pendulum at Potsdam is a larger fraction of the moon's direct attraction when it acts towards east or west than when it acts towards north or south A similar result has been found by Michelson in his observations of the lunar perturbation of water-level at Chicago A calculation is here made to ascertain to what extent the tilting of the ground caused by the excess pressure of the tide in the North Atlantic is important for the explanation of this geodynamical discrepancy Replacing the North Atlantic by a circular basin of radius 2000 kilom., taking the position of Chicago to be 1000 kilom from the coast, and the rigidity of the earth to be 6×10^{11} C G S, it is found that the attraction effect of a uniform tide per metre of height is about 0 0024″, while its tilting effect is as much as 0 0069″, the maximum of the direct lunar attraction being 0 017″. If the surface of tide is ellipsoidal, shelving towards the coast, nearly the same result is reached for the same-mean tidal height.

Note by Communicator.

[*July* 16.—The following proposition, which is implied in the paper, deserves explicit statement it is readily proved for a load localised at a point,* and follows generally by superposition A load is distributed anyhow on the earth's surface, over a region not too large to be considered as plane, for each locality it alters the level of the surface by its weight, and the direction of the vertical by its horizontal attraction; for material that is homogeneous elastically, the ratio of these two effects is in all cases the same, being $[(1-\sigma)/\mu](g/\gamma)(g/2\pi)$, where μ is the rigidity of the material and σ its ratio of lateral contraction to elongation, g is gravity, and g/γ is the mass of the earth divided by the square of its radius.* Thus, for steel, taking $\sigma = 0{\cdot}27$, $\mu = 8 \times 10^{11}$, the deflection of the surface is always and everywhere 2·1 times the deflection of the vertical, the total effect being their sum. For glass the ratio is 6·7.

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* *Cf.* Love's 'Theory of Elasticity,' para. 135

The complete results for various distributions of load can be written down at once from known expressions for cases of attraction: for example, the case of elliptic loading (over any elliptic area), which is considered at the end of the paper.—J. L.]

Historical Note on a Relation between the Gravitational Attraction Exercised and the Elastic Depression Caused by Load on the Plane Surface of an Isotropic Elastic Solid.

By C. Chree, F.R.S.

(Received August 15, 1917.)

So far as is known, the result to which the present note refers was first given in the special form which it assumes for an incompressible material in a note written by Sir G. H. Darwin entitled, "On Variations in the Vertical due to Elasticity of the Earth's Surface," included in the British Association Report for 1882 as an Appendix to the Report of a Committee appointed for the "Measurement of the Lunar Disturbance of Gravity." Sir G. H. Darwin and Lord Kelvin (then Sir William Thomson) were members of this Committee.

On p. 108 (*loc. cit.*) Darwin writes, "Before proceeding further I shall prove a very remarkable relation between the slope of the surface of an elastic horizontal plane and the deflection of the plumb line caused by the direct attraction of the weight producing that slope. This relation was pointed out to me by Sir William Thomson, when I told him of the investigation on which

---

* Since writing the above I have found that this general relation was stated by Dr. C. Chree in 1897 ('Phil. Mag.,' vol. 43, p. 177), with some particular applications to cases of rectangular loaded areas; like results had previously been given by Sir G. Darwin for the special case of a system of parallel ridges.

I was engaged, but I am alone responsible for the proof as here given. He writes that he finds that it is not confined simply to the case where the solid is incompressible, but in this paper it will only be proved for that case."

To understand the result it is necessary to mention that Darwin assumed that so far as the phenomena in question were concerned, the earth's surface might be regarded as plane, except for the presence of mountains and valleys, which he regarded as equivalent to a load varying as $\cos z/b$ Darwin used the terms "deflection" and "slope"—which I shall presently distinguish as ψ_2 and ψ_1—to indicate respectively the change in the direction of gravity due to the direct attraction, and the slope introduced by the elastic depression He also employed ν for the rigidity of the incompressible material, a for the radius and δ for the mean density of the earth, and g for the acceleration of gravity.

After obtaining certain formulæ he proceeds, "Therefore deflection bears to slope the same ratio as ν/g to $\frac{1}{3}a\delta$ This ratio is independent of the wave-length $2\pi b$ of the undulating surface, of the position of the origin, and of the azimuth in the plane of the line normal to the ridges and valleys Therefore the proposition is true of any combination whatever of harmonic undulations, and as any inequality may be built up of harmonic undulations, it is generally true of inequalities of any shape whatever."

It would appear that it was Lord Kelvin who first noticed the relation Whether he had arrived at it quite independently of and prior to Darwin's investigation, having actually solved the problem presented by compressible isotropic material and reached a definite formula, or whether he simply noticed that the result would follow for an incompressible material from Darwin's formulæ, and inferred by general reasoning that it was not confined to incompressible material, it is impossible to say If he did reach a definite formula, apparently he did not communicate it to Sir G H Darwin The latter's remarks are not explicit, but they certainly suggest that he was unaware that the rigidity is not the only elastic constant involved when the material is compressible In his numerical applications he employs for examples the rigidities of glass and steel, without any explicit warning that in actual glass and steel the results would have been widely different

The earliest publication of the result for ordinary compressible isotropic material was, I believe, in a paper "Applications of Physics and Mathematics to Seismology," read before the Physical Society of London in December, 1896

Equation (9), p. 178, *loc cit*, gives explicitly $\psi_1/\psi_2 = (1-\eta)g^2/2\pi n\gamma$, where γ is the gravitational constant, while n and η denote the rigidity and Poisson's ratio of the material. If we put $\eta = \frac{1}{2}$, the appropriate value for incompressible material, and write $g/\frac{4}{3}\pi\delta a$ for γ, and ν for n, we have Darwin's result. If, however, we take $\frac{1}{4}$ instead of $\frac{1}{2}$ for η, keeping the rigidity unaltered, we raise the value of ψ_1/ψ_2 by 50 per cent, so the question of the compressibility of the material is not unimportant. The above result agrees with Sir Joseph Larmor's (p. 14, *supra*) and with that obtained by combining Terazawa's equations (6) and (7).[*]

I was not aware of the existence of Sir G. H. Darwin's paper until long after my paper was published, and did not notice the reference in it to Lord Kelvin until the publication of Terazawa's paper led me to restudy the problem.

Experiments on Tribo-Electricity. I.—*The Tribo-Electric Series.*

By P. E. Shaw, B.A., D.Sc., University College, Nottingham.

(Communicated by Prof. E. H. Barton, F.R.S. Received May 24, 1917.)

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The term "Tribo-electricity" is used in O. D. Chwolson's 'Traité de Physique' (1). This term is a convenient equivalent for frictional electricity (τριβή = a rubbing).

I.—*Historical.*

Little exact knowledge of tribo-electricity has yet been accumulated, and this subject has certainly not been raised to the dignity of a quantitative science. In the present paper will be found an account of experiments in which the conditions of the solid bodies rubbed together have been greatly varied. Thus, temperature has been changed both before and during friction, the surfaces used have been rubbed with considerable pressure while hot; they have, wherever possible, been rubbed together when flexed; and they have been prepared before rubbing by being ground and polished in various ways. Much information has thus come to light as to the electrical surface conditions of a variety of solids.

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[*] 'Phil. Trans.,' A, vol. 217, p. 37.

Before setting forth the methods and results of this paper, let us note the information previously available Good summaries of the subject will be found in Wiedemann's 'Electricitat' (2) and Chwolson's treatise (1)

The data on the subject altogether lack coherency. The electrical effects obtained by friction had been so often found to be irregular—an irregularity which in great part I explain in this paper—that the subject was in general avoided One obvious source of uncertainty of action is that rubbing necessarily affects, perhaps destroys, the surfaces under investigation, so that friction, especially if violent, may, in effect, give rise to surfaces of a new kind

As to recent research on the subject, Hésèhous (3), in the opening years of this century (1901–5), brought to light in a series of papers, some new phenomena, and made an attempt to reduce the known facts to some sort of order He suggested certain generalisations as follows (1) a polished surface is always + to a matt surface of the same material, (2) a more dense surface is always + to a less dense one of the same material, (3) dust proceeding from a body is always – to the solid substance it leaves He also made investigations of the action of radium, and he found some materials, *e g*, glass, are made more +-forming, whereas others, *e g*, ebonite, are made more —-forming by the discharge Further, Hésèhous attempted to bring tribo-electric phenomena within the scope of the electron theory He supposed that electrons pass from the more dense to the less dense of any two bodies brought into contact, this movement of the electrons being facilitated by the change, due to contact, in the surface tension of both bodies at the point of contact Of the above generalisations, the first and second are shown below to be incorrect, the others have not been directly put to the test

A few years ago some quantitative researches (perhaps the first ever made on tribo-electricity) came from Bangor, North Wales from Morris-Owen in 1909 (4), and from W. Morris-Jones in 1915 (5) The apparatus used in the two cases was of the same type It was arranged that not only the charge (Q) produced, but also the work (W) done in producing it, and the pressure (P) applied in the friction between the solids rubbed, could be actually measured throughout an experiment Simple relations were found connecting Q, W, and P

In 1915 I investigated the anomalous effects obtained from glass after heating (6). It was found that glass, which is + to silk, becomes temporarily – to it after being raised to sufficiently high temperature Further, this anomalous effect was found in a great number of other solids, including metals and ebonite, if rubbed with suitable materials The general conclusion was reached that this behaviour, after heating, is common to all solids

The last-named research was the starting part of the present experiments. Its methods have been developed and its conclusions confirmed, while the range of enquiry has been extended to several physical changes other than thermal

II — *Various Short Tribo-electric Series*

In the early pages of most books on electricity is found a series of solid materials arranged in order in such wise that any one of the materials becomes + if rubbed with any other lower down in the series The arrangement of materials by different authorities ranging over the last 150 years may be here contrasted In the last list I take from my Table (p 32) only such materials as are mentioned in the other lists

Wilcke (1759)	Faraday (1840)	Jamin and Bouty's 'Physique' (1891)	Shaw (1917)
Glass	Cat's fur	Cat's fur	Glass
Wool	Wool	Glass	Wool
Quills	Quills	Wool	Cat's fur
Wood	Flint glass	Feathers	Pb
Paper	Cotton	Wood	Silk
Ground glass	Linen	Paper	Paper
Pb	Silk	Silk	Cotton
Sulphur	Hand	Resin	Wood, Fe
Metals	Wood	Ground glass	Ground glass
	Fe, Cu, Ag, Pb		Resin
	Sulphur		Cu, Ag
			Sulphur

If lines were drawn joining like materials in the different lists, there would be many intersections of these lines, showing how differently the solids are arranged. The many disagreements thus displayed show how necessary it is to standardise the materials used. As will be shown in the sequel, it is probable that these dissimilarities arise from the use of different methods or of different materials, or of materials whose surfaces differ in physical condition, rather than from personal errors of observation This kind of research may seem easy it is easy to get some results of sorts, but it requires considerable practice and special knowledge of surface conditions to obtain results which will stand repeated tests The series which I give in the Table (p 32) is the last of some dozen or so of lists drawn up during many months. At first, errors were made through the employment of faulty methods or from ignorance of the many factors involved It took considerable time, as one's skill increased, to sort out the materials into their correct places. When the series was considered correct, I prepared a complete new set of surfaces, and checked the series by going over the whole ground again with these fresh materials

My complete series is shown in the Table (p 32) This includes only fairly common solids and one liquid, mercury (whose peculiar action is described later, p 25)

It is desirable to have a long list, such as this, which contains about seventy-eight materials, for, by including many materials, we obtain more steps from extreme + at the top to extreme – at the bottom If the steps are many, the gradations will in general be less, and thus smaller differences between substances can be detected than would be possible if only a few steps occur, as in Wilcke's list of nine materials

In the present section we need only notice column 2, which shows the materials used in normal order These are in all cases the ordinary commercial article, but, in order to establish standard materials for future reference, particulars will be given in some cases

Furs—In rabbit's fur the hairs are clipped short, in imitation of ermine In opposum's and cat's fur the hairs are of natural length These furs have been selected from a great number, as giving convenient steps between places 1 and 12 in the Table

Vitreous Silica, Quartz, and the Glasses—In places 3 to 9 are shown no less than seven specimens of this family The quartz is the natural material to which has been given an artificially polished surface The vitreous silica used has surface smooth from fusion Three glass surfaces, including glazed porcelain, are smooth from fusion, one is artificially polished, and one surface (place 9) is the edge of thick plate glass, obtained after the plate has been cut by diamond and broken in the usual way

Wool, Silk, Cotton—Care must be taken in using fabrics purporting to be made of these materials, especially in the case of silk, which is so often adulterated with other yarn The standard materials in which I trusted are skeins of yarn The silk and wool came from yarn merchants, guaranteed pure. The silk has the natural gum washed quite free from it The cotton used is pure white cotton-wool

Mica, Calcite—The natural surfaces of fresh specimens are used after recent cleavage The surfaces of salt crystals are scraped before use

Metals and Alloys—Twenty-three materials are quoted Their surfaces are used scraped or rubbed with sand-paper, or, where possible, burnished It is important to avoid surface oxide, sulphide, etc, but no differences have been detected in any metal surface on account of the above different methods of cleaning

Felt (mixture of hair and wool and other fibres)—The specimens are of close texture, about 5 mm thick The material is such as is used for carpeting

Vulcanised Fibre—This red composite material is supplied rolled in sheets and tubes. A tube is polished by rotation in the lathe under the pressure of another tube of the material No liquid or other foreign material is used in the process

Woods—Ebony is the only wood of many tested, which does not occupy place 19 The specimens are planed up smooth and parallel to the grain

Shellac, Sealing Wax, Resin, Sulphur, are melted and allowed to solidify on glass tubes The two former must not be raised far above the melting point in the process This care is necessary to avoid disintegration

Ebonite—The solid rod is polished in the same way as vulcanised fibre

Carbon—Gas carbon rods, as used for arc lights, and charcoal of various kinds have all the same place (29) in the Table

Celluloid—This also is best in its purest form in limpid sheet highly polished. The opaque, less pure, form is higher, about place 29

Indiarubber—Under this head is implied the pure, translucent, very elastic, sheet rubber used for mending the inner tube of bicycles There are many forms of impure rubber, for instance, white rubber is much higher in the Table (place 29)

It is noteworthy that differences of behaviour are easily observable between insulating materials all of one class In order to indicate how differences in the surfaces not detectable by any other means are easily shown by tribo-electric effects, take a few cases —

(*a*) A highly insulating material, such as celluloid or mica sheet, being cut into two pieces, it will generally be found that one piece will be + and the other − when rubbed together There is nothing whatever in their appearance to show that these pieces cut from the same sheet are in any way different

(*b*) Nine different furs tested were found to take definite positions between places 1 and 13 in the Table, various hard siliceous materials falling into place between them as shown

Furs		Woods	
(Asbestos)	Black wolf	(Matt mica)	Beech
Rabbit	Caracule	Red deal	Jarra
Fox	(Vitreous silica)	Canary	Walnut
(Glass)	(Glass)	Box	Mahogany
Opossum	Astrachan	White deal.	Lignum vitæ
Mink	Cat's skin	Rosewood.	(Matt glass)
Skunk		Oak	Ebony

(c) Of 12 different woods, all except one (ebony) form one group, place 19, yet when rubbed together they can be arranged in definite sequence like furs The lists of furs and wood are given on p 20

III —*Apparatus and Methods*

The confusion which has been a feature of this subject in the past may be attributed to the uncertainty as to materials used, their past treatment, and the methods employed in experimenting on them With the attempt to standardise the methods, or at least to specify my own processes, I will give particulars throughout, even though these are admittedly simple.

The materials used may be divided into conductors and insulators The latter can, in general, be used unmounted, the former (including partial conductors, *e g*, wood, cotton fabric, asbestos) must be mounted on a non-conductor In general, those rods, tubes, or sheets, which have to be mounted are attached by sealing-wax to the inside or outside of glass tubes. It is necessary to have sufficient distance, say 10 cm, between the hand and any conductor undergoing test, for if the hand be nearer it may remove, by induction, any free charge on the material. Easily fused materials, *e g*, shellac and paraffin wax, are mounted in a solid lump at the end of a glass tube Lumps of material, such as minerals or carbon, are held in a clip specially made of ebonite Mercury is used in a cylinder into which any material can be plunged

The gold-leaf electroscope used is provided with a shallow horizontal cylinder open at both ends For charging the electroscope a wire is arranged close to the cylinder. This wire is connected with a live wire of an electric light circuit at 205 volts As a means of discharging the surfaces, a small bunsen flame is kept alight in front of the observer The flame itself should never be used as a discharging agent, since most surfaces become abnormal very quickly in the flame Round the flame is a zone, the discharge zone of air, so highly ionised that any charged body moved about at, say, 10 cm from the flame for a few seconds becomes completely discharged The gases rising from the flame are more highly ionised than the air at its side, but there is great danger, if the body be plunged in these gases, that the temperature of the body will rise so far that it will become abnormal.

In investigating critical temperatures (column 4 in Table, p 32), the substance is heated by being placed on the upper shelf of a copper oven, the oven being swaddled in sheet asbestos. The door of the oven is closed and the temperature is raised by bunsen burners underneath, and is recorded by a thermometer inserted through the top of the oven. It is easy to raise the temperature slowly to, say, 320° C, which is high enough for present purposes

As to the condition of the surfaces used, cleanliness is desirable, though apart from chemical impurity, *e.g.*, oxides, sulphides, on metals, dirt does not influence results as much as might be supposed. It is clear that if, for instance, there is a layer of dust on an ebonite rod, this, when rubbed with silk or paper, will be brushed away at the first stroke, and the true surfaces will be brought into contact But metals must be used free of corrosion by the application of sand paper or by the process of filing or scraping Apart from metals, it is well to keep such materials as ebonite, gutta-percha, and vulcanised fibre clean by frequent scraping

It is important to observe that materials, *e g*, mica, glass, or celluloid, supposed to be polished are really used in this state Polished celluloid and ebonite are easily rendered matt at the place of rubbing when pressed on some hard surfaces

We must observe a uniform method of rubbing the surfaces Consider, as an example, the case of quartz–silk If the silk be brushed over the quartz it becomes +, but if rubbed hard on it its charge becomes – These effects will generally be found when hard surfaces are rubbed with flexible ones, such as fabrics and furs To take another instance, if a mica sheet be beaten on a glass tube it becomes slightly +, but if rubbed on it strongly – The latter is the standard way, which I always use, hence in the above examples we place quartz above silk and mica below glass in the Table Furs themselves are somewhat uncertain The same specimen of opossum fur may at one end be + to vitreous silica, and at the other be – to it In such cases as this the experimenter must learn the peculiarities, if any, of the materials he uses

As will appear later the surfaces of most materials are susceptible of several states, all more or less stable Thus, if a smooth glass rod be under test it may have its position in the Table at place 3, 5, 8, or 26. But repeated rubbing, gentle heating, and the slow changes which occur with lapse of time will bring the glass to a definite position, say place 5, which is taken to be the position of glass in its most stable state

It might be thought desirable to use special devices to clear of adsorbed layers the surfaces about to be rubbed For instance, well-known methods could be suggested, such as (1) tearing off the surface by gelatine films, or (2) scraping *in vacuo* These processes have been found necessary in other surface experiments, such as surface tension and photo-electricity. But there is an essential difference between the delicate forces at work in the latter subjects and the rough impact of solid on solid, such as occurs in tribo-electric experiments In this case, the true surfaces would bear on one another The actual bearing surface of one solid on another would be so small that the pressure per square millimetre would be enormous, and would suffice to break through any thin outer foreign film, especially one of a liquid condition

IV.—*The Full Tribo-electric Series*

At the end of this paper is a Table which gives in compact form the experimental results of this investigation In the second column are shown the materials used, arranged in 36 standard places Those in capital type I have found more useful than the rest The metals are italicised. The whole list from Asbestos at the + end to Indiarubber at the — end may be likened to a staircase of 36 steps, which steps are, of course, unequal in depth to an unknown degree The list itself could be expanded apparently without limit Group B follows Group A without break

Column 3 contains such materials, bright when in column 2, as can be rendered matt by some abrasive, *e g* sand paper It will be observed what a vast disparity there is, in some cases, between the surfaces of the same material when polished and when matt. Thus, the members of the glass family, normally round about place 5, are, when matt, at place 20

In column 4 are the places of the materials when these have had their surfaces raised above the critical temperature, which temperature where it has been found is inserted in brackets I call a surface abnormal when, as a result of being heated above a certain temperature, it moves from its normal place in column 2 to some other place in column 4. This particular temperature I call the Critical Temperature (C T) There are two ways of making a surface abnormal (1) By raising the substance as a whole to the C T in the air-bath, (2) by passing the surface rapidly through the hot body of a flame In the latter case (which, of course, is not applicable to all materials), though the interior of the substance may remain cool, the surface particles will be raised to the C T in a few seconds or less. Experience shows that the abnormal place of a body in column 4 is always the same by whichever of the above processes this condition is obtained Suppose a body has been raised to the C T, and on being rubbed at that temperature is found to be abnormal at place 26 It will remain abnormal when cold and still occupy place 26 in the Table

It will be observed that the critical temperatures range from 300° for quartz (a hard substance) to 70° for vulcanised fibre (a soft compost) In general, the minerals have high critical temperatures, while in metals the C.T ranges from 240° to 130°. As to the accuracy of these figures, they are trustworthy to 5° in the case of glass and vitreous silica, which have received special attention For the other materials, the figures are probably reliable only to 10° or 15°. There is some doubt about the case of tin It melts at 230°, and when raised to 210° it certainly took the place shown in column 4, but as at that temperature the surface was yellow with oxide, this particular result must be considered doubtful

Column 5 has only six entries Many more have been observed, but as they are not quite consistent, they are omitted. Suppose a sheet of cork or ebonite be pressed and rubbed by an ordinary "flat-iron" when at a temperature, say, about 150°, it will descend in position as shown in the Table Another method is, after warming a sheet, to place it between two sheets of its own material, and press the whole together while the hot sheet is pulled from between the others This is called the Pressed-Hot (P H) state It can be removed, in the case of hard bodies, such as glass, by passing the surface dexterously through the flame for a short time If the surface be heated too long, it will go further and become abnormal

Columns 6 and 7 should be considered together In experiments to be described later (p 26), it is shown that, under certain conditions of stress, a convex surface and a concave surface, both of one material, become charged in opposite senses It will be seen that mica behaves in one way, paper, ebonite, and celluloid in the contrary way

Thus the Table (p 32) shows how readily the surfaces of a great variety of materials change their places in the series. A scrutiny will reveal that these changes occur according to very simple laws Divide the second column in two groups, A, B, by drawing a demarcation line below place 14. Then it is observed that materials in Group A all move one way; those in Group B all move the contrary way. The laws of these movements may be tabulated thus —

	Matt	Abnormal	P H	Convex	Concave
Group A	Down	Down	Up	Down	Up
Group B	Up	Up	Down	Up	Down.

Another peculiarity in this grouping is that all the animal materials occur in Group A, all the vegetable materials in Group B.

The many empty spaces occurring in columns 3–7 are not blank through inadvertence. Practically all the suitable materials have been tested for these effects, but it has not been possible to perceive effects for all the diverse substances in the Table In no case has any distinct result been suppressed. But there have been a few cases where effects have been detected, but have been subsequently ignored, as they would not repeat Thus glass, silica, and quartz have sometimes been obtained P.H at place 2—but this effect is not consistent The contrary influences found in the two groups I call *duality*

Notes on the Behaviour of Mercury

Before concluding the description of the Table (p 32), mention should be made of the anomalous behaviour of mercury It will be seen that this metal is placed with platinum and gold (place 34), and also high up (place 2) The effects observed are as follows —

(1) When rods of various materials are gently pushed into mercury, every one in the series, except celluloid and indiarubber, will become charged + Thus we assign to mercury place 34 If, however, the rods be plunged quickly into the mercury or stirred in it, they all, except asbestos, become — Let us denote mercury when the friction is violent as (mercury′). Its place is 2 Possibly violence causes the rods to become extremely abnormal in contact with mercury Lenard's theory (9) regarding "waterfall" electricity may be recalled In the case of water, he supposes there is an electrified double layer, the external one being — and the internal + If there be a similar double layer in mercury, the external one would be + and the internal —

(2) Abnormal glass, polished or matt, is — to mercury even when immersed in it gently Yet the same specimens of glass are + to gutta-percha, which itself is + to mercury. The same effect will be found for abnormal mica and other materials Supposing the charges obtained when bodies are plunged into mercury are really due to friction, then the effect just recorded seems to mean that the abnormal surfaces are in reality below place 34, for such gentle friction as mercury provides, but that rubbing with solid surfaces so flattens, or otherwise affects, the surface particles as to bring the abnormal surface up, in the case of glass, to place 26

(3) A glass rod plunged into mercury becomes highly charged On repeating the process several times, however, the + charge becomes very small But the original excitability is restored by rubbing the glass by hand or by a cotton duster It is possible that the glass is rendered abnormal by mercury, and made normal again by rubbing.

(4) If mercury be raised in temperature, its power of exciting charges on solids immersed in it gradually lessens as temperature rises, returning again entirely at normal temperature Thus, at 110°, the charge produced on ebonite is small, whether immersion be gentle or violent. Glass in mercury at the above temperature receives no charge for gentle immersion, but

receives a large charge when immersion is violent When the mercury is at 200°, it will give no charge to any body under any conditions tried

Notes of the Peculiar Behaviour of some Flexible Sheets

(1) Take two strips, cut from the same sheet of transparent, polished celluloid Place one sheet on the other, and draw them between thumb and first finger, bending the strips as they pass between the fingers Now place the strips, still in contact, for a second in the discharge zone of a flame to remove the — charges on their outer surfaces due to friction of the material on the fingers On separating the strips, that one of the surfaces in contact which was convex when flexed is +, the other surface in contact, the concave one, being — An effect of this sort was pointed out by Jamieson (7) Hence opposite charges arise from complex strains on the opposed surfaces There is no special significance in using the fingers, identical effects being found if the fingers are replaced by other rubbers, say pieces of wood

The above effects are recorded in the Table (p 32), columns 6 and 7

(2) Similar effects are found for two sheets of ebonite or cork, of filter paper or mica But it is of interest to notice that mica, which is in Group A, acts in one way, whereas the other materials, which are in Group B, behave in the contrary way for convex and concave surfaces respectively

(3) A variant of the above experiment is curious Place the two celluloid strips in contact, and draw them between the fingers as before, but in this case leave them unflexed in the process After removing the external — charges as before in the discharge zone, the strips will be found to cling together strongly, and, when tested, are found to be charged, one +, the other — This, of course, is a simpler experiment than (2), for now the two strips are treated identically throughout The two surfaces in contact must differ in composition or condition, so as to act on one another, in effect, as different materials It is a commonplace in tribo-electricity to find two surfaces apparently identical which, when rubbed together, excite one another with opposite charges The present effect seems to be merely an example of this kind

In experiment (1) the convex surface becomes always + to the concave, however the surface of the two strips be changed about But in experiment (3) the effect is peculiar, not to the material, but to a given specimen. if the surface is + in one position, it will be + always, whatever its position relative to the other surface. Thus, the effect in (1) depends on flexure, whereas in (3) it is simply characteristic of the materials used.

V.—*Theory*

On the simple electronic theory of tribo-electric phenomena there are free electrons close to the surface of solids which readily leave the surface when rubbed. The materials high up in the series are supposed to be characterised by having electrons specially free to leave. This theory, as it stands, does not account for the presence in the Table, mixed in among insulating bodies which are poor in free electrons, of metals and alloys, all peculiar for their richness in free electrons. Nor is it evident how it can explain the effects in columns 3–7 in the Table. These are striking and consistent enough to call for an explanation. A theory is wanted. Moreover, a definite concept of surface conditions is needed for continued successful investigation of the subject.

The outstanding results are (1) the order (not unmethodical) in which materials arrange themselves in column 2, (2) the duality found in columns 3–7 in the behaviour of the top and bottom sections of the series, (3) the principle of the critical temperature.

In commencing an enquiry into the basis of these effects the question arises as to any possible relation of chemical composition of the material to its position in the series. In the following list is given the composition of the materials used. They are arranged roughly in the order of column 2, except in those cases where like bodies are grouped together.

Material	Composition
GROUP A	
Asbestos	Magnesium silicate.
Furs	Compounds of C, H, O, N, possibly P, S. The surface of hairs consists of flattened dried cells, containing a dehydrated proteid, ceratin ($C_{27}H_{58}OH$)
Wool	
Silk	
Felt	
Hand	
Silica	SiO_2, with (in glasses) sometimes Na_2O, K_2O, CaO or Al_2O_3
Glass	
Quartz	
Mica	K, Al, SiO_2
Calcite	$CaCO_3$
Borax	$Na_2B_4O,10H_2O$
Fluor spar	CaF_2
Pb, Al, Zn, Cd, Mn, Cr	
GROUP B	
Vulcanised fibre	?
Filter paper	Cellulose $(C_6H_{10}O_5)_n$ with traces of mineral matter
Cotton	
Woods	
Cork	
Magnalium	(70 Al, 30 Mg)

Material	Composition
GROUP B—(*contd*)	
Paraffin wax	Hydrocarbons, *e g* $C_{30}H_{62}$
Potassium bichromate	$K_2Cr_2O_7$
Amber	Mixtures of compounds of C, H, O, of unknown formula Sealing wax sometimes has mineral matter in traces
Shellac	
Sealing wax	
Resin	
Potash alum	$KAl(SO_4)_2,12\,H_2O$
Chrome alum	$KCr(SO_4)_2,12\,H_2O$
Iron alum	$NH_4Fe(SO_4)_2,12\,H_2O$
Slate	Aluminium silicate (hydrated)
Ebonite	Hydrocarbons and sulphur
Indiarubber	
Gutta-percha	Similar to indiarubber, but without sulphur
Iron pyrites	FeS_2
Galena	PbS
C	
Cu, Sn, Sb, Co, Ni, Bi, Pd, Ag, As, Te, Pt, Hg, Au	
Celluloid	Nitro-cellulose (C, H, N, O) and camphor

It is easy to detect characteristics special to each group

Group A contains —

(1) Silicates, including the glass family, asbestos and mica

(2) All the animal materials

(3) The notably + metals and lead

Group B contains —

(1) No silica compounds, but a great variety of carbon compounds

(2) All the vegetable materials

(3) Many metals, including those notably −

There is no apparent gradation in density, but the hard silica compounds in Group A are in striking contrast to the soft bodies, especially the waxes, in Group B Hydrated bodies are found in the lower rather than the upper group The three alums are seen to be widely separated from one another in the order we should anticipate from the electro-potentials of their distinguishing metals.

Peculiar interest attaches to the elements, metallic (20 in number) and non-metallic (3 in number), in the series. Of the metals, all but two, lead and mercury, are arranged according to the electro-potential series of Ostwald (7A) Allowance must be made for the fact that owing to their conductivity metals rubbed together show no charges detectable by the present means, hence the metals are grouped in many cases as shown, no less than nine occurring at place 29. Mercury with electro-potential −1·03 is so nearly the same as silver (−1 05) and palladium (−1·07) that its displacement in the series relative to these two metals may be ignored This leaves lead as the only exception to the electro-potential order Lead is undoubtedly peculiar, for with potential −0 13 it is above, for instance, aluminium with potential +1 00 Three distinct specimens of commercially pure lead behaved alike, and no change in behaviour was found after the metal had been melted and when cool cleaned on the surface Commercial sheet lead is so pure that it is not considered necessary to try a specimen of special purity An independent reason for supposing lead to be in Group A is that when abnormal it descends in the series to about the position of undoubted Group A metals, *e g* aluminium and zinc

Apart from the metals, only three elementary solids, carbon, sulphur, and tellurium, have been tried These highly negative elements occur, as we should expect, near the bottom of the series One supposes that the other non-metallic elements, if obtainable in suitable solid form, would also take their place at or beyond the bottom of the list, and that the most strongly electro-positive elements, potassium and sodium, would occur high up in the series In general the basigenic elements come above those with oxygenic qualities, yet the property of forming very stable oxides, sulphides, etc, would seem to be a better criterion of tribo-electric position than base- or acid-forming power Thus if calcium or manganese be rubbed with their natural oxide on the surface they will take places far down in Group B, if cleaned, and the pure metal exposed, they are found in Group A Again, silica is a prominent compound at the top of the series, whereas the inert metals, platinum and gold, are at the bottom. Now, if acid-forming power were considered, silicon should be low in the series and silica also very low, certainly not far above platinum and gold These matters appear to be vitally important from the theoretical standpoint

As to the critical temperature, the change from the normal to the abnormal state appears to be sudden. A rise of 10° or so suffices to take the surface from one state to the other But glass at 220° (*i e* 23° below C T) has been found at place 20 in the series, as if it were in a transition state from normal to abnormal In the cases of some metals, *e.g.*, silver and copper, the abnormal state has been looked for, but not detected. It is clear that in most metals it would be hopeless to look for a critical temperature higher than, say 300°, as oxide would then form readily on the surface. Closer and more extensive research may reveal the significance of these critical temperatures

In scrutinising the series, one can detect nothing in the ordinary physical or chemical characteristics to elucidate the phenomenon of duality mentioned on p. 25.

The conditions at the actual surface of a solid are, of course, peculiar The particles there are more free than below the surface They are, in general, unprotected and, so to speak, raw and liable to strain, orientation, and disintegration by outside influences such as the impact of other solids Tribo-electric charges are derived from the actual surface , but whether from constituent parts of the atom or from interatomic or intermolecular sources is not decided The theory will be very complicated if, as is possible, the charges are derived from two or more of these sources. The atom is the simplest unit, and would appear to be the most likely source, but the question must be left open at present

Now, if two groups (*e g*, Groups A and B in the Table) of bodies behave in ways exactly contrary to one another, one might imagine each group to be characterised by its own kind of unit (whether atomic or molecular).

Suppose the characteristic of Group A is an α unit, that of Group B a β unit If we assign certain qualities to the units, then in any surface the units would decide how the surface would change its place in the series when its condition is changed According as the α or β units predominate in the material the action will be that of Group A or Group B in the Table on p 32 The case has been stated here in the most general way It would be premature to go further and outline the model which I have had in mind throughout the experiments, and which has successfully indicated the course to pursue in these investigations

Something should be said about the principle of the electric double layer by which many physicists, *e g* J J Thomson (8) and P. Lenard (9), have interpreted surface effects Amongst others, O W Richardson (10) has explained the fact that in thermionic experiments a definite amount of work must be spent to enable an electron to escape from a heated solid surface The view is generally held that the − layer is always outermost of the two layers Again, R. A Millikan (11) has given new support to the principle by proving the exactness of Einstein's formula $\frac{1}{2}mv^2 = h(\nu-\nu_0)$ for photo-electric effects on certain metals

Recently, J. Frenkel (12) has developed the idea Assuming that all positions and all orientations of an atom at the surface are equally probable, he explains the double-layer principle on the simplest basis. For if Rutherford's nuclear atoms be placed in all positions and all orientations at the surface, the electron orbits will in the aggregate project beyond the + nuclei, and there will be established a − layer outside a + layer.

Frenkel carries on the argument, successfully applying the principle to a great variety of surface effects. In short, the evidence in favour of the double-layer effect seems overwhelming.

However, the results of the present paper may be taken to show that whereas all orientations at the surface are equally probable for the abnormal state, and less so for the matt state, yet for the normal and pressed states the surface particles are orientated unequally, a quasi-mechanical strain being imposed (by rubbing) on the surface particles

As an analogy, imagine a row of pivoted magnetic needles all pointing N, but their centres in line E and W If the pole of a bar magnet be taken along the line near but not touching the needles, these will, if at suitable distances from one another, be left at the end of the operation pointing in line E and W The bar magnet has orientated them, and they are left in a stable condition of strain Jolting the system would cause a release from the strained state of the needles, just as heat causes the release from the normal to the abnormal state in the case of the tribo-electric experiments

Orientation of surface atoms would be tantamount to bringing the outer — layer and the inner + layer nearer to one another

Another type of atom, though not apparently one of the Rutherford pattern, might produce at the surface of a solid a double layer with the + layer outermost If this were so, we could explain the effects of Group A, by imagining the new type of atom predominant in those materials, and the effects of Group B by having the Rutherford atom predominant in the materials of this group

Whether, in the double layer, the — layer or the + layer be outermost at the surface, the layers would be most separated from one another in the abnormal state and least separated in the pressed state.

VI —*Conclusion.*

The endeavour is made to direct attention to a long-neglected field of research The results revealed are considered sufficiently novel for publication as they stand. With more refined experimental methods the subject will advance no doubt from its present stage, but the process will be slow Quantitative work on materials *in vacuo* cannot easily be done when, as here, considerable mechanical force has often to be applied to the materials But this is not the only reason for expecting slow progress The materials possible to be tested are unlimited, and generalisations should not be based on a few instances only

In the last few years the surface conditions of solids have been investigated from new standpoints. To mention two instances bearing on our experiments · H. L. Curtis (13) has observed enormous change (as much as one hundred million-fold) in the surface conductivity of insulators as the relative humidity of the air over the surfaces varies from 20 to 90 per cent

This brings in the moot subject of adsorption, which one would expect to have a bearing on tribo-electric effects Mr C Hayes and I have observed a direct connection between dampness of a surface and the sign of the charges it gives when rubbed. Again, J W French (14) has discovered that the surface of artificially polished glass to a depth of 5 μ is of a peculiar nature, differing in cohesion from the main underlying material Mr Hayes and I have found that the surface of polished glass has under some conditions a neutral state In this state it yields neither + nor − charges This effect is found only in the artificially polished material

Any theory on this subject to be acceptable should embrace not only the effects included in the Table, p 32, but also several others such as the varied results obtained (see p 22) according as one surface is brushed, rubbed, or struck by another The theory should take cognisance, also, of the peculiar behaviour of mercury (p 25), though as mercury is the only liquid used in these experiments its case is singular

This paper is meant to be the first of a succession on the subject of tribo-electricity Some quantitative work has already been done on various branches of the subject, *e g*, the influence on the surfaces about to be rubbed of (1) low vacua, (2) electric high-potential discharge, (3) various liquids Mr C Hayes, M Sc, has worked with me for a considerable time on these subjects I am glad to acknowledge valuable help in obtaining data for the present paper from him and from Miss H M Browning, B Sc I also wish to acknowledge my indebtedness to Prof G A Schott, D Sc, for valued advice as to theory

VII.—*Summary.*

(1) The tribo-electric series, in which solid materials are arranged in order according to the charge they acquire when rubbed together, is reliable with due precautions In the Table are shown the common solid metallic and non-metallic elements, interspersed with a great variety of compound materials.

(2) Most solids are found to alter their place in the series if heated above a certain temperature, which is specific for each material. This temperature is called the critical temperature The surface in its new condition is termed abnormal

(3) The series may be divided into an upper Group A and a lower Group B It is found that these groups have tendencies contrary to one another as the surfaces of the materials are rendered (*a*) matt, or (*b*) abnormal, or (*c*) pressed, or (*d*) flexed If under any of these agencies Group A becomes more +-forming, Group B becomes more −-*forming*, and *vice versâ*. This principle is here called duality It is possible that the

Table showing the Tribo-electric Normal Series, Column 2, with Variations for Matt, Abnormal, Pressed and Flexed States in Columns 3, 4, 5, 6, and 7 respectively The more important materials are in capitals Metals are italics

GROUP A

Place	Materials in normal series	Prepared matt surfaces	Abnormal (and critical temperature in ° C)	P H (p 24)	Flexed	
					Convex	Concave
	+					
1	ASBESTOS (sheet)					
2	RABBIT'S FUR Hair (Hg)'					
3	GLASS (Combn Tube)					
4	VITREOUS SILICA, OPOSSUM'S FUR					
5	GLASS (fusn)			MICA		MICA
6	MICA					
7	WOOL				MICA	
8	GLASS (pol), QUARTZ (pol), glazed porcelain					
9	GLASS (brn edge), ivory					
10	CALCITE		Wool	Zn		
11	CAT'S FUR					
12	*Ca*, *Mg*, *Pb*, fluor spar, borax					
13	SILK					
14	*Al*, *Mn*, *Zn*, *Cd*, *Cr*, FELT, hand, wash leather	CALCITE	Silk Cd (200)		F paper	

GROUP B

Place	Materials in normal series	Prepared matt surfaces	Abnormal (and critical temperature in ° C)	P H (p 24)	Flexed	
					Convex	Concave
15	FILTER PAPER					
16	VULCANISED FIBRE	V fibre	Ca (240) ?			F paper
17	COTTON		Calcite(270)			
18	Magnalium	MICA				
19	Potash alum rock salt satin spar	WOODS, V SILICA				
20	WOODS, *Fe*	Quartz		F paper		
21	Unglazed porcelain, sal ammoniac	GLASS COMBN TUBE	V fibre (70) Ebonite(100)	Woods		
22	Potassium bichromate PARAFFIN, tanned iron		Pb (240)			
23	CORK, ebony		Al (240) Zn (220)		Cork	
24	Amber	SEALING WAX	Fe (200)			Cork
25	Slate, chrome alum	RESIN SHELLAC		Cork		
26	SHELLAC, RESIN, SEALING WAX		V. silica 245 Glass (245) Mica (270) Quartz(300) Brass			
27		EBONITE	Ni (200) Sb (180)		Ebonite	
28	EBONITE		Sn (210) ?			
29	*Co*, *Ni*, *Sn*, *Cu*, *As*, *Bi*, *Sb*, *Ag*, *Pd*, Carbon, Te, eureka straw, copper sulphate, brass			Ebonite		Ebonite
30	PARA RUBBER, iron alum		Celluloid Au (180)			
31	GUTTA PERCHA		Pt (180) Indiarubber			
32		CELLULOID				
33	SULPHUR	SULPHUR				
34	*Pt*, *Hg*, *Au*				Celluloid	
35	CELLULOID					
36	INDIARUBBER			Celluloid		Celluloid
	—					

phenomena observed by Héséhous with radium may be another case of this duality of action, for glass is in Group A and ebonite in Group B

(4) Anomalous effects are observed when liquid mercury is used as one of the materials, its behaviour being quite unlike that of solid surfaces

(5) As to theory, it is suggested that the prevalent idea that the electric double layer, existing at the surface of solids, has the − layer outermost in all cases is incorrect Normally, the materials in Group A would have − outermost, those in Group B having + outermost Orientation of surface atoms would give rise to changes in the disposition of the two electric layers, and so account for observed effects

(6) Tribo-electricity undoubtedly affords a means, of extraordinary delicacy, of discriminating between materials apparently alike

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Revolving Fluid in the Atmosphere

By Sir Napier Shaw, F R S

(Received June 7, 1917)

In a paper read on December 8, 1916,* Lord Rayleigh makes an important contribution to the dynamics of revolving fluids, taking as "the starting point of part of his investigation," the paper by Dr Aitken on "The Dynamics of Cyclones and Anticyclones."† After setting out the general equations of motion of an inviscid fluid Lord Rayleigh says "for the present purpose we assume symmetry with respect to the axis of z so that u, v, w and P (assumed to be single valued) are independent of θ" I take that to be the definition of a revolving fluid for the purpose of the subject under consideration, and it is that form of motion, that is to say, motion which is symmetrical with regard to a vertical axis, that Lord Rayleigh had in mind when he wrote the opening sentence of the paper "So much of meteorology depends ultimately upon the dynamics of revolving fluid that it is desirable to formulate as clearly as possible such simple conclusions as are within our reach."

For most ordinary readers, meteorology in its dynamical aspect is quite rightly regarded merely as a useful synonym for "Cyclones and Anticyclones" Dr Aitken illustrates his views of the nature of cyclones and anticyclones by many interesting experiments in the dynamics of revolving masses, and when Lord Rayleigh is moved by Dr. Aitken's paper to set down clearly the conclusions that can be drawn from the theory of revolving fluids it is apparently with the hope that those conclusions may find their application in the phenomena exhibited by cyclones and anticyclones.

I must now confess that in the 20 years during which I have been professionally as well as personally concerned for the progress of dynamical meteorology and the explanation of the observed phenomena of cyclones and anticyclones I have deliberately avoided the discussion of the theory of revolving fluid, because I could not find in the weather maps, which are the basis of practical meteorology, any real case of revolving fluid, with suitable details of pressure, temperature, rain, etc, to which the theory might be applied The literature of meteorology contains so many examples of the application of elaborate dynamical and physical reasoning to systems of supposed facts which, as applied to the atmosphere, were nothing better than guessing, that a more searching analysis of the actual facts seemed to be the primary and most urgent demand A close examination, especially in the period between 1900 and 1905, of many typical groups of isobars as set out in a paper before the Royal Meteorological Society on the "Meteorological Aspects of the Storm of February 26–27, 1903,"* and in "The Life-History of Surface Air Currents"† made it manifest that the cyclones and anticyclones of our latitudes are not examples of revolving fluid in the particular sense referred to and therefore cannot be explained on the hypothesis that they are

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* 'Roy Soc Proc.,' A, vol 93, p. 148

† 'Proceedings of the Royal Society of Edinburgh,' vol. 36, p. 174 (1916).

I confess, too, that the conclusion arrived at was disappointing and depressing, for meteorologists in common with others had been, and, I fear, still are, in the habit of assuming from the obviously circular appearance of a well-developed cyclone on the map that there could be no mistake in supposing a cyclonic depression to be a case of motion of revolving fluid, perhaps a little distorted locally but still retaining the characteristic features, and in fact dynamically similar if not identical Uncertain whether the actual motion might not prove to be circular motion superposed upon some form of motion of translation, I have not hitherto felt sufficiently sure of my ground to meet that view with a categorical negative, but I think I may safely do so now

Making use of more recent conclusions I can perhaps best explain the situation by saying that if the motion of the air had really been motion of a revolving fluid symmetrical with regard to a vertical axis it would not have appeared in circular form on the map If it looks circular it is not a case of revolving fluid The explanation of this apparent paradox is that all the depressions of well-marked circular form in our latitudes are found to travel with a speed which is of the same order as the velocity of the wind of which it is composed. If the depressions are stationary they are ill-formed and irregular, the characteristic cyclonic features of strong winds with a definite centre are not to be found there But when the motion of translation is of the same order of magnitude as the winds the instantaneous motion is round a moving centre, and the actual motion with reference to the centre is the apparent uniform motion round the centre shown on the map combined vectorially with a velocity equal and opposite to that of the translation of the cyclone. Consequently the appearance of uniform and symmetrical instantaneous motion in a cyclone is in itself proof that we have not in that case symmetrical motion, about a centre, of a mass which travels as a whole. In other words, the motion of air in cyclones is not the motion of revolving fluid in the special sense referred to.

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* 'Q. J. Roy. Met. Soc.,' vol. 29, p 234 (1903).

† M. O. Publication, No. 174, 1906.

In "The Life-History of Surface Air Currents," with the assistance of Mr R G K Lempfert, this conclusion was minutely examined by the construction of a large number of trajectories, or actual paths of moving air, in the case of fast-moving depressions, slow-moving depressions, secondary depressions, V-shaped depressions, straight isobars, and less minutely but with ample confirmation of the conclusions by the tracing of the trajectories of air for periods of several days over the Atlantic and adjoining land areas Incidentally we found and formulated the phenomena of the "line squall" but found no case of revolving fluid.

Assuming that the symmetrical motion of a revolving fluid with regard to a centre implies the travelling of the centre and its accompanying fluid in the same way as a vortex-ring of smoke travels through the air, we may safely say that cyclonic motion is not that kind of motion, and in further justification of that proposition it is only necessary to point out that if the instantaneous motion of the various parts of a horizontal section of a vortex-ring were plotted in the same way as we plot the winds on a weather-map the resulting chart would be very different from a series of concentric circles On the other hand the trajectories of a moving cyclonic depression show that the air flows through the depression If it approaches the centre it, or its substitute, subsequently comes away again The actual paths of the air under various conditions were discussed in 1905 by the late Professor W H H. Hudson under the title of "Anemoids," and at the time I had a machine constructed, designed by Mr H Darwin, for drawing the curves automatically. If the motion were really symmetrical with regard to the moving centre there could be no return therefrom unless the vortex actually breathed or pulsated

The conclusion is the more discouraging for meteorologists because any person, even the least informed in meteorological theory, is able to point to visible examples of revolving fluid in the atmosphere which no meteorologist can deny, the eddy of dry leaves in a gusty wind at a street corner, the "dust-devil" of Eastern deserts, the water spout, the whirlwind, and something, at any rate on the scale of the cyclonic depression, namely, the tropical revolving storm, a very real phenomenon which is treated in all books on meteorology as a gigantic eddy, with the knowledge that an eddy always finds its typical analogue in the vortex-ring

To these examples we can only reply that in truth they may be cases of revolving fluid, but they are not the cyclonic depressions of middle latitudes with which we are concerned in dealing with the meteorology of the British Isles That sounds very like quibbling, and it seems still more so when it is pointed out that, in an official publication for which I am myself responsible, there is cited a case of a tropical revolving storm, an acknowledged eddy, first identified on August 3, 1891, in the mid-Atlantic in latitude 12° N, which had an eventful journey of 38 days, of which, perhaps, the first week is rather dubious, but the remainder well authenticated First it travelled westward along the Northern West Indian Islands to the Coast of Florida, then skirted the Atlantic Coast of America, passing round the western side of the permanent Atlantic "high," turned eastward over the Gulf Stream as an intense cyclonic depression, with a little hesitation about longitude 40° W, it made for the mouth of the English Channel, and, missing that, crossed to the Mediterranean, and lost itself there on September 9 There are various other examples of a similar character, in which, so far as maps enable us to judge, what began as revolving fluid in tropical latitudes passed by gradual transition into the form of a travelling cyclonic depression, and yet we have to deny the identity of the two types of phenomenon

On the other side, it must be remarked that the tropical revolving storm is a special phenomenon of very rare occurrence, the average number recorded in the various localities where they sometimes originate being about one a year, and that only in certain months, whereas on our own maps, in the ten years 1907–16, no fewer than 1200 cyclonic depressions were tracked, a new one on the average every three days

Nevertheless, whenever it occurs, the gradual transition from the revolving storm of tropical latitudes to the cyclonic depression of our own is of great interest, and may justify a short digression.

If the hypothesis of the strophic balance of the pressure and wind-velocity in a cyclone of any kind be true, in accordance with the well known equation*

$$\gamma = 2\omega v\rho \sin\lambda + \frac{v^2}{E}\rho\cot r$$

(where γ is the barometric gradient, ω the angular velocity of the earth's rotation, v the velocity of the air, ρ its density, λ the latitude, E the earth's radius, and r the angular radius of the small circle osculating the path of the air), we have at first an eddy with winds of, say, 50 metres per second (about 100 miles per hour), which travels bodily, with a velocity of about 4 metres per second (or 10 miles per hour), in a latitude for which $\sin\lambda$ is very small, and, in consequence, the pressure-gradient is almost entirely balanced by the spin in the small circle, and is almost independent of the earth's rotation; the whole eddy travels as a mass of revolving fluid, with very little disturbance of its character or shape by the translation. It is carried slowly along in an easterly wind, turns round the western boundary of the Atlantic anticyclone, and, as it reaches more northern latitudes, sin λ becomes more important, and a greater part of the gradient is balanced by the earth's spin, leaving less for the rotational spin to balance In consequence, one of two things must happen, either the velocity of the wind which defines the spin in the small circle must diminish, or the pressure-difference must deepen What actually happens, apparently, is the former alternative, and the range of pressure is distributed over a wider area In the outer regions of the extended area the centripetal action becomes practically negligible, in consequence of the large value of r and the more moderate value of v, and we get a cyclonic depression, in which the greater part of the pressure-difference is borne by the rotation of the earth (the geostrophic component), with perhaps a small core of revolving fluid, in which the small-circle-spin (the cyclostrophic component) remains predominant The combination is kinematically possible if the inherent velocity of progression of the cyclonic margin is such as to fit in with that of the translation of the vortical core It would necessitate surrounding the discs of figs 2, 3, 4 by a series of concentric circles, and using parts of the inner circles to join up the ends of the lines of the revolving fluid It is needless to say that the conditions under which this complex dynamical structure can maintain its stability for days together are not at all simple, but, however complicated they may be, they have occurred From the frequency with which an ordinary cyclone can be observed, we may conclude that the conditions which are necessary for it are much more easily realised in practice than those necessary for the cyclostrophic balance of a revolving fluid mass, and it is quite possible that, in the end, the conditions necessary for maintaining a core of revolving fluid disappear, and the whole system degenerates (if that is the correct expression) into cyclonic motion, without a core of revolving fluid Certainly, there is nothing in the maps to contradict such a suggestion

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* 'Barometer Manual for the Use of Seamen,' No. 61, 8th Edition, 1916

As a matter of common experience, I may add that the tropical revolving storm is a very unsatisfactory example of revolving fluid for scientific purposes, because practically always it destroys or disables all the recording instruments in its path, together with the buildings in which they are housed, and sometimes also the persons who attend to them, so that we cannot tell to what extent Lord Rayleigh's theoretical conclusions or Dr Aitken's practical illustrations are exemplified in its behaviour If we want examples that lend themselves to numerical calculation, we must look elsewhere Accepting the implicit challenge of Lord Rayleigh's statement that much of meteorology depends upon the dynamics of revolving fluid, I have approached the subject in a different way, by endeavouring to meet the questions "Let it be granted that cases of permanent or quasi-permanent revolving motion, symmetrical about a moving axis, do occur in the atmosphere, are they shown in the weather maps which are used by meteorologists as the basis of dynamical meteorology, and, if so, in what manner? If cyclones and anticyclones do not furnish examples of the motion of revolving fluids in that sense, as certainly in our latitudes they do not, what characteristics should be looked for instead?"

I start from the position which, as I have said, meteorologists cannot deny, that at any rate temporary whirls are actually visible The most picturesque example which has come to my knowledge is a case described to me verbally by Mr R H Curtis, formerly of the Meteorological Office He said that on the occasion of an open-air entertainment on a sultry afternoon in the south of London, when the members of the band had temporarily left their places, there came a puff of wind in a little whirl which carried off the sheets of music from the music-stands and whirled them away in a spin which carried them to a great height and dropped them ultimately some miles away Similar accounts have been given of the whirling away of loose hay, and so forth The precise particulars are of no consequence We have obviously two elements the current which carries the whirl, and the whirl within the current There must be something approaching to discontinuity, because the onset of the whirl itself is sudden Let us therefore suppose it possible that we may have a mass of revolving fluid with a discontinuity of velocity at the circle bounding the revolving mass It is possible that there may be a relation between the original velocity at the margin of the whirl and the wind in the current which carries it, but, as Lord Rayleigh's reasoning and Dr Aitken's experiment make clear, the velocity in the whirl at any time depends upon dynamical or physical processes within the whirling mass itself, we cannot therefore make any general assumption of a relation between the velocity in the whirl and the rate of its translation

Whirls on the small scale of those here referred to would not be shown on the weather map for the day, even if the hour fitted There may be many whirls in the atmosphere which find no record on a map The scale of the larger map of the Daily Weather Report is 1 inch to 300 miles, so that anything smaller than 50 miles would hardly be noticed in the isobars The local wind produced by a whirl might be given, but without anything to suggest its true significance. The scale of the working charts of the Office is twice that of the published map, so that, perhaps, a disturbance with a diameter of 20 miles might be detected The thickness of the atmosphere available for meteorological purposes may be set at 10 miles, perhaps, if one allows that the main features of the distribution of pressure arise from differences of temperature of the air of the stratosphere, a revolving column of air is therefore probably not more than 7 miles high. Hence, at the most, the height of a cylinder of revolving fluid would be less than one-half of its diameter, and whereas the cases of revolving fluid which can be seen are cases in which the height is very great compared with the diameter, the rotating fluid masses to be found in weather maps must belong to the other extreme, and have the shape of comparatively thin discs. The vortex must be more like a penny than a pin, if it is to be found on the map.

It is therefore without any misapprehension as to the number of vortices likely to be caught, in relation to the number which may have at least a temporary existence in the atmosphere but elude the map, that I proceed to the method of identifying them. I have constructed graphically the instantaneous velocity lines in the horizontal section of a mass of revolving fluid which is being carried along by a current of air, the velocity of which is taken successively as zero, $\frac{1}{2}$, 1, and $\frac{3}{2}$ of the rotational velocity at the edge of the circle. The section is supposed to be taken at a height of, say, 500 metres, where the effect of the surface friction may be neglected. The section will be represented as bounded by a circle, and as some arbitrary assumption must be made as to the distribution of the velocity for different distances from the centre, it will be taken, to begin with, as uniform from centre to edge. I shall make no trouble at this stage about the discontinuity of velocity at the edge. The pressure will be continuous, but there will be discontinuity in the space-variation of pressure. This will, of course, not be rigorously possible in nature, but it will be better to find out what the practice of accommodation is than to try to anticipate it.

In order to obtain illustrative cases, I suppose the revolving fluid to be represented by velocity-lines drawn on the understanding that the velocity is proportional to the number of current-lines per unit length of cross-section in the same way as the number of lines of force indicates the strength of a field. The currents which carry the revolving fluid along are represented in the same way, but in that case the velocity lines are taken as straight and parallel instead of being circular, as they are in the section of the revolving fluid. The resultant velocity lines representing simply the kinematical combination are shown in figs 1–4. The problem lends itself easily to algebraical solution. The curves to which the resultant velocity is everywhere tangential are conic sections, of which the centre of the disc is the common focus. The eccentricity is the ratio of the velocity of translation to the velocity in rotation, assumed uniform. For the special case of equality between the velocity of translation and the velocity in rotation (fig. 3) the

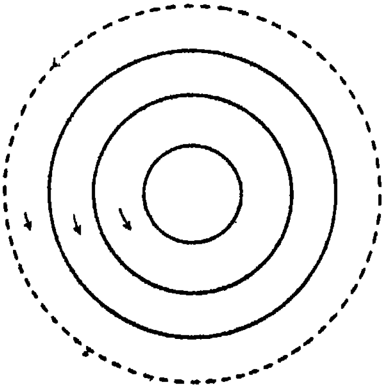

FIG 1 —Rotation without translation

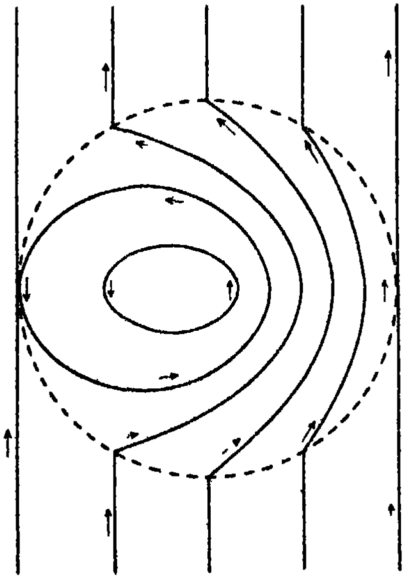

FIG 2.—Translation equal to $\frac{1}{2}$ × rotation.

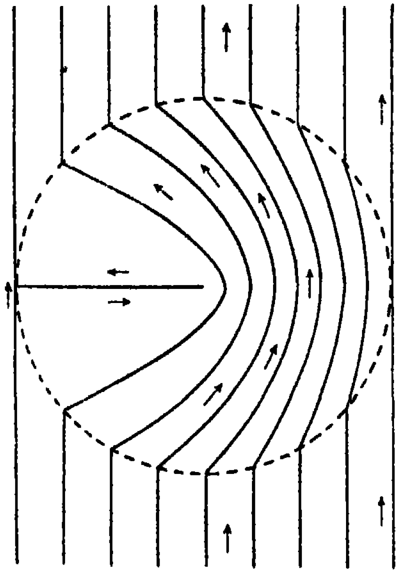

FIG 3 —Translation equal to rotation.

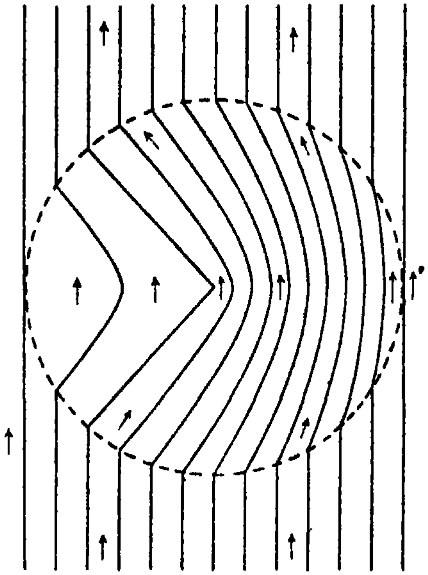

FIG 4 —Translation equal to $\frac{3}{2}$ × rotation

curves are parabolas, for slower travel (fig 2) they are ellipses, and for travel faster than rotation they are hyperbolas Four examples, including the circular form for the case of a stationary whirl, have been drawn for illustration, in order to give a general indication of the shapes of isobars on a map which can be looked for as representing travelling revolving fluid.

Fig 1 represents the effect of velocity, supposing the spin to be developed in the atmosphere at rest and the velocity everywhere to be V

Fig 2 the same with the superposition of an external velocity-field of $\frac{1}{2}$ V

Fig 3 the superposition of an external velocity-field of V

Fig 4 the superposition of an external velocity-field of $\frac{3}{2}$ V

In each case the dotted circle shows the boundary of the revolving fluid

Figures representing the intermediate stages can be constructed in like manner

These diagrams are very suggestive, not of the typical circular form of cyclonic depression but of various well-known forms of secondary depression as indicated by the distortion of the isobars of a large cyclonic system They explain at once the difficulty which has hitherto been felt in recognising cases of revolving fluid carried along by an atmospheric current We have been looking for them in the wrong place. In fig 1 the boundary of the revolving fluid coincides with a velocity line which could easily be identified by a sufficient number of observations, it is also identical with an isobar, and therefore easily recognised on a map, but in the other cases the boundary circle cuts at a finite angle all the velocity lines except those at the top and bottom, to which it is tangential There is discontinuity all along the bounding circle, in the magnitude of the velocity at the top and bottom and in the direction as well as the magnitude in all other parts of the circle. The boundary cannot, in any circumstances, be supposed to be an isobar. So, when we have looked for circular motion to be defined by a group of circular isobars, very naturally we have not found it

The effect of the superposition of successive external fields of velocity may be briefly described as follows If we begin with a circular field which has four velocity-lines below the centre and four above, then with each successive superposition of an external field of one-quarter of the velocity in rotation, one velocity-line is bent downwards, and its point of passage across the transverse central line is transferred from above the centre to below it, until, when the external field is the same as the internal field, all eight velocity-lines pass below the centre Omitting the intermediate steps there are four closed velocity-lines in fig 1, two in fig 2, none in fig. 3, because the closed curve becomes a straight line of "no velocity" connecting the boundary line at the top with the centre and in fig 4, when the external field is dominant, there are ten velocity-lines crossing the lower radius as hyperbolas in a direction which, for the sake of brevity, we may call from west to east, one velocity-line crossing the upper radius in the same direction, and one consisting of a pair of radii of the disc, so that the wind in every part of the revolving fluid has a westerly component, and nowhere has an easterly component The rotational velocity expresses itself merely in the variation of direction and magnitude of the generally eastward-moving current

The next step in realising the expression of these conclusions upon a map is to convert the velocity-lines into isobars, because the pressure is the continuously varying element that is represented on maps, and it is by means of the isobars that we get the best available indication of the velocities of atmospheric motion at a height of 500 metres, where the flow is practically undisturbed by the frictional effects of the surface In this step there is a very serious difficulty arising from the fact that the relation of steady wind to the distribution of pressure is dependent partly upon the earth's spin which gives the geostrophic component of the barometric gradient depending upon the first power of the velocity, and partly upon the centrifugal effect of the motion in a small circle which balances the cyclostrophic component of the barometric gradient depending upon the square of the velocity and the angular radius of the small circle osculating the path

If we could neglect the cyclostrophic component and consider only the geostrophic component, the velocity lines drawn in the figures would at once become isobars drawn on a uniform scale, and the translation into isobars would consist simply in writing suitable pressure values against the lines, when the latitude and the numerical values of the velocity are assigned, but that simplification is not permissible When we are dealing with cases of revolving fluid the cyclostrophic component cannot be neglected In the equatorial regions, where sin λ is a ratio of negligible magnitude, the geostrophic component might be disregarded and the lines of such a diagram as fig 1 could have its pressure values written against them by inserting values for v and r in the cyclostrophic formula because the velocity lines are obviously also path lines and the curvature is that of the circles as drawn In all the remaining diagrams the assignment of pressure values to the velocity lines is a complicated process even if the balance of pressure and velocity be assumed, because the velocity lines are not path lines.

In any particular case the actual path curves can be determined when the data are obtained. If we take a point on the outer rim in the case of fig 3 the path is a cycloid. For the upper part the cyclostrophic component may be neglected because the velocity is small and near the bottom the radius of curvature is large, so for a large outer circle the curvature correction may be disregarded without affecting the general shape of things, but near the core of any revolving fluid, unless the velocity is very small, the rotational term is dominant The assumption of uniform velocity in rotation that was made for drawing the velocity diagrams is an arbitrary one, and without any obvious departure from known facts we may substitute for it the assumption that the distribution of velocity is such that what we have obtained as velocity lines shall now represent pressure lines and the velocity may be adjusted in accordance therewith Whether that assumption would give a distribution of velocity symmetrical with regard to the centre I have not fully examined, but, if not, we may assume that the shapes of the lines could be adjusted to give the necessary symmetry without altering the general appearance as represented on a map which cannot give the details minutely.

Let us, therefore, now regard figs 2–4 as diagrams of isobars of that part of a map which contains a mass of revolving fluid carried along in a main current The next step is to deal with the ends of the isobars outside the circle of revolving fluid In the atmosphere an isobar cannot have loose ends, they must be joined We have drawn the isobars outside the circle as parallel straight lines Straight isobars are not at all an unusual form of grouping on a map, there must be low pressure on the left and high pressure on the right and the ends will be joined sooner or later by enclosing a low-pressure area or a high-pressure area. They belong in fact to the marginal region of a cyclone or anticyclone The position of the revolving fluid is thus identified not as the centre of a cyclone or anticyclone but as a distortion of the isobars in the main current controlled by the general distribution of pressure of which the cyclone and anticyclone are elements It is the velocity of this current which carries the revolving fluid along, and if our diagnosis is true the velocity of translation of the revolving fluid is the velocity of the main current in which it is formed, and that velocity can be inferred from the distribution of the pressure in regions near to but undisturbed by the revolving fluid. It was merely for the purposes of simplicity that we drew the continuations of the isobars as straight lines on the diagram, what amount of curvature can be allowed in the isobars of the current in which a revolving column or disc is formed must be left to experience to decide.

Two more elements must be brought into consideration connected with the possibility of permanence of a column or disc of fluid carried along in a main current of air The first is to what height in the atmosphere does the column of fluid extend? One of the great difficulties of imagining the maintenance for days together of a tropical revolving storm or any other example of revolving fluid which is supposed to demand unlimited supplies of air from the bottom is why it does not take in air from the top and fill up Without going into particulars, which are really necessary, it has been customary to mention the word convection and leave it at that; but it requires a very finely organised scheme of convection to keep a hurricane in being for a week Convection, we know, stops short of the stratosphere, which is low over low pressure In the application of a formula for the variation of wind with height in a cyclostrophic system for a lecture at the Royal Institution in March, 1916, I made out that when the stratosphere was reached and, in consequence, higher pressure corresponds with colder air, the velocity of rotation and consequent pressure-difference would fall rapidly with height, and so a cap in which rotation gradually falls off would automatically protect the column of revolving fluid below If then the main current which carried the revolving column extended into the stratosphere and so was available for carrying the cap along with the revolving column the combination would have a fair chance of permanence or persistence

The best a meteorologist can say at this stage is that there seems no possibility of maintaining quasi-permanently a column of revolving fluid in the atmosphere unless it extends from its base up to the boundary of the troposphere It is quite possible to imagine a secondary "lid" formed by an "inversion of temperature gradient" which would take the place of the stratosphere at a lower level, but there is no definite evidence about the bounding of a quasi-permanent eddy in that way

That brings us to the second point in connection with the persistence of a column of revolving fluid, namely, the necessity for the extension of the main current carrying the revolving fluid without change of its velocity up to the top of the column in our case apparently right up to the top of the troposphere and a little beyond it. That is a matter which certainly requires consideration One of the commonplaces of meteorological statistics is that the wind generally increases, sometimes decreases, and hardly ever remains the same at different levels Egnell's law derived from the observation of clouds, and accounted for on a physical basis, in a paper on the interpretation of the results of soundings with pilot balloons* gives the velocities at any height as inversely proportional to the densities of the air at those heights. This would mean that with a velocity of 20 metres per second near the surface there will be a velocity at 10 kilom. height of 60 metres per second Such a distribution of velocity would very soon make an end of a column of rotating fluid; after an hour of it the head would be 90 miles away from the foot, after a day, 2000 miles away. It would be literally torn to pieces. So we have got to imagine the conditions in which Egnell's law does not apply, and in which the velocity of the wind suffers neither increase nor decrease in the 10 or 12 kilometres of height over which the meteor stretches In the lecture already referred to, I gave a formula for the variation of wind with height under geostrophic conditions, and Mr W. H. Dines* has put similar reasoning into a much simpler form He shows that the wind velocity remains invariable with height if the isobaric surfaces in the atmosphere are also isothermal

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* 'Q. J Roy Met. Soc.,' vol. 40, p. 111 (1914).

I do not know exactly how to interpret this condition in general terms. I can imagine the atmosphere to be stratified in layers defined by isobaric surfaces, of which the parts of each layer are isentropically related without consecutive layers being so related The entropy-relation of different parts of the atmosphere is so complicated by water and its vapour that I have not yet succeeded in reducing the reasoning about it to an easily manageable form, but in any case this is true, that if the churning of the atmosphere and consequent isentropic equivalence of the several parts is complete, then Mr Dines' condition for uniform velocity at all levels is satisfied. In the upper regions of the atmosphere convection is the chief, if not practically the only, agent for promoting isentropic mixture, and if we want to find an example of a current of air with practically equal velocity at all levels it would be well to look for it in a region where convection has been active and ubiquitous There may be other examples covered by the formula which I cannot picture to myself, but that one is certain. Now, from an entropy temperature diagram of atmospheric air, not yet published, which Mr E V Newnham has recently constructed for me, it is clear that the average condition of air in a cyclone very nearly corresponds with an adiabatic line for saturated air, from which I conclude that the rainy region of a cyclone (that is to say, the region where convection has been and is active) is the proper place to look for an example of a current that is generally isentropic, and, consequently, has a uniform velocity throughout its height and is in the condition necessary for carrying a revolving fluid without pulling the top from the bottom.

To be effective, for the purpose of carrying along a column of revolving air, this current must extend upwards somewhat beyond the base of the stratosphere The southern side of a cyclone, so far as we can judge, does that The pressure-difference between the outer region on the south of a cyclone and the region near the centre may be said to originate in the stratosphere† In this case the isentropic condition gives a relation between p and θ, which gradually diminishes the pressure-difference as one goes upwards, but it is only when the stratosphere is reached that the pressure-difference and the velocity begin to fall away rapidly, in consequence of the slope of temperature being opposite to the slope of pressure the westerly current on the south side of a cyclone is one of the most definite meteorological entities, and most certain of great extension in the vertical direction

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* 'Nature,' vol. 99, p 24 (1917)

† See 'Geophysical Memoirs,' No 2, 1912.

If, therefore, we desire to find an example of the motion of revolving fluid in the atmosphere of these latitudes, we shall not look for it at the centre of a cyclone with circular isobars, nor shall we expect its own isobars to be circular when we find them. It will be represented by distortions of the isobars of the general distribution of pressure similar in shape to one or other of the figures (1–4), according to the relation between the velocity with which it travels and its own speed of rotation The speed for any circle will be one-half of the difference between the velocity indicated on the map for the two points on the diameter of the circle transverse to the motion of translation, and the speed of translation will be one-half of the sum

I propose to give two examples which illustrate these views. In the first (fig 5), the shapes of the isobars suggest the existence of a rotating system such as that described. It is shown on the map for 8 A.M on March 24, 1895, and, with very little change of character, also on the maps for 2 and 6 P.M. on the same day The map for 8 o'clock the following morning suggests that by that time the system had degenerated into a cyclone. The second is the case of a tornado in Devonshire and South Wales on October 27, 1913,[*] the worst damage of which was done about 6 P.M., although the isobars on the map for that epoch (fig. 7), as drawn in the working-charts of the office for the Daily Weather Report, show no sign of any exceptional atmospheric activity, so that in this case the revolving fluid must be regarded as being carried forward with the stream represented by the isobars without producing any interference with the general run of those lines.

The first case, that of March 24, 1895, has always been regarded as a noteworthy example of the destructive effect of a "small secondary," and for that reason it is figured in my book on 'Forecasting Weather.' It was the most destructive gale within my experience of the weather at Cambridge between the years 1872 and 1900. It began there about 2 o'clock in the afternoon, and by 6 o'clock many of the oldest and strongest trees had been uprooted, some buildings had been demolished, and a great deal of minor

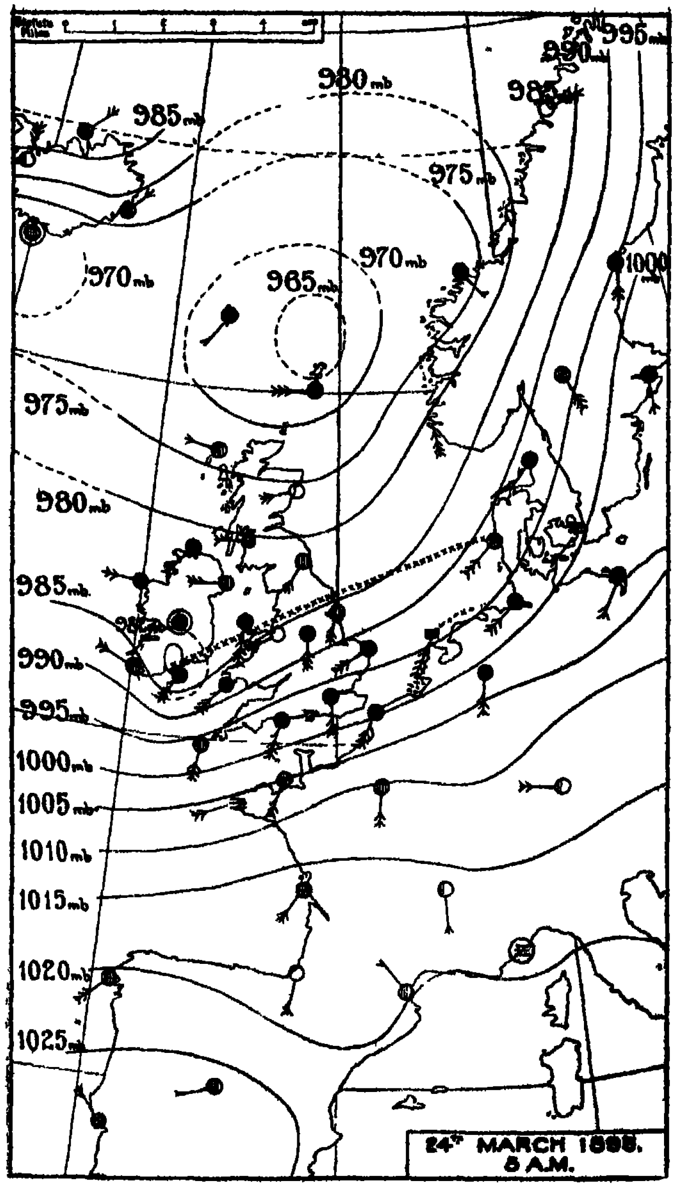
[*] 'Geophysical Memoirs,' No. 11, 1914.

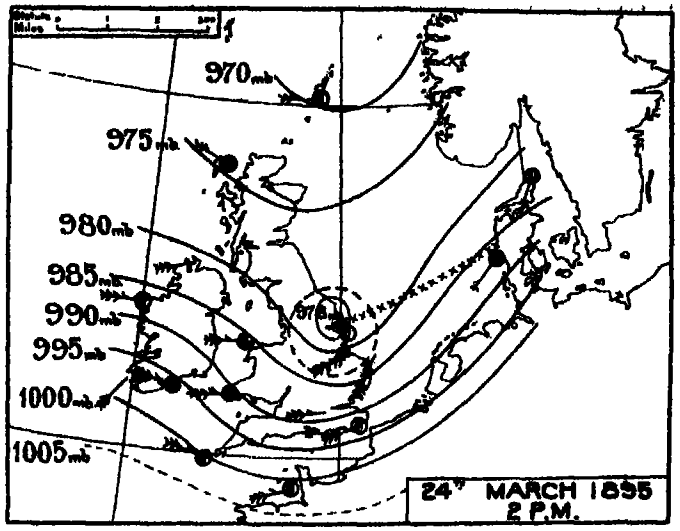
damage done The remarkable feature of the gale was that, in the Eastern Counties, it was unattended with any rainfall, either before, or during, or after the strong wind Except for the wind, it was a fine afternoon, and remained so The scene of destruction caused by the wind in a few hours will probably always remain in the memory of those who were in Cambridge at the time On the same afternoon many trees were uprooted or broken in Norfolk, one particular plantation near Sheringham was completely wiped out A remarkable feature of the case was that, although the phenomena were locally suggestive of an occurrence that happens only once in a lifetime, the Daily Weather Report of the following morning seemed to regard it only as an ordinary gale, and no winds of exceptional force were noted at the observing stations. In illustration of what was noted at the time, I have had the observations reported to the Meteorological Office for 8 A.M, 2 P M, 6 P M, on March 24, and 8 A M on March 25, recomputed and plotted on maps showing isobars and wind, with others showing weather and temperature for the four epochs and past rainfall for each of the two days at 8 A M The two strongest winds, each of force 11, on the map for 2 P M are taken from the logs of lightships off the Norfolk coast.

The locality of the centre of the revolving fluid is easily identified on the three maps of the 24th, and in each case a dotted circle has been drawn, to indicate the probable boundary of the revolving mass. The guiding idea in selecting the radius is the extension of the line of calm on the northern side of the centre in comparison with the extension of the intensified wind on the southern side The revolving fluid seems to have travelled with great rapidity, but with little change, between 8 A M. and 6 P M The actual speed of translation, according to the maps, between 8 A.M and 2 P.M, is 55 miles per hour, the geostrophic wind computed from the 8 o'clock map 63 miles per hour if taken from the separation of the isobars south-east of the revolving mass, or 40 miles per hour if a line is taken between the isobars on either side of the path. Between 2 P.M and 6 P.M the velocity computed in a similar way is either 81 miles per hour or 67 miles per hour, and the actual velocity of translation is 82 miles per hour. There is a curious congestion of the isobars south of the path, which suggests that the current which carried the whirl eased off to the north of the whirl.

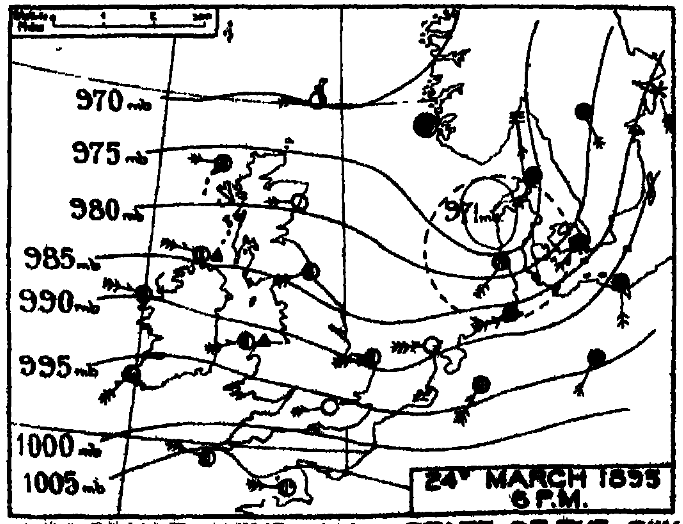
After 6 P M. the rate of travel of the centre was much slower, and the characteristic features of rotating fluid had probably been lost. This may, therefore, be cited as a good example of revolving fluid carried along in a main current, the velocity of translation being governed by the spacing of the isobars of the main depression outside the area affected by the local rotation, and, in virtue of this, the air which formed the mass of revolving

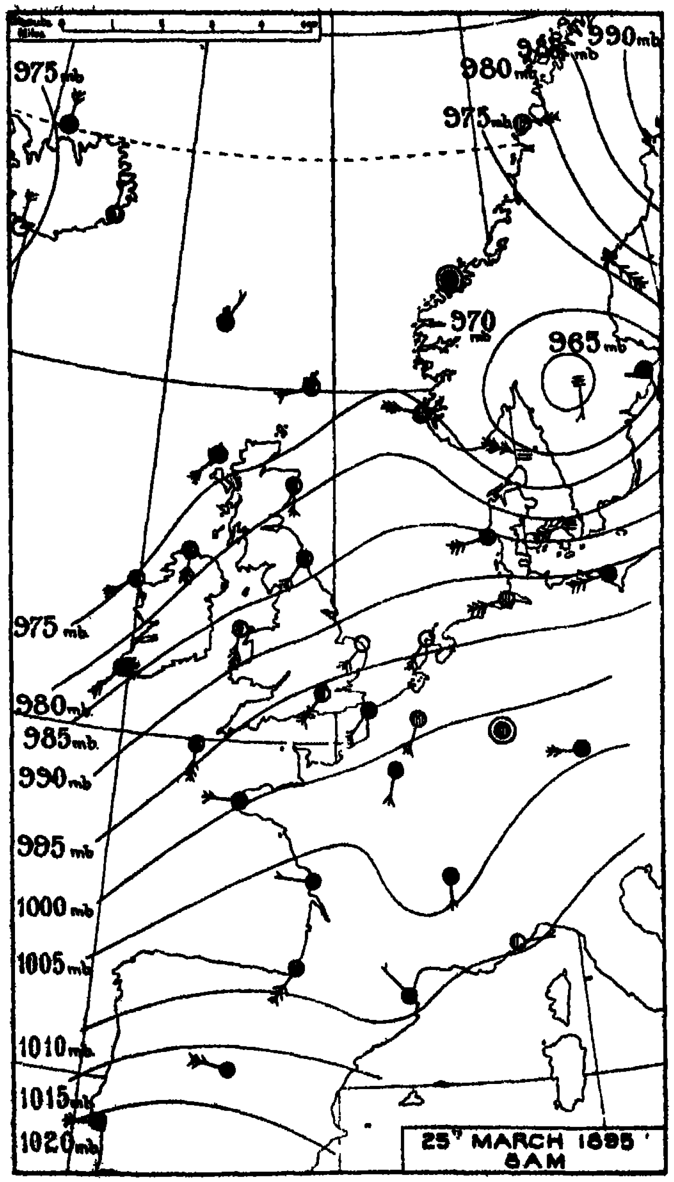
FIG 5.—Charts of isobars, wind and state of the sky at 8 A.M., 2 P.M., and 6 P.M. on the 24th and at 8 A.M. on the 25th March, 1895, illustrating the passage of a small "secondary" from the south of Ireland to Denmark, which probably consisted of a column of revolving fluid, indicated in the dotted circles on the maps for the 24th





Explanation.—Arrows fly with the wind feathers show the force on the Beaufort scale. Small circles denote the positions of the stations, the lines crossing them the number of quarters of the sky covered A larger concentric circle indicates *calm* Three horizontal strokes denote *fog*





PRESSURE, WIND AND STATE OF THE SKY.

The line of crosses indicates the path of the centre of the secondary

fluid over the south of Ireland at 8 A.M was carried along bodily to Denmark by 6 P M The spin was maintained, and the damage done proves that the velocities of the air were exceptionally great The local variation of intensity may be attributed to the alteration in diameter of the revolving rings The absence of rain shows that very little additional physical energy was given to the revolving fluid during its passage across the Eastern Counties In fig 6 is given all the available information about temperature and rainfall. It shows that the differences of temperature were very slight, both from the geographical and chronological points of view The origin of the rotation in the first case may, perhaps, be attributed to the line of heavy rainfall along the south of Ireland, Valencia 26 mm, Roche's Point 28 mm, and Pembroke 18 mm, before 8 A M on the 24th The line is more or less parallel to the general run of the isobars, and the suggestion of parallelism between the rainfall lines and the pressure lines is borne out by the records at the other stations on the map The existence of the rainfall is *primâ facie* evidence of a good deal of convection in the atmosphere, so that it would be reasonable to assume that the conditions specified for the maintenance of a column of revolving fluid are satisfied, namely that the wind velocity in the surrounding medium should be uniform at all heights This case therefore agrees with the theoretical considerations that have been put forward in respect of the velocity of travel in relation to the isobars of the air in which the secondary is formed, the shape of the isobars in the immediate locality of the secondary, and in the physical conditions of the atmosphere necessary for the persistence of a travelling column of revolving fluid

The second case, that of October 27, 1913, is interesting as illustrating what may certainly occur, because in this particular instance it did occur, without any noteworthy deformation of the isobars actually drawn on the map, and explaining the possibility of immense structural damage in those circumstances by the existence of a column of rotating fluid carried along by a general current of wind The phenomena are thus summarised by Mr Billet in the memoir referred to —

" On October 27, 1913, a severe thunderstorm swept the West of England and Wales, from the South of Devon to Cheshire, and developed locally into a tornado of exceptional violence Such phenomena were experienced for a distance of about 11 miles up the Taff Valley in Glamorganshire, for about an equal distance in Shropshire, and for about 5 miles in Cheshire

" Violent meteorological phenomena were confined approximately to a straight line running almost due North from the South of Devon to Cheshire They did not exhibit themselves with equal intensity throughout the length of this line, but there appear to have been four main regions of action :—(1) South Devon, (2) South Wales, (3) Shropshire, (4) Cheshire"

The map which forms the frontispiece to the memoir, and which has been redrawn in fig 7, shows a cyclonic circulation round a centre to the west, over the Atlantic The relation of the geostrophic wind in the region of South Wales to the motion of translation of the revolving fluid is not quite satisfactory The direction accords perfectly It appears from the text of the memoir that the velocity with which the violent disturbance travelled over its whole course from Devonshire to Cheshire varied from 34 miles an hour between Devon and South Wales to 37 miles an hour between South Wales and Cheshire The geostrophic wind computed for South Wales from the original chart of 6 P M, when the tornado was in that neighbourhood, was 39 miles per hour, which gave also quite good agreement for the speed But on the map, as redrawn for this paper in fig 7, the speed computed from the pair of isobars which cross South Wales is 48 miles per hour, which is certainly too fast for the tornado The diameter of the area of the damage varied apparently from 150 yards to 300 yards, and the barometers were not affected during the passage of the tornado by as much as 1 mb at a greater distance than two or three miles from the line of destruction, a little further out only by a fraction of a millibar. Close to the centre the effect upon the barograph was a sudden fall and recovery of 10 mb which took place within less than one minute There must therefore have been a very definite line of discontinuity marking off the area of rotational velocity from its environment In this case the localities where damage was done were localities of very heavy rainfall, arranged as shown on the map in a line nearly identical with the line which would have been followed by a column of revolving fluid carried by the general wind along isobars from Devonshire across South Wales to Cheshire. The concentration in South Wales with its accompanying destructive progress seems to have been due to the convergence of the air within the moving column of revolving fluid and to have furnished a good example of the progress of revolving fluid although the radius within which the whole operation was taking place at any moment was not large enough to affect the general run of the isobars on the map It is noteworthy that the period of destruction was preceded by a period of ominous calm during which the wind from the south appropriate to the isobars was arrested by the effect of the approaching whirl.

Thus it appears that it is possible occasionally to identify examples of revolving fluid in association with phenomena registered on the maps, although in some cases the region affected may be altogether too small to show on the map, and in other cases the revolving fluid may only be identified

THE REGIONS MARKED BY SPECIAL VIOLENCE ARE INDICATED BY SHORT BLACK BARS WITH CIRCLES SHOWING A DIAMETER OF 10 MILES.

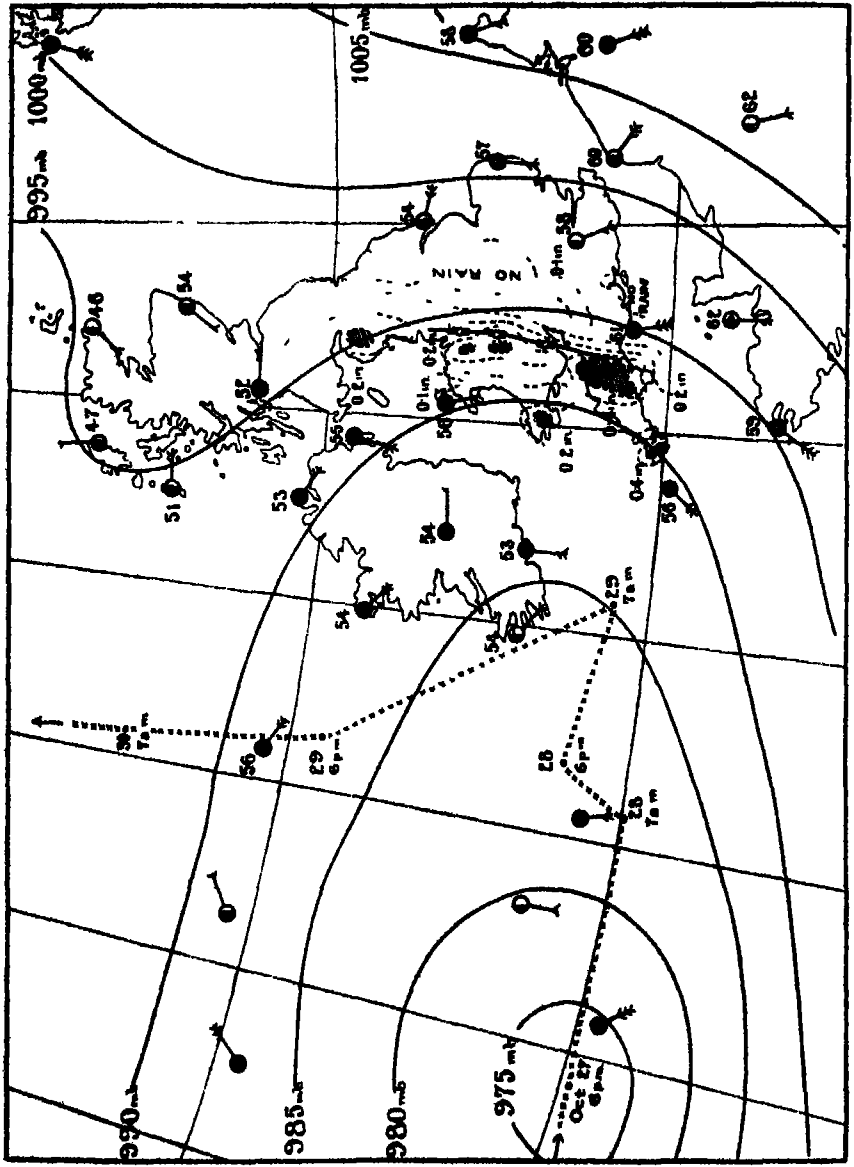


FIG. 7

MAP SHOWING THE METEOROLOGICAL CONDITIONS OF 27TH OCTOBER 1913 6 P.M. DURING THE OCCURRENCE OF A TORNADO IN SOUTH WALES

TEMPERATURES ARE GIVEN IN DEGREES FAHRENHEIT RAINFALL IN THE NEIGHBOURHOOD OF THE TRACK OF THE DESTRUCTIVE WHIRL FOR THE "RAIN DAY" OCT. 27TH IS SHOWN BY DOTTED LINES MARKED IN INCHES.

as a small secondary The identifiable cases are certainly not very frequent, but there may be others less conspicuous in their effects which are less easily found

We are thus able to draw a definite distinction between the cyclonic or anticyclonic motion as exhibited in the atmosphere and cases of revolving fluid motion In the latter there is rotation about the vertical axis and the whole mass of revolving fluid is carried along bodily in a current of air which has a velocity corresponding with its position in the main cyclonic system and which can be related to the run and the distance apart of the isobars. There is practical discontinuity of velocity between this current and the rotating mass, and there is no relation between the velocity of wind in the whirl and the velocity of translation in which the whirl travels over the country In the case of a cyclonic depression the travel over the country is a part of the motion of the wind in the cyclone itself There is no discontinuity of velocity The wind so moves that each part of it revolves about a centre which has a motion of translation The rate of progress of the centre depends upon the curvature of the path of the air in the cyclone, and the motion of the centre is apparently due to the fact that the cyclostrophic component of the pressure is governed by the radius of curvature of the path and not by the radius of the circle of instantaneous motion The motion in a travelling cyclonic depression is, therefore, of a different type from that of motion in a circle with a superposed velocity of translation.

[*Note added August* 3, 1917 —Since this paper was written the author has had the advantage of seeing a proof of a paper on Tornadoes in the United States, by Prof R de C Ward, of Harvard University, which is to appear in the 'Quarterly Journal of the Royal Meteorological Society' for July. There is a remarkable similarity between the terms used by Prof Ward to describe the meteorological conditions in which tornadoes are found to occur and those used in this paper in respect of the occurrence of columns of revolving fluid · so much so as to leave little or no doubt that tornadoes are columns of revolving fluid developed locally in the southern portions of large cyclones and carried along with the ordinary wind of the cyclone in which they are formed.]

Investigation into the Imbibition Exhibited by some Shellac Derivatives.

By A P Laurie and Clerk Ranken.

(Communicated by Prof J Walker, F R S Received July 14, 1917)

If shellac is boiled up with carbonate of soda, and the solution allowed to cool, a considerable part of the shellac separates out as a solid cake, resembling gutta-percha in appearance, and easily cut with a knife If slices of this cake, or portions of the thin film deposited on the sides of the beaker, are placed in water, they rapidly expand, at the same time changing in appearance, becoming opaque, and of a weak purple colour The colour is doubtless due to the dye which the shellac still contains. Ultimately, after swelling to several times the original size, the whole mass begins to break down into a flocculent precipitate The time when this occurs is somewhat irregular, and seems to depend upon the conditions of preparation If a small portion is cut from the solid cake and immersed in water and allowed to expand, and then replaced in strong carbonate of soda solution, it rapidly contracts, on being replaced in water expands again, and so can be alternately expanded and contracted

In order to make a rough measure of the expansion of these fragments small glass cells were laid on graph paper, and a series of fragments cut so as to measure five-tenths of an inch each way, and roughly about one-tenth to two-tenths of an inch thick, were placed in these cells, which were then filled with sodium carbonate solution of different strengths, and the expansion of the fragments measured by means of the graph tracing cloth The following results were obtained —

Table I

Strength of soda solution	Original area	New area.	Expansion in tenths of an inch
4 normal	25	25	0
2 "	25	25	0
1 "	25	30	5
0·5 "	25	36	11
0·25 "	25	49	24
Pure water	25	81	56

These results will be found to plot into a fairly regular curve In the case of the water, the 0·25 and the 0·5 solutions, the cakes, after expanding to the registered size, began to break down. In the case of the normal solution, no such breaking down took place. The original strength of solution in which the shellac had been dissolved was twice normal, but had become concentrated during the solution The shellac dissolved in potassium carbonate seemed to behave in the same way, and there was no obvious distinction between the two salts If pieces of shellac are left for some days in a cold twice normal solution of sodium carbonate they darken in colour, but otherwise there is little alteration If placed in water they will absorb water and expand just as in the case of the other preparation, though more slowly.

The next experiments were made with a preparation of shellac in borax, as it was found that on expansion in water the borax compound resisted disintegration, a skeleton being left retaining the shape of the original fragment. After experiments made with solutions of different strength, the following method of preparation was adopted:—57 grm of shellac were gradually added to 100 c c of a solution of borax containing 8 grm of borax. As the shellac dissolved the whole mass was continually stirred, the final result being a thick treacly mixture The boiling of this was continued until on dropping a portion into cold water it expanded, but did not disintegrate This preparation differs from the sodium carbonate one in that the whole of the solution is involved, there being no separation of the solid body from the liquid

If a little of the hot mixture is poured on a glass plate and another glass plate pressed upon the top, it can be pressed out into thin sheets which readily separate from the glass when cold Portions from these thin sheets expand when placed in water, and at the same time the water becomes stained, of a brownish yellow colour, the ultimate result being an insoluble skeleton, and a brown coloured solution out of which, on addition of an acid, the shellac is precipitated If these squares are introduced into salt solutions of different strength they will expand up to a certain limit, depending on the strength of the salt solution, and at the same time the salt solution becomes slightly stained with the soluble compound

The following Table, obtained by the expansion of small squares, while not to be regarded as of quantitative value, gives a sufficiently accurate indication of the nature of this process

It was found in practice that no two preparations expanded to exactly the same extent, and that also the amount of expansion varied according to the length of time between the preparation and the use of the substance. The substance evidently loses water quickly by evaporation at ordinary temperatures, as the thin sheets rapidly curl up, and, as it loses water, it becomes less capable of expansion.

Table II

Strength of salt solution	Sodium sulphate	Sodium chloride	Ammonium acetate
W	55	40	40
⅛ N	40	36	33
¼ N	33	30	30
½ N	25	25	25
N	25	25	25

It will be noted, for instance, in the figures given, that the final expansions of water are somewhat irregular At the same time, the general indications are in favour of the hypothesis that we are dealing here with something of the nature of an osmotic expansion, which is controlled by the strength of the salt solution, the results being similar for different salts

With a view to obtaining more exact results, a new method was used in the next experiments A series of 5 c c glass measuring vessels were taken, and a series of brass rods about ⅛ inch in diameter Brass instead of glass was selected, because it was found that the substance which it was desired to examine clings better to a metal surface The volume of the brass rods up to a mark having been determined by lowering them into the graduated vessels, they were then coated with a thin layer of the mixture, and placed in the vessels hanging from corks, and the displacement volume again noted In this way a rough determination of the volume of the shellac mixture was made Expansion was then allowed to proceed, and the rods lifted out from time to time, and the volume of liquid remaining noted As in the former cases, considerable irregularities were found in individual experiments, but these irregularities tend to disappear when the means of separate experiments are taken.

The following Table contains the mean of the three results obtained for sodium sulphate and the three results for potassium chloride

Table III.—Table of Results with Brass Rods. The Total Volume of the Shellac Preparation being taken as 1.

Strength of salt solution.	Mean of three measurements for sodium sulphate	Potassium chloride
N	1	1
½ N	1 2	1 2
¼ N	1·7	1 8
⅛ N	2·0	2 1
W	2 4	2 8

If the order of sodium sulphate and potassium chloride is remembered in the Lyotrope series, it is evident that this substance behaves quite differently to gelatine.

It will be seen how closely they agree, and how much they confirm the results from the expansion of the small squares In one of these experiments, the expanded rod taken from one-eighth normal solution was placed in a normal solution, which was occasionally renewed. In course of a week it had contracted, the solution having increased in volume to within 15 per cent of the original water absorbed, so that, as in the case of the other experiments earlier described, it is possible to expand in a weak, and to contract and remove water in a strong solution If, however, instead of taking the substance expanded in a salt solution, we take the ultimate results of expansion in water, it is impossible to get this substance to contract on putting into a salt solution.

In order to get over the irregularities caused by this method of preparation, it was decided to prepare the borax solution in another way A solution of borax was prepared containing 10 grm in 500 c c. borax, and shellac was added to the boiling liquid until no more dissolved In this case, owing to the greater dilution of the borax solution, the process did not go on sufficiently far to obtain the thick mixture of the other method This borax solution was then filtered and evaporated down, and when a thick consistency was obtained it could be poured out and cast into sheets or into any other convenient form. These sheets are very similar in appearance to those already described but are no longer affected to the same extent by being kept in the air, and show greater regularity of expansion They are at the same time more sensitive to expansion than the substance prepared in the former way, which contains a larger proportion of shellac, and consequently an expansion begins in a solution of twice normal, although it is, at that saturation, very slight The general form of the curve obtained agrees with that in the earlier experiments, but the breakdown of the material and free delivery to the solution of the soluble contents takes place at about ⅛ to ¼ normal

It is evident that these phenomena are of the nature of those classified under the head of imbibition which are to be noticed in the case of substances like gelatine and gum tragacanth, but at the same time there are indications that there are differences from those already recorded for gelatine. As is well known, the connection between the expansion of gelatine and water and the strength of the surrounding salt solutions is very complicated, and is not in any way proportional to the strength of the solution.

In the case of this preparation of shellac there are, I think, distinct indications that such a classification is possible. If we imagine that we are dealing with a soluble organic molecule surrounded by a diaphragm which is impermeable to the soluble organic nucleus, and which is at the same time elastic and capable of expansion, the easiest way to explain the results is to assume the diaphragm to be also impermeable to the salt molecules. Then, as soon as the internal osmotic pressure produced by the passage of the water into the nucleus is sufficient to balance the resistance of the elastic wall plus the external osmotic pressure of the salt solution, the equilibrium is obtained and the expansion and contraction can be varied according to the strength of the external solution. If, however, the substance is introduced into pure water, the expansion goes on indefinitely until the osmotic cell walls are broken, and the soluble constituent escapes into the water. At the same time, even in the case of a salt solution, some of the external osmotic cells are probably broken, and hence some of the soluble contents escape, the percentage of the soluble content being, however, very small in the case of a salt solution as compared with the complete disintegration that takes place in the presence of water.

To test this theory a brass rod was lowered into a burette containing 15 c.c. of 0·56 N sodium sulphate. The volume of the rod up to the scratch as measured by displacement = 1·9 c.c. Rod coated with shellac preparation = 2·35 c.c.

After the shellac had finished expanding it was removed from the solution, the volume diminishing by 3·95 c.c. Therefore the volume of solution removed = 1·6 c.c. On analysis the original solution contained 0·09156 grm. sodium sulphate per cubic centimetre. After the experiment it contained 0·09232 grm. per cubic centimetre.

There was, therefore, a slight concentration of the solution due to the absorption by the shellac, the shellac taking up 0·0852 grm. per cubic centimetre instead of 0·0916 grm. It is evident, therefore, that the diaphragm is not impermeable to salt solutions.

It is evident from these experiments that the explanation of the control of the expansion of this substance is not to be found in a balance of osmotic pressure, and must be looked for elsewhere.

Now if a solution of the soluble nucleus is prepared by allowing several pieces to expand to their full in cold water and filtering off the skeleton, and a strong salt solution is added to this, the soluble matter is at once precipitated. This is not like the precipitation of colloids by very weak salt solution. The solution has to be strong, approaching saturation, to produce such a precipitate. When diluted little or no precipitation occurs. The explanation, therefore, of the control of the expansion by the strength of the salt solution seems to lie in this.

As the salt molecule passes freely through the diaphragm the soluble nucleus is dissolving in the presence of the salt solution, and therefore the amount that can dissolve and the consequent osmotic pressure is thus controlled

Among a fair number of ordinary salts tried, only two exceptions have so far been found. In sodium carbonate the borax mixture swells much more than in water ultimately disintegrating, and ammonium chloride, even if dilute, precipitates the soluble portion A systematic investigation of the behaviour of salts requires to be made

At first sight the behaviour of this substance seems to differ very markedly from the behaviour of substances like gelatine and gum tragacanth, but the insolubility of the soluble portion in strong salt solutions very largely explains this difference.

In order to make a fair comparison, a medium must be used, containing a constituent in which gum tragacanth, for instance, is insoluble The following experiment illustrates this —

We took a series of graduated glass cylinders, and placed 0 2 grm of powdered gum tragacanth in the bottom of each, and filled up each cylinder with 5 c.c. of a mixture of alcohol and water The following Table gives the expansion of the gum tragacanth —

0 2 grm of Gum Tragacanth in 5 c c Solution

	Space occupied by expanded gum
	c c
90 per cent alcohol	0 5
25 per cent of water by volume	0 8
50 " " "	1 4
75 " " "	2 5
Water	3

It is here behaving in exactly the same way as the shellac mixture, but there is this difference on replacing the weaker by stronger alcohol, it did not contract again The interest of these experiments in shellac, in the light of the accepted view of the two-phase character of gelatine solutions and the continuance of this in the solid, and the view held as to a soluble nucleus surrounded by a network of less soluble material, is obvious.

If the shellac mixture, prepared in the way first described, is soaked for a considerable time in ether, a small quantity of a resin is extracted, but it retains its properties. Solution in alcohol and subsequent evaporation has no effect on it, and the skeleton left dissolves on raising the temperature, while the soluble portion, if evaporated down again, expands when put in water. In the case of the preparation with sodium carbonate, as already stated, the ultimate result is complete disintegration, most passing into solution with a slight amount of flocculent precipitate.

Determinations of the heat of imbibition of the shellac derivative were made by mixing in a calorimeter 5 grm. of the ground shellac derivative with 10 c.c of water, and noting the change of temperature. No change in temperature was recorded. Of course, this does not preclude the possibility of heat being developed by the imbibition The heat so developed might be counterbalanced by a lowering of temperature due to the solution of the inorganic portion in the water.

In conclusion, this preparation of shellac seems to be of interest as throwing fresh light on the behaviour in presence of water of substances of the nature of gelatine and gum tragacanth, and also may possibly throw further light on the problems connected with the movements of sap.

Two Cases of Congenital Night-Blindness.

By Sir W de W Abney, K C B, F R.S.

(Received July 21, 1917)

In the recent communication of my paper on a "Fourth Sensation in Colour Vision," in April last, attention was called to the case of night-blind eyes as throwing light on the question of the functions of the rods and cones in the retina in regard to colour and colourless vision I cited two cases of congenital stationary night-blindness, which, through the kindness of Mr. Nettleship, were brought to my laboratory for examination in view of certain researches in which Prof (now Colonel) Watson and myself were together interested. As Colonel Watson has been long at the Front in France, and as he conducted a principal part of the only partially completed examination, I hesitated to give the details without his collaboration. I have now obtained his acquiescence in my request that I should communicate the results obtained to the Royal Society. I now do so. As a matter of fact, I had worked out the observations before my colleague left me for the Front some 2½ years ago, and these results I have put in the following communication:

I may say that any other form of night-blindness than those to which the late Mr. E. Nettleship introduced us would have been useless for the investigation on which we were engaged, as, if it were not so, night-blindness due to disease might call in question some of the deductions to be made Mr. Nettleship's investigations of the family to which they belonged, and its history, left no doubt that we were dealing in our two cases with genuine cases of congenital stationary types.

The subject of congenital night-blindness (sometimes called moon-blindness) has been left with several obscure points unexplained, and the present communication, it is hoped, may throw light on some of them

The late Mr E Nettleship collected a large number of pedigrees in which the characteristics of night-blindness are shown His papers on the subject contain, it is believed, nearly everything that is worth knowing The papers were published in the 'Royal London Ophthalmic Hospital Reports'* He has amply proved that congenital night-blindness is hereditary He commenced with the work that the late Florent Cunier† published in 1838 on a family in which night-blindness existed Cunier gave long genealogical pedigrees with the names, domiciles, and dates of birth and marriages of the members The total number of persons in seven generations was 629, of whom 56 were affected by night-blindness. By subsequent researches Mr Nettleship was able to enlarge the pedigree, which as it stands now reaches 2121 persons, 1001 males and 960 females (of the remainder, sex not known), of which 72 males and 62 females, and one (sex not stated) are known to have been night-blind, and all of whom sprang from a common ancestor, Nougaret by name, who lived about 1600

This is only one of the several family pedigrees which Mr Nettleship gives, but all show that there is heredity in congenital night-blindness There is a great distinction between the night-blind and the congenital monochromatic vision, of which cases are rare The one can see in daylight all the spectrum colours as ordinary normal vision does, the other does not. A quotation from one of Mr Nettleship's papers will give an idea of how congenital night-blindness manifests itself. After a description of the retina and refraction, he says "(The patient) is extremely night-blind, but can see some of the brighter stars, and can do well by bright moonlight and artificial light On a moonless night has a great difficulty in finding his way, cannot see the street before him, but guides himself in keeping to the middle of the road, and looking up and recognising the sharp and black line of the house tops against the sky.

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* See vol. 17, Part III, and vol 27 of the 'Ophthalmic Society's Transactions,' besides others.

† Fl Cunier, 'Médecin Militaire,' "Histoire d'une Héméracopie Héréditaire depuis Deux Siècles dans une Famille de Vendemair près Montpellier"

"A poor man, vine labourer, slow and awkward by training, but intelligent and most graphic in his description if allowed time to find words and gestures. In walking up and down the partially darkened room" (in which he was examined), "he would put his hands out and sometimes stop altogether; indeed, he had the aspect of one blindfolded He was quite unable to see the fingers or even the back of the hands at 0 3 metre with an illumination that was ample for a normal person. He had been like this as long as he could remember (he was 46 years old), but had, nevertheless, been obliged to serve five years in the (French) Army His sight is not getting worse

"This man's son, 16 years old, is as night-blind as his father"

In the case of the monochromatic vision to which I have alluded, they all saw well in low illumination, and but moderately in bright daylight These cases have almost exactly opposite kind of vision. For this reason a physical examination of the vision of the night-blind seems desirable, and though this examination was not completed in some points such as colour field, it is believed that we have indications of marked differences in the perception of colours and light when tested by the spectrum method

The clinical aspects of night-blindness, of course, my colleague and myself are unable to discuss It suffices to say, I think, that these two cases fulfilled the object we had in view, viz, to examine two of the congenital cases which (the late) Mr Nettleship included in one of his pedigrees

The first patient, Mr. E, a clergyman, came to the laboratory on February 25, 1913, when he was examined in the writer's apparatus and presence for his extinction of colour from the red to the blue of the arc spectrum

The luminosity of the D line coming through slits to the screen was the standard of the intensity of the luminosity of the different rays of the spectrum This was gradually diminished by an annulus placed in the path of the ray until he just saw no light in the illuminated small square in the darkened camera attached to the spectroscopic apparatus He said that all light had vanished when the colour was extinguished He repeated the observations in a reverse manner, noting the advent from darkness of the spectrum colour.

This series of observations was repeated in a second apparatus in which the source of light was a Nernst lamp, the thread of which was rendered incandescent by an ampère of current from a battery of 100 volts, the current was kept constant. Similar observations were made by this apparatus and recorded, and afterwards converted to the arc scale Finally, the luminosity curve of his spectrum was taken and compared at the time with that of my colleague.

In his observations he invariably said that, when the extinction was noted by the reduction of the said intensity, whenever the colour was gone all the light had also gone In other words, the same reduction in intensity of the light was the threshold for both light and colour In the paper, "The Threshold of Vision for Different Coloured Lights,"* the same identity of extinctions of light and colour at the fovea of Class I retina is to be noted, as given by all observers, though the extinction of light outside the fovea is nearly 0·0001 times less than it is for colour at the fovea.

Comparing the loss of colour for my own eye (Class II) when the intensity of the spectrum D is one candle-foot, the colour is extinguished at an intensity of about 0 0016 candle-foot, and the extinction of light at about 0 000035 candle-foot intensity For "E" the extinction of both light and colour in both cases takes place at about 0·0015–6. This indicates that the extinction of the feeble white light (or, as I have called it, of the fourth colourless sensation) is dependent on some other retinal perception that the normal eye possesses beyond that possessed by the night-blind. The same is probably the cause of the difference in the fovea of No I retina compared with that of No II retina If the rods and cones fill the places in reference to the light sensations which have been allotted to them, as stated in my last communication on the fourth sensation, then there is an absence of sensitive rods in the whole retinæ of the night-blind The same remarks may apply to the observer B. and his measures, though they differ slightly in shape for those of E, but not more so than do the curves of various observers in the 'Philosophical Transactions' paper just referred to

On May 20, 1913, B came for his examination at the colour laboratory He is a clergyman, and is, we believe, a cousin of E His examination was conducted on the same lines as that of E, the arc light alone being used. In addition, he was made to match the colour of the D sodium line with mixed colours of thallium and lithium blue, in what is called the anomaloscope, a very useful instrument, though only partially indicative of any defect in colour vision

[In Colonel Watson's and my paper, jointly published in the 'Philosophical Transactions,' it was shown that there seemed to be two classes of retina one in which only the colours of the spectrum were recognised in the spectrum and no colourless rays affected the fovea, and the other class in which both colour and the feeble colourless rays were equally effective. As we soon gathered the retinæ of the two patients at any part apparently were not stimulated by the colourless rays, we determined to compare them with the No I class of retina, for which purpose we had to reduce the arc results into the Nernst coloured spectrum This was done, and the night-blinds' extinction of colour coincided practically with the threshold of No I retina at the fovea.]

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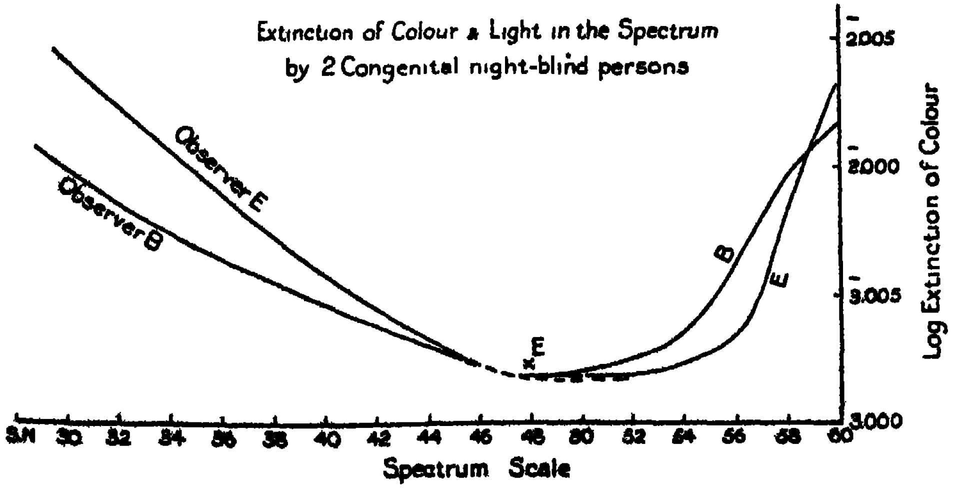
* 'Phil. Trans.,' A, vol. 216.

It should be mentioned that Mr Nettleship divides congenital night-blindness into two classes, both showing the same night-blindness. The one is myopic, but the other apparently normal, as far as refraction is concerned The two observers appear to be in the first division

The night-blind luminosity curves, taken with a luminous spectrum of the arc, compared with that of the normal eye, show that the intensity of colour is the same for both Allowing for the difficulties of the shadow test, and a certain error in observation which is found in the flicker test, the curves of luminosity of both may be said to be the same The luminosity curves can be best compared by making the night-blind measures the numerator and the normal measure the denominator of a fraction. The following are the fractions for E and B ·—

S S N	58	56	54	52	50	48	46	44	42	40	38	36	34
$\frac{E}{N}$	1 12	1 12	1 06	1 00	1 00	0 98	1 00	0 92	0·90	0 93	0 96	1 02	1 05
$\frac{B}{N}$	0 97	1 00	1 00	1 02	1 02	1 00	1 08	1 08	1 08	1 08	1 10	0 94	1 05

Allowance has to be made for a difference in pigmentation of the foveal region It has been shown in my recent communication on the fourth sensation that the added luminosity of the sensation of this fourth sensation is



negligible when the spectrum is of normal brightness (say when the D ray is of ½ candle-power brightness at foot distance from the screen) These measures show that the luminosity of such a spectrum is, within the limits of error, the same in both cases

Annulus 315 Observer E

S S N	Corrected annulus reading	Absorption factor	Product of 2 and 3.	Or	Reduction to one candle foot + $\bar{1}$ 155 (log)	Limit of vision
60	66	−0 ·01170	−0 772	$\bar{1}$ 228	$\bar{2}$ 383	0 0242
58	106	−0 ·01184	−1 255	$\bar{2}$ 745	$\bar{3}$ 900	0 0079
56	143	−0 01197	−1 712	$\bar{2}$ 288	$\bar{3}$ 440	0 ·00276
54	158	−0 01212	−1 915	$\bar{2}$ 085	$\bar{3}$ 240	0 00174
52	153	−0 01229	−1 979	$\bar{2}$ ·021	$\bar{3}$ 176	0 00150
50	124	−0 01244	−1 966	$\bar{2}$ 033	$\bar{3}$ 188	0 00154
48	136	−0 01259	−1 926	$\bar{2}$ ·074	$\bar{3}$ 229	0 ·00170
46	144	−0 01273	−1 833	$\bar{2}$ 167	$\bar{3}$ 322	0 00210
44	136	−0 01288	−1 752	$\bar{2}$ 248	$\bar{3}$ 403	0 00253
42	126	−0 01302	−1 641	$\bar{2}$ 359	$\bar{3}$ 514	0 00374
40	115	−0 01318	−1 516	$\bar{2}$ 484	$\bar{3}$ 639	0 00436
38	105	−0 01334	−1 401	$\bar{2}$ 599	$\bar{3}$ 754	0 00560
36	91	−0 01354	−1 232	$\bar{2}$ 768	$\bar{3}$ 923	0 00838
34	78	−0 01374	−1 072	$\bar{2}$ 928	$\bar{2}$ 083	0 ·0121
32	66	−0 01390	−0 917	$\bar{1}$ 083	$\bar{2}$ 238	0 0173

Annulus 315. Observer B

S S N	Corrected annulus reading	Obstruction factor	Product of 2 and 3.	Or	Reduction of D to one candle foot + $\bar{1}$ 155 (log)	Limit of vision
60	73	−0 01170	−1 ·088	$\bar{2}$ 912	$\bar{2}$ 097	0 00125
58	102	−0 01184	−1 208	$\bar{2}$ 792	$\bar{3}$ ·947	0 00607
56	135	−0 01197	−1 156	$\bar{2}$ 484	$\bar{3}$ 639	0 00436
54	147	−0 01212	−1 782	$\bar{2}$ 218	$\bar{3}$ 373	0 00236
52	155	−0 01229	−1 ·905	$\bar{2}$ 095	$\bar{3}$ 250	0 00175
50	157	−0 ·01244	−1 953	$\bar{2}$ 047	$\bar{3}$ 202	0 00159
48	157	−0 ·01259	−1 972	$\bar{2}$ 028	$\bar{3}$ 183	0 00152
46	152	−0 ·01278	−1 ·935	$\bar{2}$ 065	$\bar{3}$ ·220	0 ·00166
44	144	−0 01288	−1 ·855	$\bar{2}$ 145	$\bar{3}$ 300	0 ·00200
42	136	−0 01302	−1 771	$\bar{2}$ 229	$\bar{3}$ ·384	0 ·00242
40	128	−0 ·01318	−1 687	$\bar{2}$ 313	$\bar{3}$ 468	0 ·00294
38	120	−0 ·01334	−1 601	$\bar{2}$ 394	$\bar{3}$ ·549	0 ·00354
36	112	−0 ·01354	−1 516	$\bar{2}$ ·484	$\bar{3}$ ·639	0 ·00436
34	103	−0 ·01374	−1 415	$\bar{2}$ 585	$\bar{3}$ 740	0 ·00550
32	93	−0 01390	−1 ·293	$\bar{2}$ ·707	$\bar{3}$ ·862	0 ·00727

B, 61 3; Li, 59 ·8; C, 58 1; D, 50 6, E, 39 ·8; *b*, 37 ·7; F, 30 ·05; Blue Li, 22 ·8; G, 11 ·2. The above give the positions of Fraunhofer lines on the (arc) spectrum scale

A Contribution to the Study of the Magnetic Properties of Manganese and of some Special Manganese Steels.

By Sir Robert Hadfield, F R S, C Chéneveau, and Ch. Géneau

(Received May 4, 1917.)

I —General and Detailed Consideration of the Apparatus and Method of Measuring Magnetic Properties

The object of this research has been to find a connection between the magnetic susceptibility per unit mass (called the mass susceptibility) of different manganese steels and their chemical constitution All measurements were made with the help of the magnetic balance of P Curie and C Chéneveau * In this instrument the body to be tested is attached to one arm of a torsion balance A horseshoe magnet can be moved horizontally, so that the direction of motion is at right angles to the line joining the poles, and the body is gradually brought into a stronger and stronger field The observed deflection of the torsion balance increases at first, then diminishes, and becomes zero when the body lies between the poles of the magnet If the magnet be moved still further, a deflection in the opposite direction is observed, which again passes through a maximum at the point where the variation of the field of force is greatest Fig 1 illustrates the deflections observed in a particular case. The difference between the readings of the torsion balance when the deflections are greatest on either side is proportional to the mass susceptibility, so that this quantity may be determined by comparison with a specimen whose susceptibility is known.

Throughout the paper the mass susceptibility is denoted by the symbol χ.

Glass Envelopes

Unless otherwise indicated, all specimens were included in glass tubes of very different shapes, and generally with very thin walls These are shown in fig 5. They gave an inappreciable deflection in the most intense magnetic fields, even when the balance was at its greatest sensibility, *i.e* when the platinum wire was the thinnest procurable The use of this glass, the search for which has entailed great trouble, is exceedingly advantageous, in that it does away with the necessity of correcting for the magnetic effect of the envelope.

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* 'Journ. de Phys.,' Series 4, vol. 2, p. 796 (1903); 'Phil Mag,' vol 20, p. 367 (1910).

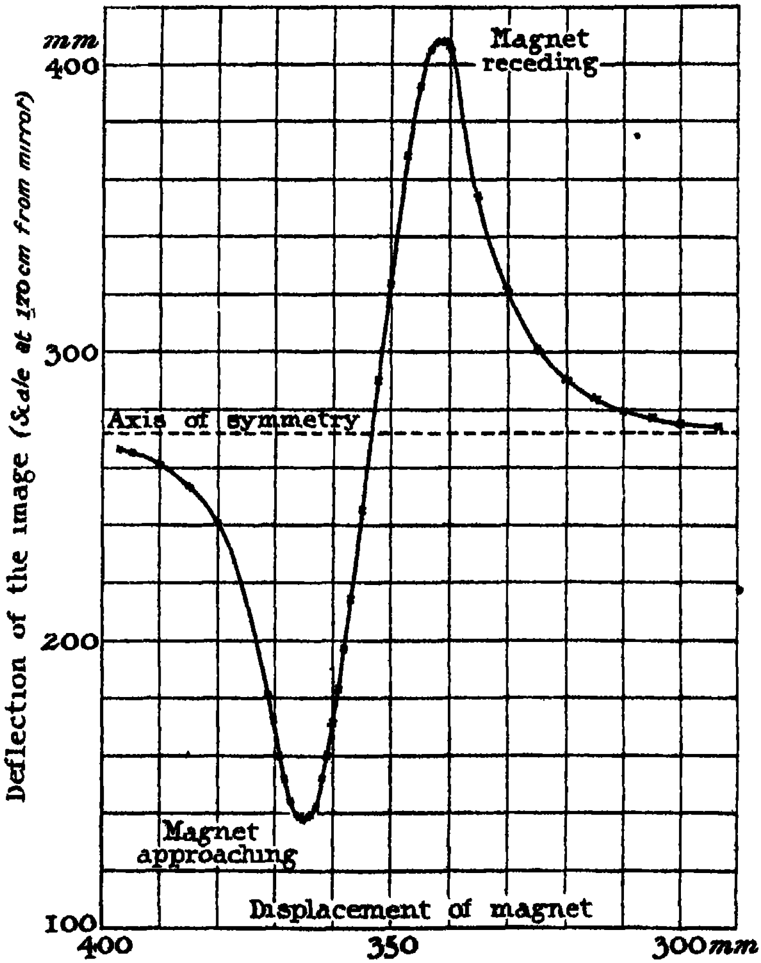

FIG 1.—Variation of deflection shown as function of displacement of magnet.

Standard of Comparison

P. Weiss and Foex* have devoted great attention to the determination of the mass susceptibility of salts in solution and in the solid state. We have adopted, as standard of comparison, anhydrous cobalt sulphate, to which these workers assign the value

$$\chi = +58 \times 10^{-6} \text{ at } 20^{\circ} \text{ C.}$$

We have prepared some of this salt, adhering strictly to their instructions. It was divided into three portions, of which one was placed in a tube, which was then sealed and weighed. This served as a standard. The following Table shows that this sample, placed in fields of different strengths and under different conditions of sensibility, always gave concordant results

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* 'Journ de Phys.,' Series 5, vol. 1, p. 274 (1911).

Field	I	II	I	II
Sensibility, calculated	7200	1440	180	36
	mm	mm.	mm	mm
Cobalt sulphate (sealed tube m = 0·914) deflection for 1 grm at 1 metre	—	71	-	1 9
Cobalt sulphate (open tube m = 0 941) deflection for 1 grm at 1 metre	342	69	10	2 2
Cobalt sulphate (open tube m = 0 807) deflection for 1 grm at 1 metre	—	70	—	—
Mean	342	70	10	2 1
Ratio of sensibilities	5 0		5 0	
Ratio of deflections	4 9		4 8	

Magnetic Fields

Two permanent magnets (Allevard steel), of the following dimensions, were used to establish the field of the Curie balance. The ratio of the fields was about 5

	Magnet I	Magnet II
Exterior diameter	15 cm	15 cm
Interior "	12 "	12 "
Thickness	1 45 cm	1 45 cm
Thickness of pole pieces	0 45 "	0 65 "
Height of polar space	3 0 "	3 0 "
Breadth of " "	1 25 "	2 5 "
Field	320 gauss	1600 gauss

The values of the field strengths, which are only approximate, were calculated from the general formula of the apparatus by the aid of the known value of χ for cobalt sulphate and from the constants of the torsion balance. An excellent verification of the validity of this procedure was obtained in Case II by the aid of a Grassot fluxmeter This gave 25,500 maxwells. Taking an approximate value of 12 for the permeability of steel, and 1 235 sq. cm. for section of polar space, a field of 1750 gauss was calculated

To obtain more feeble fields the magnets were shunted, but the results of such tests were very irregular

Sensibility

Varying sensibilities were obtained not only by interchanging the magnets, but also by altering the diameters of the platinum suspending wires. The relation of the sensibility to the diameters is given in the following Table :—

Wires	1.	2	3	4
Length Diameter	35 4 cm 0 01 „	33 8 cm 0·02 „	33 1 cm 0 05 „	33 5 cm 0 01 „
Sensibility with—				
Magnet I	115,000	7,220	180	—
„ II	—	1,440	36	23,000

Wire No 1 broke very early on in the work, and was replaced by No 4 No 3 gave very irregular results. The most trustworthy results have been obtained with Nos 2 and 4

The effect of the susceptibility of the air was found to be negligible in all cases

Influence of Temperature—The following experiments made on steel 1010 N when placed in a glass tube showed that in these experiments a variation of 26° C had but little effect on the results

Steel No 1010 N. Mass, 5 782 grm Distance of scale, 236 cm

	Temperature	Deviation.	Deviation per grm. at 1 metre
	° C	mm	mm
Ordinary condition	26·3	97	7
Tube dipped in ice	13	90	6 5
Tube retained in ice	0	84	6
Allowed to heat up for one hour	21	85	6
Steel rusted and washed in hydrochloric acid	20	105	7

Accuracy of Results.

In all cases, on account of experimental difficulties, and the many sources of error which had to be overcome, the steel results can only be trusted to within 5 per cent.

The manganese results are a little more accurate. Their precision is indicated in the results

II —STUDY OF THE MAGNETIC PROPERTIES OF MANGANESE.

Through the kindness of M P Weiss, we have been able to work on the specimens of manganese used by M Kamerlingh Onnes and himself in their work on the magnetic properties of manganese, chromium, and vanadium [*]

(1) *Powdered Manganese*

The first specimen of manganese was in the form of a grey powder It was prepared from an amalgam of manganese, which, when heated to a sufficient temperature in pure dry hydrogen, yielded the metal in the form of the powder mentioned above This powder was pressed into a glass tube in order to form a rod, and then sealed

The two previous workers have shown that this form of manganese is para-magnetic

After a qualitative confirmation of this view we proceeded to a quantitative test by determining the deviation of the sealed tube in fields I and II Unfortunately, we were obliged to break this tube (naturally with great care) in order to determine the mass of the manganese and the magnetic quality of the glass

The following results were obtained, a very small glass correction having been allowed for

Field	Sensibility	$\chi \times 10^6$	Temperature
			° C
1600 gauss	7200	+10 6	18 6
820 „	1440	+10 9	17 6
Average		+10 8	18

Owen[†] gives the number $+8\,9 \times 10^{-6}$ at 18° C Our number agrees with that found by H du Bois and Honda, while Gebhardt's value, quoted by Weiss and Kamerlingh Onnes[‡] is five times too large Without doubt their samples were oxidised

We have repeated the experiment upon the manganese powder after it had been kept for 10 days in a closed tube We have obtained

$$\chi \quad 10^6 = 11\,2$$

There was therefore no sensible oxidation

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[*] Pierre Weiss and Kamerlingh Onnes, 'Comptes Rend. de l'Acad. des Sci,' vol 150, p. 687 (1910).

[†] 'Ann. Physik,' vol. 37, p 657 (1912)

[‡] 'Journ. de Phys.,' Series IV, vol. 9, p 555 (1910)

We have assured ourselves by the sensitive potassium thiocyanate reaction that there was no trace of iron in the sample.

We conclude, then, that the mass susceptibility of powdered manganese at 18° C. is

$$\chi = 11.0 \times 10^{-6}, \text{ within 2 per cent.}$$

(2) *Cast Manganese.*

The second sample of manganese was composed of small black pieces, in parts coloured a metallic grey or yellow. It was prepared by the fusion of the powder in an electric resistance furnace, the atmosphere being hydrogen. Weiss and Onnes state that this form is ferro-magnetic.

With these pieces we carried out a series of experiments which showed that the substance was either attracted or repelled, according to its position in the field. This observation, which was verified for each separate piece, indicates that this sample of manganese was magnetised. A slight, but sensible, effect upon a compass needle confirmed this.

Five years and five months after our first experiments we again tested this cast manganese, and found that it still retained its magnetism—which shows a strong coercive force. We then carefully heated two fragments to a bright red—one a fragment of the black (No. 1), the other a fragment with metallic lustre (No. 2). The following results were obtained. Initially the highest values of χ were of the order $+2000 \times 10^{-6}$.*

Black fragment	$\chi \times 10^6 = +26.5$
Brilliant fragment	$\chi \times 10^6 = +27.1$

The influences of heating, of temperature, and of the size of the sample, are of some importance. With a fairly large fragment (No. 3) we obtained

$$\chi \quad 10^6 = +130$$

---

* M. Kuh (Thèse, Zurich, 1911) has given for the magnetic moment M of cast manganese:—

M	H	$\chi \times 10^6$
	gauss	
0.7080	875	2130
1.622	1500	1060

It is to be observed that for a field strength of about 320 gauss we have found numbers of the same order for the highest values of $\chi \times 10^6$ of ferro-magnetic manganese, viz., 2200 and, after an interval of five years, 2700.

Several washings in pure hydrochloric acid gave successively

1st heating	1st washing	$\chi \times 10^6 = +130$
„	2nd „	$\chi \times 10^6 = +116$
„	3rd „	$\chi \times 10^6 = +110$
„	4th „	$\chi \times 10^6 = +109$

Reheating and again washing in hydrochloric acid gave

2nd heating	1st washing	$\chi \times 10^6 = +34$

It is therefore clear that sufficiently prolonged heating considerably lessens the magnetic properties of magnetic cast manganese whilst the oxidation of the specimen slightly raises its susceptibility

In heating specimen No 1 above in order to oxidise it, without washing in acid, we obtained

$$\chi \times 10^6 = +53,$$

while with specimen No 2, after washing in acid, we obtained successively

Initial value	$\chi \times 10^6 = +27\ 1$
1st washing	$\chi \times 10^6 = +18\ 1$
2nd washing	$\chi \times 10^6 = +16\ 9$

The effect of heating and washing is clearly to cause an approach to the value for powdered manganese (+ 11 0) Heating produces demagnetisation, subsequent washing in acid the disappearance of the oxide formed

Desirous of saving the remainder of the manganese, further experiments were discontinued

If the manganese demagnetised by heating and deoxidised by the action of hydrochloric acid were ferro-magnetic, it should easily re-acquire its magnetism when placed in the field of magnet No 1, or even by simply bringing into contact with another magnet That this is not so, is shown by the following results, giving identical values before and after the attempted magnetisation

Specimen	$\chi \times 10^6$	
	Before magnetic action	After
No 2	+ 16 9	+ 169
No 3	+ 109	+ 109

The high coercive force of manganese fused in hydrogen (analogous to that of iron under similar conditions) and the preceding experiments lead us to the belief that its ferro-magnetism is due to hydrogen. The difficulty of completely disengaging a dissolved gas from a liquid is well known, and *a fortiori* the same difficulty must arise with a solid.

It is hardly to be expected that the heat applied to our samples was sufficient to drive off all the hydrogen, and so to lower the magnetic properties of the substance to that of the pure metal But the gas was driven off sufficiently to prevent the metal being permanently magnetised This conclusion is verified by an experiment of Seckelson,* who has obtained electrolytically ferro-magnetic manganese, which necessarily contains hydrogen just as did the electrolytic iron of Cailletet and d'Houllevigue.†

Conclusion

Manganese is para-magnetic, whatever its condition Its mass susceptibility (χ) at 18° C is $+11\,0 \times 10^{-6}$ The ferro-magnetic properties which it possesses after being melted or deposited by electrolysis are without doubt due to occluded hydrogen

It is perhaps in place to point out that the presence of occluded gases in metals or alloys which have been prepared by melting should not be neglected when examining the magnetic properties

III —STUDY OF THE MAGNETIC PROPERTIES OF SOME MANGANESE STEELS

The first experiments made with steel (in the form of small cylindrical rods) gave very irregular results For instance, they showed the steel 2084 A as ferro-magnetic, which was subsequently confirmed by experiments But they also gave the steel 1010 N as ferro-magnetic, whereas it is in fact para-magnetic. They showed, however, in all cases, in spite of the irregularities, the cause of which we shall try to find, that all the steels save the one mentioned above were only slightly magnetic, and the employment of the type of apparatus used was justified in the overwhelming majority of cases

In what follows we confine ourselves to para-magnetic steels, later we will deal with the ferro-magnetic variety

(1) *Influence of Chemical Reagents*

The paradoxical result obtained with the steel 1010 N may easily be explained on the hypothesis that some ferriferous impurity gives to the specimen its ferro-magnetic character. In fact, if the cylinder is gently washed with hydrochloric acid, the mass susceptibility passes rapidly from a value of χ too large to be measured to $\chi = 100 \times 10^{-6}$ The following experiment lends weight to this assumption When the cylinder 1379 D_2 was cut by an iron tool without subsequently washing in acid, χ rose in value from 76×10^{-6} to 1960×10^{-6} As the steel bars were worked into shape, and as it is often necessary to test small pieces cut from them, it is therefore essential, before using them in the balance, to wash them in hydrochloric acid, in order to remove traces of iron left by the tools An examination of the curves in fig 2 shows that for the steel 1010 G, whatever the form of the specimen, the deflections which, before chemical treatment, were not proportional to the weight became so when the sample was so treated *

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* 'Wied. Ann.,' vol 67, p. 37 (1899).

† 'Journ. de Phys ,' Series 3, vol. 6, p. 245 (1897)

Steel 1010 G (Magnet 2, Sensibility 1440)

	Dimensions		Before attack by HCl		Time of attack	After attack	
	Diameter	Height	Weight	Deviation per gramme at 1 metre		Weight	Deviation per gramme at 1 metre
	mm	mm	grm	mm	min	grm	mm
Small cylinder	4 5	2	0 252	35	15	0 247	24
Sphere	4 5	—	0 377	30	30	0 367	22
Medium cylinder	4 5	4	0 407	30	45	0 478	23
Large cylinder	4 5	6	0 744	24	60	0 708	23

Finally we mention here the observations of A C Jolly† and of one of us‡ upon copper In this case the mass susceptibility only became constant after a first washing with hydrochloric aid Subsequent washings had very little effect This is also shown in the curve for the steel 1010 N (fig 2), although the mass of the specimen was reduced in the process from 4 510 grm to 1 833 grm

Although it would seem that in many instances the ferriferous impurity resides at the surface, yet there are others in which this does not seem to be entirely the case. We find it difficult to decide, however, whether we are really dealing purely with superficial effect, with an internal effect, or with a combination of the two. We simply estimate that the diminution in weight due to washing is at its maximum about 1 or 2 per cent

Other experiments upon the steels 1010 G_2, 1010 F, 1010 O, 1010 M, 1010 E, 1339, 1343 A, 1343 B, 598 and 620, and the curves for the specimens 1010 D

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* The hydrochloric acid used was strictly diamagnetic with $\chi = -0\ 58 \times 10^{-6}$ If the true value (from Quincke's work) is $-0\ 66 \times 10^{-6}$, then the acid above does not contain more than 1/100,000 per cent. of iron

† 'The Electrician,' vol 65, No 25, p 1013 (1910)

‡ C. Chéneveau, 'Journ de Phys.,' Series 4, vol. 9, p. 163 (1910).

and 1798 H_2 (fig 2) lead equally to the conclusion that, after a fairly prolonged washing in hydrochloric acid, the deflections become practically proportional to the mass of the specimen

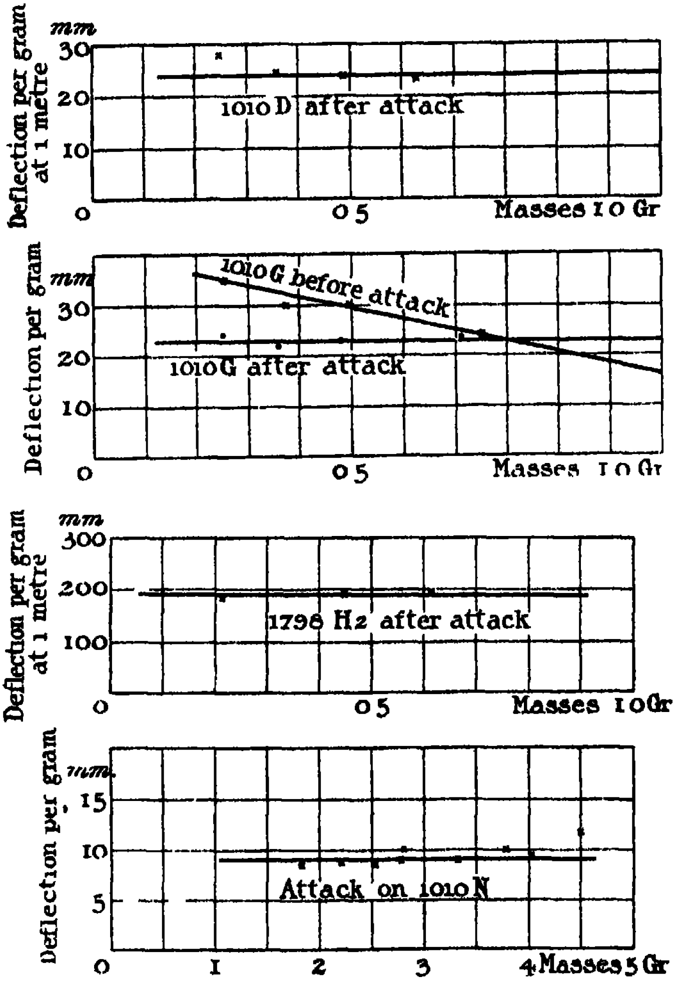

Fig 2 —Influence of hydrochloric acid on sensibility

(2) *Influence of Size of Specimen*

For para-magnetic bodies in which the susceptibility is not a function of the field strength, the shape of the specimen should be immaterial and, in consequence, its dimensions should not affect the measurements, otherwise than through the change of mass

Certain of the steels were very hard to work and in the absence of diamond dust we shaped them with ordinary tools. But for our experiments we chose fairly workable steels which were somewhat easily attacked by the acid. The washing was continued until strict proportionality was obtained between deflection and weight.

With the steel 1798 H_2 we find that when the sample is properly placed between the pole pieces the deflection is the same whether the specimen is placed parallel to the line joining the poles or perpendicular to it

	Number of observations	Mean deflection
		mm
Parallel	6	271
Perpendicular	4	272

With the same steel we carried out a series of experiments under different conditions of sensibility, in which we varied one of the dimensions of the cylinder, keeping the other constant The curves in fig 3 show the relationship between the deflection and the height and diameter of the specimen

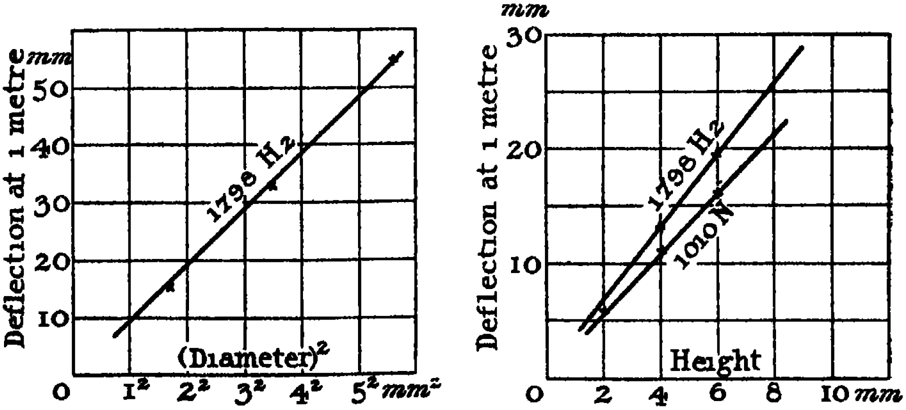

FIG 3 —Variation of deflection with dimensions.

Experiments were also carried out on a very different type of steel (1010 N) The curve (fig 3) shows, in agreement with our previous results, that, provided certain limits are not exceeded, the deflection is proportional to the dimensions of the specimen, *i.e.*, to its weight.

The conclusion to be drawn from these preceding series of experiments is that serious error can only be avoided by working on specimens giving deflections proportional to their masses. It is, therefore, advantageous to work with comparatively small samples.

(3) *Influence of the Position of Specimen.*

The same conclusion is reached in studying the effect of the position of the specimen in the field on the accuracy of the results The following experiments were carried out with a small cylinder of steel 1798 H_2, having the following dimensions Height, 2 90 mm , diameter, 3 30 mm , mass, 0 197 grm It was washed in acid and placed in the tube sketched in fig. 4

(a) *Variation of the deflection with the vertical displacement of the metal in the field, the specimen being at equal distances from the pole pieces and its centre on the line of symmetry of the polar interstice*—The results are shown in Curve I, fig 4

(b) *Variation of the deviation with the distance of the metal from the pole pieces, the height of centre of specimen being* 16·5 *mm.*—The results are shown in Curve II, fig. 4

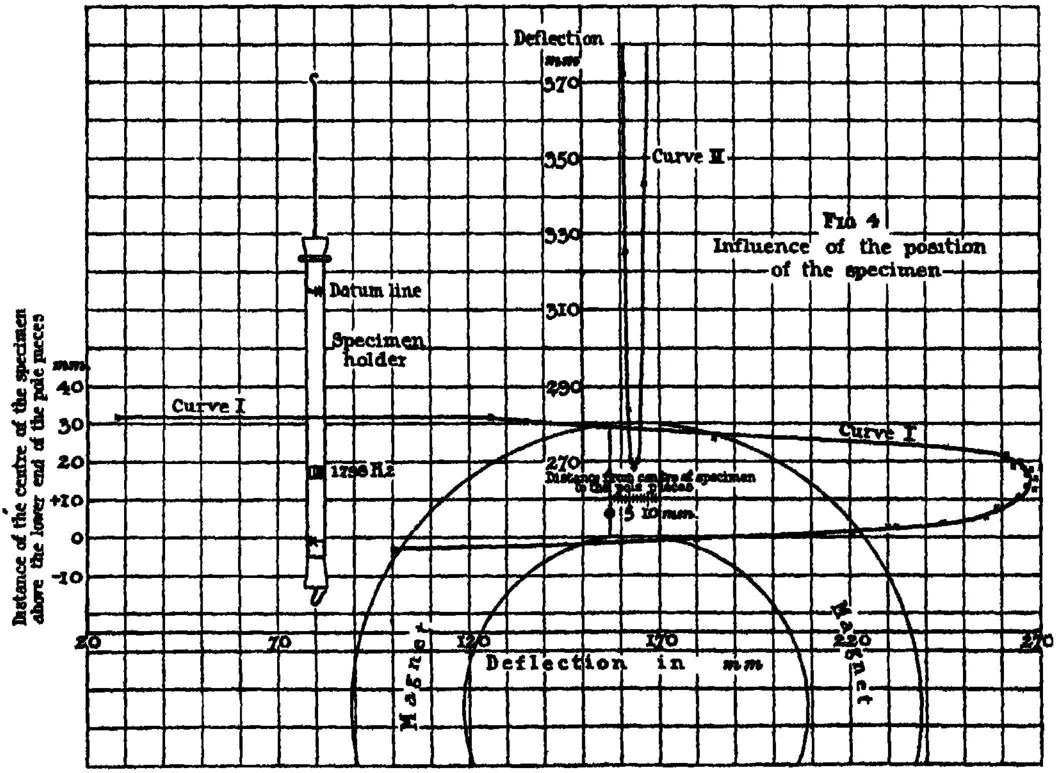

FIG 4
Influence of the position of the specimen

(c) *Variation of deviation with the position of the magnet.*—This is shown in fig 1.

It is easy to deduce from these experiments the position for which small displacements least affect the accuracy of the results.

(4) *Best Experimental Conditions.*

It has been shown that it is advantageous to use fairly small specimens having a well defined geometrical shape (cylindrical) The specimen must be sufficiently washed in hydrochloric acid, and accurately placed with its centre at the centre of the polar interstice—that is, at equal distances from the pole pieces, and vertically, so that the centre of the specimen lies in the central line joining the pole pieces

This adjustment, which is somewhat delicate, can, nevertheless, be easily accomplished by the employment of tubes having the forms shown in fig 5 The specimen is retained in position by means of lightly packed cotton-wool

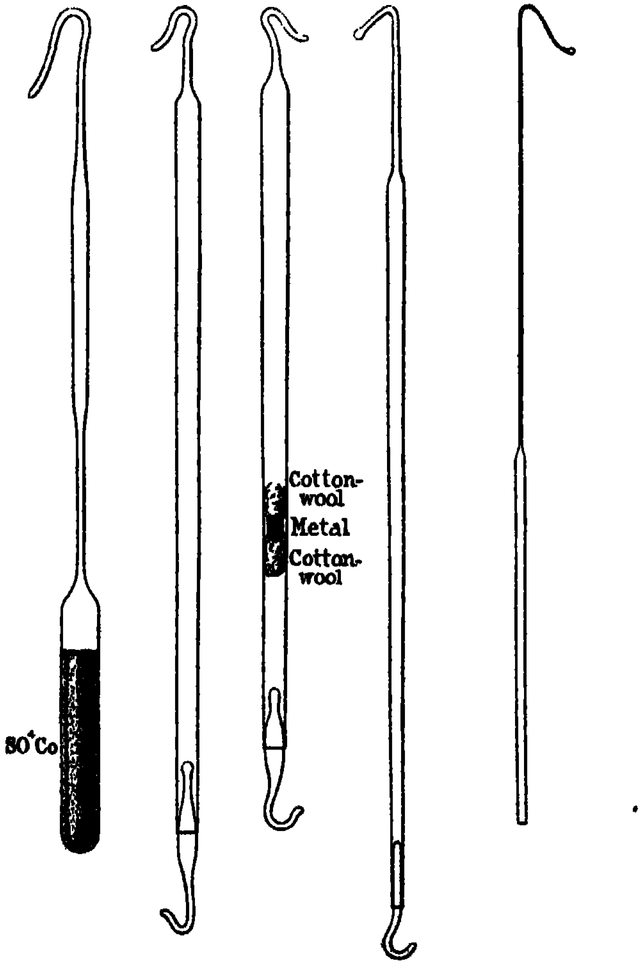

Fig. 5.—Various experimental tubes.

The following numbers for the specimen 1798 H_2 show that error caused by small variations of height is slight, if the average of several experiments be taken

Height	Number of experiments	Deflection	Mean
mm		mm	mm
17.	1	271	
	2	263	266
	3	265	
16	1	268	267
	2	266	
15	1	268	268
	2	268	

(5) *The Case of Ferro-magnetic Steel*

The magnetic torsion balance only lends itself to the determination of the mass susceptibility of diamagnetic and paramagnetic bodies For the successful study of ferro-magnetic bodies, it is essential that the rod should be placed parallel to the field, and that its length should be great in comparison with its diameter *

We have only been able to work on very small specimens, weighing a few milligrammes, and, under these circumstances, it was impossible to eliminate the demagnetising field due to the magnetisation In all cases, however, we have endeavoured to correct this error by approximate mathematical computation

Such a correction is naturally somewhat uncertain, and we tested and verified its approximate validity by comparing the specimen 2084 A with a sample of pure iron wire (piano wire) Washing in hydrochloric acid was, as usual, carried out before testing

To prevent the tube coming into contact with the magnet of the instrument when it contained ferro-magnetic matter, we attached a weight from a hook fixed to the lower end of the tube, as shown in fig 4. We ascertained that this did not affect the deflections when the weight did not exceed 10 grm

Weight	20 grm.	10 grm.	5 grm.	2 grm.
Deflection at 164 cm . .	172 mm.	178 mm.	178 mm.	180 mm

The following are the results (after all corrections had been applied) obtained in two perpendicular directions, with the sample placed either vertically and at right angles, or horizontally and parallel, to the field. The different values obtained in these cases is typical of the ferro-magnetic state.

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* *e.g.*, P. Curie, 'Œuvres,' 1908, p. 289.

Body	χ (corrected) Horizontal direction parallel to field H = 450 gauss *	χ (not corrected) Vertical direction perpendicular to field H = 820 gauss
Steel No 2084A	0·24	0·017
Pure iron	0·47†	0·022

Thus, the steel 2084 A is slightly ferro-magnetic compared with iron. Its mass susceptibility is of the same order as that of pure iron.

A curious point arises in connection with this specimen. Before being sent to Paris in 1911, it was roughly examined with a hand magnet, and showed only slight magnetic quality, certainly not more than 4 or 5 per cent. of that of pure iron. The above test, however, which was made about six years later, indicates distinct ferro-magnetic properties of the order of 50 per cent. of that of pure iron. This marked difference was so striking that it was thought well to make a further examination of the specimen in the original approximate manner, with the result that this considerable increase of magnetic susceptibility with progress of time was fully confirmed. The specimen had not, during an interval of several years, been subjected to heat or mechanical treatment, and it can only be inferred that a gradual secular and internal change in magnetic character, apart altogether from any extraneous influences, had taken place in this material.

(6) *Study of Powdered Steels*

These experiments were carried out on filings from samples 1010 and 1109 D. The chemical composition (which is very similar to that of the rods) is given below, together with the results of the magnetic tests.

These filings are so magnetic that we found it necessary to use a stouter torsion wire (diameter 0·5 mm.), and to fix a magnetic shunt to magnet No. 2. In a field of about 18 gauss (1/18 of its previous value) we obtained the following results —

Steel No	C.	Si	Mn	Ni	Deflection per gramme at 100 cm
					mm
1010	1·26	0·45	12·00	—	38
1109D	0·60	0·34	5·04	14·55	61

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* The magnet used here was slightly stronger than that in the other case. For the same torsion wire, we deduced from deflections made with cobalt sulphate that the field in the position of greatest deflection was 450 gauss.

† In a field of 450 gauss it can be shown from Curie's experiments that $\chi = 0·46$.

These deflections are approximately 130 times those given by the same material in bars

Due precautions were taken to place the sample at the centre of the tube (fig 6), and we also tested the strict proportionality between weight and deflection (fig 6) up to the limits to which the experiments were confined.

It is certain that these filings contain a considerable quantity of iron Washing in acid was resorted to, but we never reached a final value nor got rid of all the iron With steel 1010 we even found an increase of magnetisation (fig 6), suggesting that the constituents of the powder were unequally attacked by the acid

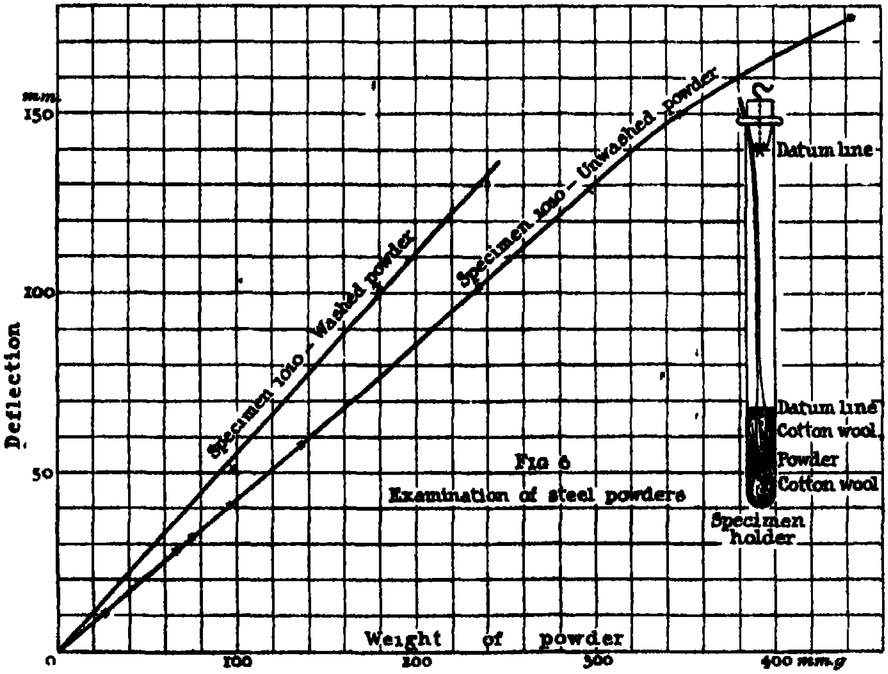

Fig 6

Examination of steel powders

We conclude that filings cannot be used for the determination of the mass susceptibility of manganese steels

IV.—SUMMARY OF EXPERIMENTAL RESULTS

In the following Tables the chemical and physical results for the special manganese steels are collected together so as to determine the relations between chemical composition and the magnetic properties.

(1) *Influence of Manganese*

Within the range examined, it appears that in the case of manganese steels of constant carbon content this metal has only a feeble effect on the magnetic properties

(2) *Influence of Carbon*

The Table also shows that for steels of constant manganese content the susceptibility increases as the carbon content diminishes.

These two conclusions are summarised in the column giving χ as a function of the quotient C/Mn, and also in the curve representing the figure which is reproduced in fig 7 The mass susceptibility of manganese steel is seen to become smaller as the ratio C/Mn increases

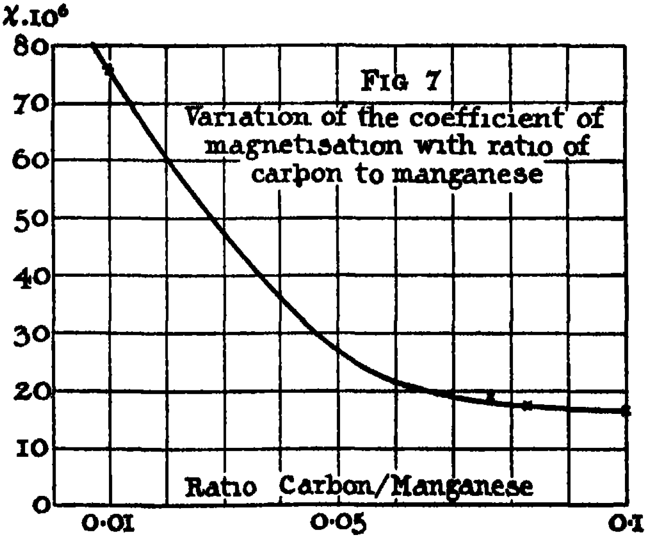
FIG 7

Variation of the coefficient of magnetisation with ratio of carbon to manganese

Recapitulative Table, showing Influence of Manganese and Carbon.

Steel No	C	Mn	$\chi \times 10^6$	Steel No	C/Mn	$\chi \times 10^6$.
80	1 50	15 25	17 −	1879D_2	0·01	78
598	1 54	18 50	17 +	1010G	0·076	19
1879 D_2	0 15	15 20	76	598	0·088	17 +
80	1 50	15·25	17	80	0 1	17 −

General Table of Results.

Steel No	Treatment.	Chemical analysis.							Densities				Mass susceptibilities x 10⁶				
											Barrett, Brown and Hadfield		1600 C G.S units.	320 C G S units.		Mean value	
		C	Si	Mn	Ni	Cu.	Cr	W					Sensitiveness 7200	Sensitiveness 1440	Sensitiveness 23,000		
									°C			°C					
1379 D₂	Quenched in water from 1050° C	0·15	—	15·2	—	—	—	—	15	7·94	7·86	From 12 to 18	—	76·2	—	76	Para-mag
30	Ditto	1·50	—	15·25	—	—	—	—	15	7·83	7·83	Ditto	16·6	16·6	18·2	17	Dittó
598	Ditto	1·54	—	18·50	—	—	—	—	19	7·89	7·88	Ditto	16·4	16·6	18·2	17+	Ditto
620	Ditto	0·88	—	17·5	—	—	3·5	—	19	7·76	7·72	Ditto	18·6	19·0	19·9	19—	Ditto.
1339	Ditto	1·21	—	8·0	2·57	—	—	—	16	7·91	7·95	Ditto	21·9	24·0	22·4	23—	Ditto
1313 C	Ditto	1·40	—	10·25	9·0	—	—	—	15	7·93	7·84	Ditto	32·4	35·6	38·1	35+	Ditto.
1424 B	Ditto	0·83	—	5·9	14·44	2·25	—	—	19	7·98	7·91	Ditto	61·4	63·8	65·4	63—	Ditto
1109 D	Ditto	0·80	—	5·04	14·55	—	—	—	15	7·96	7·84	Ditto	48·7 ?	51·4	55·5	52—	Ditto
1414 A	Ditto	0·60	—	5·04	19·0	—	—	—	16	7·96	7·93	Ditto	65·6	64·6	69·6	67—	Ditto.
1343 A	Ditto	1·34	—	11·1	—	—	—	2·85	15	8·06	—	—	26·3	29·8	30·6	29—	Ditto.
1453 B	Ditto	1·08	—	10·2	—	—	—	2·11	14	8·00	—	—	17·5	17·4	19·0	18—	Ditto.
1798 E₂	Cooled in furnace from 550° C	0·48	—	1·35	19·98	—	—	—	14	8·08	8·12	Ditto	142·4	157·4	161·6	154—	Ditto
2103	Quenched in water from 945° C	1·08	—	0·95	12·88	—	3·12	—	15	7·97	—	—	248·5	248·6	280	259	Ditto
2084 A	Quenched in water from 1050° C	0·19	6·15	12·35	—	—	—	—	14	7·55	—	—	—	240,000 H = 450	—	240 000	Ferro-mag
1010	Ditto	1·21	0·33	10·56	—	—	—	—	15	7·86	—	—	20·7 ?	19·0	19·0	19	Para-mag
1010 D	Ditto	1·34	0·47	13·34	—	—	—	—	15	7·80	—	—	18·6	20·7	20·7	20	Ditto
1010 E	Ditto	1·70	0·35	16·07	—	—	—	—	14	7·77	—	—	19·9 ?	19·9	22·4	21—	Ditto.
1010 F	Ditto	1·94	0·40	19·61	—	—	—	—	14	7·68	—	—	21·9	20·7	21·5	21+	Ditto
1010 G	Ditto	1·24	—	16·23	—	—	—	—	14	7·78	—	—	18·5	19·0	19·9	19—	Ditto.
1010 G₂	Ditto	1·22	0·17	16·80	—	—	—	—	19	7·88	—	—	17·8	16·6	18·2	18—	Ditto
1010 M	Ditto	1·25	0·33	18·35	—	—	—	—	14	7·78	—	—	17·3	16·6	17·4	17+	Ditto
1010 O	Ditto	1·06	0·28	18·65	—	—	—	—	14	7·84	—	—	18·6	17·4	19·0	18+	Ditto
1010 N	Ditto	1·04	0·27	18·90	—	—	—	—	19	7·79	—	—	18·0	16·6	18·2	18—	Ditto

The results obtained in the two following series of experiments are in good agreement with the conclusions arrived at In the first the manganese and carbon contents increase approximately in the same ratio.—

Steel No	C	Si	Mn	C/Mn	$\chi \times 10^6$
	per cent	per cent	per cent		
1010	1 21	0 83	10 58	0 11	19−
1010 D	1 34	0 47	13 34	0 10	20
1010 E	1 70	0 35	16 07	0 10	21−
1010 F	1 94	0 40	19 61	0 11	21+

It is seen that χ remains approximately the same

The second series contains steels of constant carbon content and increasing manganese content As the increment of manganese is only 2 7 per cent, and as we have seen that an increment of 3 5 per cent does not affect χ, it follows that these steels should have the same magnetic quality This is confirmed by the results —

Steel No	C	Si	Mn	$\chi \times 10^6$
1010 G	1 24	—	16 23	19−
1010 G_2	1 22	0 17	16 80	18−
1010 H	1 25	0 33	18 35	17+
1010 O	1 06	0 28	18 65	18+
1010 N	1 04	0 27	18 90	18 −

(3) *Influence of Nickel*

If the nickel-manganese steels, containing approximately the same carbon-manganese ratio, are arranged in order of their nickel content, it will be seen either from the following Table, or from the curve of fig 8, that the mass susceptibility of such steels increases with the proportion of nickel —

Steel No.	C/Mn	Ni	$\chi \times 10^6$
1889	0 15	2 57	23
1818 C	0 14	9·0	35
1109 D	0·15	14·56	52
1414 A	0 12	19 0	67

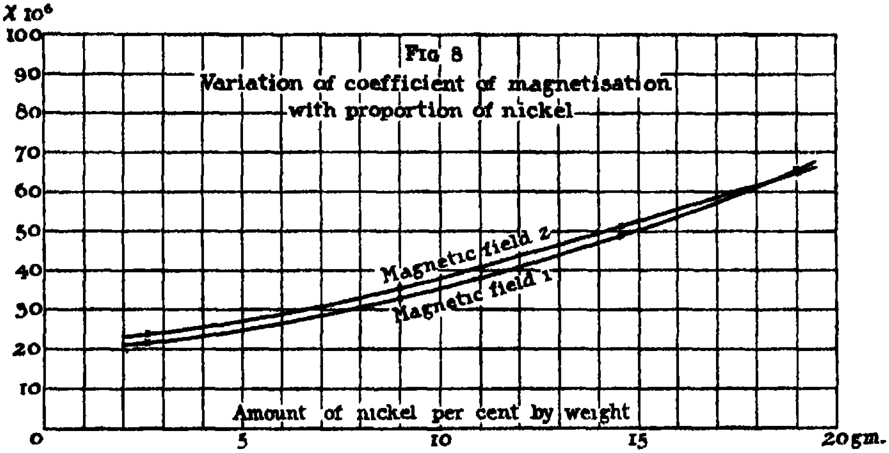
Fig 8
Variation of coefficient of magnetisation with proportion of nickel

(4) *Influence of Copper*

The addition of copper to a manganese-nickel steel appears to raise the mass susceptibility, as shown below :—

Steel No	C/Mn	Ni	Cu.	$\chi \times 10^6$
1424 B	0 14	14 44	2 85	64
1109 D	0 15	14 45	—	52

(5) *Influence of Chromium*

Chromium also seems to raise slightly the mass susceptibility :—

Steel No	C/Mn	Cr	$\chi \times 10^6$
598	0 ·08	—	17 +
620	0 05	3 5	19 −

(6) *Influence of Tungsten.*

Tungsten seems to act more strongly, for the addition of 0 74 per cent has a marked effect on the susceptibility :—

Steel No	C/Mn	W	$\chi \times 10^6$
1848 A	0 126	2 65	29
1848 B	0 120	2 11	18

(7) *Influence of Silicon*

The addition of a considerable proportion (6 per cent) of silicon to a manganese steel makes it ferro-magnetic when the temperature of quenching is high —

Steel No	C/Mn	Si	Temperature	$\chi \times 10^6$
			° C	
1879 D_2	0 010	—	1050	76
2084 A	0 015	6 15	1050	240,000

We have already seen that this steel resembles iron in its magnetic properties

V—RELATIONS BETWEEN MAGNETIC PROPERTIES AND OTHER PHYSICAL PROPERTIES

(1) *Density*

As was to be expected, there is no simple relation between density and mass susceptibility, just as there is none between density and chemical composition

(2) *Magnetic Field*

For certain of the steels, especially those of high susceptibility, there are small variations between the values in fields of different strengths In general, the highest susceptibility corresponds to the weakest field In view of the big difference between the fields and the smallness of the variation, it would not be safe to conclude that this is due to feeble ferro-magnetism

On the contrary, it can be stated that, within the errors of experiment, the mass susceptibility is in the majority of cases independent of the field strength. The curves of fig 8, referring to manganese-nickel steels, in fields 1 and 2, prove this. Apart from the manganese-silicon steel, which is decidedly ferro-magnetic, the other steels are para-magnetic

(3) *Heat Treatment*

To show the effect of the temperature at which the steel was cooled, we give a comparison of similar steels —

Steel No	C/Mn	Ni	Thermal treatment	$\chi \times 10^6$
1798 H_2	0 35	19·98	Cooled in furnace from 550° C.	154
1414 A	0 12	19·0	Quenched in water from 1050° C	67

The nickel-content in these steels is approximately constant, and, as we have seen that independently of the temperature of quenching the susceptibility should increase as the value C/Mn diminishes, it appears from the experiments that the susceptibility is increased by a low temperature of heating.

As chromium has only a feeble effect on the magnetic properties of manganese steels, its presence does not explain the difference found below between steels 2103 and 1109 D.

Steel No.	C/Mn	Mn/Ni or (Mn + Cr)/Ni	Thermal treatment	$\chi \times 10^6$
2108	1.11	0.32	Quenched in water from 945° C.	259
1109 D	0.16	0.35	Quenched in water from 1050° C.	52

The conclusion to be drawn is the same as before. The lower the temperature from which the steel is cooled, the more are the magnetic qualities accentuated.

VI.—GENERAL CONCLUSIONS.

We have determined, with the magnetic balance of P. Curie and C. Chéneveau, the mass susceptibility of manganese and of some of its special para-magnetic alloys.

1. *Manganese.*

We have studied manganese under two forms, powdered and fused, in an atmosphere of hydrogen. The specimens were very kindly lent us by M. P. Weiss, who had prepared and examined them.*

We believe ourselves to be justified in concluding that whatever its condition, so long as there is no occluded gas, manganese is para-magnetic; its mass susceptibility at 18° C. is $11.0 \times 10^{-6} \pm 2$ per cent. This value is a little higher than that given by H. du Bois and Honda (8.9).

With regard to the ferro-magnetism of cast manganese, we attribute it to the presence of hydrogen. This agrees with the observation that this type of manganese, after being heated red, cannot be remagnetised, and with the well known analogous properties of electrolytic manganese and electrolytic iron.

In this connection we desire to recall the effect which certain occluded gases have upon the magnetic properties of cast metals and alloys.

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* Pierre Weiss and Kamerlingh Onnes, 'Comptes Rendus,' vol. 150, p. 687 (1910).

2 *Special Manganese Alloys*

We have shown that the mass susceptibilities of these alloys, which vary between $+17 \times 10^{-6}$ and $+259 \times 10^{-6}$, are independent of the magnetic field. The strength of the field ranged from 320 gauss to about five times that value These alloys are therefore para-magnetic

(1) *Manganese Steels*—

Within the range examined, the mass susceptibilities are only slightly influenced by the amount of manganese present, but depend considerably on the carbon content Speaking generally, we may say that for these special steels the mass susceptibility increases as the ratio C/Mn diminishes

(2) *Steels containing Manganese and other Metals*—

If the carbon-manganese ratio is kept constant, the addition of other metals always increases the mass susceptibility

Manganese-nickel Steels.—An increase of the nickel (from 2 57 to 19 per cent) raises the mass susceptibility in the ratio of 23 67

Manganese-tungsten Steels—An increment of 0 74 per cent of tungsten increased the mass susceptibility in the ratio of 18 29

Manganese-chromium Steels—Chromium seems also to cause a slight increment. An increase of 3 5 per cent of chromium seems to correspond to an approximate increase in the mass susceptibility of 10 per cent

Manganese-nickel-copper Steel—The comparison of two steels each having the same manganese, nickel and carbon content, one containing no copper, and the other 2 25 per cent, shows that this metal, in spite of its diamagnetism, increases the mass susceptibility by about 19 per cent

All the preceding steels were quenched in water from 1050° C They are all of the well known para-magnetic Austenitic type of steel The addition of a substance like manganese, which is also para-magnetic, will have but little effect in altering the properties of these steels, which remain para-magnetic

Manganese-silicon Steel.—The addition of 6 per cent of silicon to a manganese steel, where the C/Mn ratio is about 0 01, makes the steel much more ferro-magnetic. A secular change in the magnetic properties of this steel has also been observed, the specific magnetism increasing, in a period of several years, from the order of 4 or 5 per cent of that of pure iron up to nearly 50 per cent.

Duration of Luminosity of Electric Discharge in Gases and Vapours Further Studies

By the Hon R J STRUTT, F R S

(Received July 19, 1917)

[PLATE 1]

In several previous papers,* I have described experiments showing that the luminosity of the spark or arc discharge through gases and metallic vapours lasts for an appreciable length of time after the discharge has ceased to pass, and that the luminous vapour can be removed from the region where the electric current is passing, and still remain luminous, showing the same spectrum as before, though in certain cases some lines of this spectrum fade out sooner than others The present paper is a continuation of these studies

§ 1 *The Luminous Jet at Very Low Pressures Effect of a Transverse Electric Field*

The first experiments to be described were designed to observe the effects at very low gaseous pressures, and with small currents. The advantage of this is that the luminous particles in the jet are then comparatively free to move under the action of an external force, and their motion in an electrostatic field can be conveniently examined Moreover, since the ions are few, they do not disturb the uniformity of the electric field applied for this purpose

On the other hand, the luminous effects are very faint under these conditions, this difficulty can partly be met by photography with long exposures, but it is difficult to use the spectroscope

The discharge passes through the globular vessel on the right (fig 1), which was varied in size in different experiments Usually, it was 16 cm. in diameter, though in some cases much more. The cathode is the iron disc A, which is drawn up close to the wall of the globe.† The anode of the discharge is the iron disc *b*, perforated with a 1-mm hole in the centre. This iron disc covers as accurately as may be the end of the narrow glass discharge tube *d*, sealed on to the globe. The arrangements for holding the iron disc in position, and making contact to it, are a little complex at first sight The disc is soldered into the end of a brass tube *c*, and the glass tube *d* fits into *c*, and is continued up to the very end, so that the gaseous discharge can pass to *b*, but not to any part of *c* Electrical contact is made to the further end of *c* as shown, and airtightness is achieved by the rubber sleeve *l*, which is in practice sufficiently tight

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* 'Roy. Soc Proc,' A, vol 88, p. 110 (1912), vol. 90, p 364 (1914), vol. 91, p. 92 (1914)

† Sometimes a mica washer was interposed to diminish risk of cracking the glass

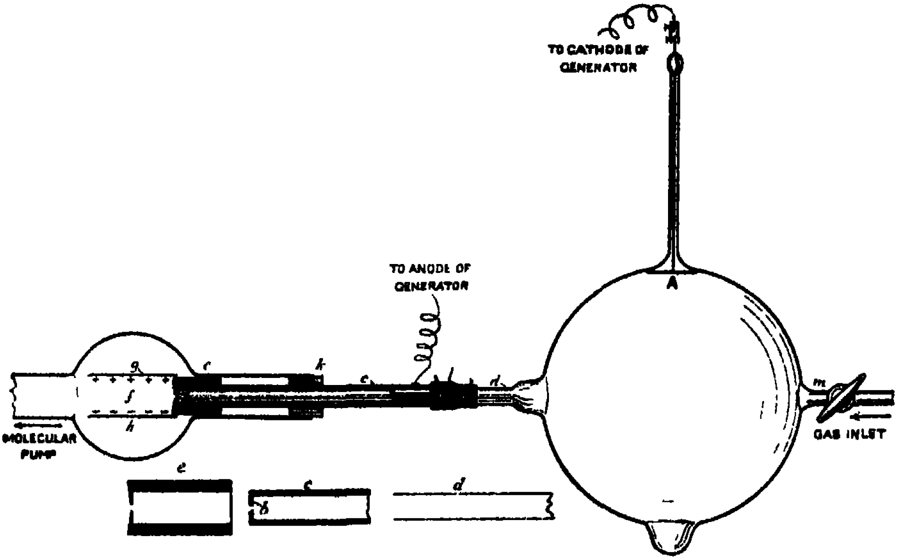

FIG 1

So much for the discharge vessel, in which the luminous gas is generated Now for the observation vessel, into which the luminous gas is allowed to stream, and in which it can be subjected to a transverse electrostatic field This is a glass tube *f*, 5 cm in diameter, and with its axis perpendicular to the plane of the diagram It is prolonged for a distance of about 8 cm in front, and about the same distance behind, the diagram The front is closed by a plate glass window, and, behind, the electrical connections to the condenser plates *g*, *h*, are brought out These plates are carried by an ebonite support behind, which affords a black background *

To supply the current, a continuous source is necessary. It would be very desirable to have a battery of small storage cells, giving 5000 volts No such battery was available, and I used a continuous-current generator, supplied by Messrs. Evershed and Vignoles This consists of three high-tension magneto generators, in series, direct driven by a motor The commutators of the several machines are so arranged round the axis as to compensate fluctuations of voltage, and oscillograph tests by the makers show that the variation is not more than ±0·22 per cent But even this gave trouble, and a battery would be much preferable. The generator can yield 10 milliampères at 5000 volts, and the full current output was usually employed

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* It has not been thought necessary to show the details of this.

When the molecular pump was running, a stream of gas was admitted through *m*, so as to maintain the pressure in the discharge vessel nearly constant The pressure was not measured, but the length of negative dark space was used to control it The dark space varied in different experiments from ¼ inch (6 3 mm) to 2 inches (51 mm) The large discharge globe was filled with negative glow, and the cathode rays struck the bottom of it, which became somewhat warm The tube *d* was filled with positive light, right up to the orifice at the far end *b*

Owing to the small size (1 mm) of the orifice at *b*, the outrush from it was rapid, and the outrushing jet of gas remained visibly luminous as it traversed the space between the plates *g* and *h* The luminosity was, however, faint, and, as in the analogous case of the nebulæ, its form could be determined far better by photography than by visual observation Accordingly, a camera with a rapid lens was mounted in front of the tube *f*. Rapid photographic plates were used, and exposures up to one hour were given, both with and without an electrostatic field between the metal plates *g*, *h*

The chief difficulty of the experiments is to eliminate stray electric discharges passing through the orifice *b*, and going to earth by the plates *g* and *h*, or down the exhaust pipe to the pump There is no difficulty in recognising these stray discharges, and distinguishing them from the genuine effects The stray discharges are much brighter, and they generally flicker in sympathy with any luminosity that can intermittently find its way round to the back of the cathode The particular arrangement of cathode shown, with the disc in contact with the walls of the bulb, was found effective in preventing this The trouble is connected, I feel almost certain, with the want of perfect steadiness in the E M.F. of the generator, which may be regarded as yielding a minute alternating current superposed upon the continuous one It is to this minute alternating current that the trouble is attributed

An additional precaution found useful was to carefully insulate the battery used to maintain an E.M F. between *g* and *h*. The observation tube *f*, in which these are mounted, should not be made of metal The ebonite cap *e* served to prevent the main discharge coming through the orifice and passing to the outside of the disc *b* and the tube *c* which carries it

These various precautions proved perfectly successful in confining the discharge to its proper channel between cathode and anode, and preventing any part of the current from the generator penetrating into *f*

The gases investigated in this apparatus include hydrogen, oxygen, nitrogen, carbon dioxide, and mercury vapour The shape of the luminous jet was very similar in all in the absence of an electric field. It is seen in Plate 1, No 1, which was obtained with hydrogen (dark space 1 inch)

When, however, the electric field was applied, the behaviour of the jet depended very much on the nature of the gas and on the pressure

The voltage employed was 160 volts between the plates *g* and *h*, which were 13 mm apart The traces of these plates can be distinguished on the photographs, the positive plate being in all cases the lower one So far as was observed, voltages less than 160 gave similar results, but less well marked Larger voltages were apt to start an arc in the ionised air between *g* and *h*, particularly at the lowest pressures The current between *g* and *h*, when the voltage was on, was not systematically measured, but was of the order of 1 microampère This is very small compared with the currents which can be passed through the much more luminous jets distilled from the mercury vacuum arc, as in former experiments* These latter are of the order of 1 milliampère

Taking the photographs in order, No 2 was obtained with hydrogen, dark space ¼ inch (6 mm) Here the bulk of the luminous particles are attracted to the positive plate and are, therefore, to be regarded as negatively electrified A few, however, pursue their course undeviated, which indicates that they are electrically neutral.

In No 3, hydrogen, dark space ½ inch (13 mm), the luminous particles are practically all negative and the neutral particles have disappeared.

No. 4, hydrogen, dark space 1 inch (25 mm) here a few positive particles have made their appearance.

No 5, hydrogen, dark space 2 inches (51 mm.): the positive particles have markedly increased, though the negative ones still predominate It was hardly practicable to work at much lower pressures than this last

No 6, oxygen, dark space, ¼ inch (6 mm.) the negative particles are here somewhat more conspicuous than the positive, but the difference is not great

No. 7, oxygen, dark space ½ inch (13 mm) and No. 8, oxygen, dark space 1 inch (25 mm.). in these the positive and negative particles produce about equal effects.

No. 9, carbon dioxide, dark space 1 inch (25 mm)· positive and negative about equally conspicuous.

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* 'Roy Soc. Proc.,' A, vol 91, p 92 (1914).

No. 10, nitrogen, dark space 1 inch (25 mm) positive particles rather in excess

The remaining photographs relate to mercury vapour A small quantity of mercury was placed at the bottom of the discharge bulb, and warmed by the cathode rays or by heat externally applied, which was regulated so as to keep the dark space at the desired length It was found difficult to maintain steady discharge conditions after long running, when mercury vapour only was present, and all traces of other gas had been removed Some trace of other gas had to be present, therefore, though the spectrum of the discharge was always that of mercury, pure and simple For this reason it was sometimes necessary to admit a small quantity of air, or to interrupt the exposure, let air into the apparatus for a time, and then re-establish the proper conditions

No. 11, mercury, dark space ¾ inch (19 mm) negative particles only

No 12, mercury, dark space 1¼ inch (32 mm) negative particles, with a trace of positive

No 13, mercury, dark space 1½ inch (38 mm) negative particles, with some positive

The general result then, of these experiments is to show that the issuing stream of gas always contains luminous particles, negatively electrified, which may be deflected sideways by a transverse electrostatic field Sometimes also it contains luminous positive particles which may carry as much of the luminosity as the negative ones, or even more The tendency of a lowering of pressure in the discharge vessel is to increase the number of positive particles. But the positive particles come into prominence much sooner in oxygen, nitrogen or carbon dioxide than they do in hydrogen or mercury vapour

It is difficult to see how these experiments can be reconciled with the idea that the light is due to recombination of positive and negative ions. If this were so, it would be expected that a separation of the two kinds of ions in a transverse field would lead to the extinction of all luminosity Instead of this the two kinds of ion, separated by the field into distinct streams, show the same aggregate luminosity as before. Sometimes the luminosity is all in that stream which goes to the positive plate Sometimes both streams are about equally luminous. The streams lose their luminosity when they are discharged by contact with the electrified plates.

For these reasons I regard the luminous centres as being excited before they have left the discharge tube. For the only process that can be supposed to excite them afterwards, namely, ionic recombination, seems to be excluded by the considerations just given This view, to which I have inclined before in the interpretation of kindred experiments,* no doubt presents some difficulties. The evidence brought forward in my former paper has been criticised.† I hope to return to the discussion later in the light of further experimental evidence.

The electrostatic conditions require that positive and negative particles should be present in equal numbers in the jet, if, as in these experiments, no current is carried down to the pump. In cases where one, say negative, sign shows in excess in the luminosity, it is not to be concluded that the equivalent of positive particles are not present but rather that they are not luminous.

§ 2. *High-Pressure Experiments. Jet in Hydrogen. Thinning of the Broadened Spectrum Lines.*

The experiments which have been described so far in this paper are carried out at very low gaseous densities, and the luminous effects are very feeble—too feeble for easy spectroscopic examination. I come now to some kindred results obtained by working at much higher pressures, pressures high enough in fact to give what may be called a spark rather than a vacuum discharge.

The method is a slight modification of one which I described formerly.‡ A spark passes between iron wires 1 cm. apart in a silica tube of 1·5 mm. inside diameter. Half way between the iron wires a lateral hole through the silica wall allows of the outflow of gas from the region of the spark to the space outside, which is in connection with a power air pump (see fig. 2). Any desired gas can be admitted at a regulated pressure to the spark region flowing in through the annular space between the silica tube

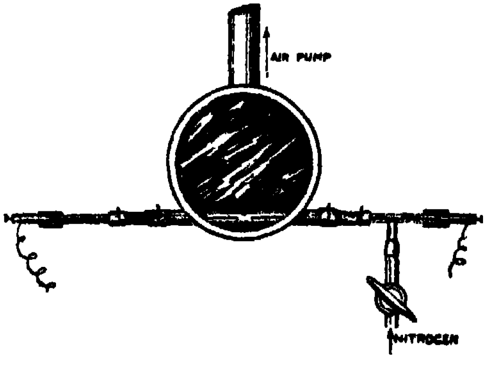

FIG. 2.

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* 'Roy. Soc. Proc.,' A, vol. 91, p. 98 (1914).

† Child, 'Phil. Mag.' [VI], vol. 31, p. 139 (February, 1916).

‡ 'Roy. Soc. Proc.,' A, vol. 88, p. 110 (1913).

and one of the iron wires, as shown The lateral hole in the silica tube is so small that it allows a considerable difference of pressure to be maintained between the space outside the tube and the spark-gap inside The hole usually was 0 7 mm diameter at the smaller end, tapering slightly outwards, as it passed through the walls of the tube 1 mm thick.*

The most interesting results with this arrangement were obtained with hydrogen and nitrogen

Hydrogen gave far more striking effects with the condensed than with the uncondensed discharge The best result was obtained with about 17 cm pressure inside the tube, the pressure outside being reduced to 2 or 3 mm In this case a tongue of flame of a beautiful rosy colour, and about 9 mm long, was seen issuing from the hole The photograph, Plate 1, No 14, shows the spark itself, together with this exuding tongue of flame, the latter is, however, to a great extent lost in the glare of the spark itself By covering the silica tube with a thin brass tube provided with a hole to correspond with that in the silica, as shown in fig 2, this glare can be suppressed, and the exuding luminosity can then be photographed alone (see Plate 1, No 15, where the outline of the tube has been ruled in by hand, in its proper position relative to the photographed jet)

The spectrum of this exuded jet is of interest Where the spectrum of a hydrogen discharge is examined directly, the Balmer series is invariably mixed up with lines of the secondary spectrum, which produce a luminous background, specially confusing at the end of the series, where the series lines are faint and crowded. In the exuded jet of luminosity, the background of the Balmer series is quite black, and the series has been photographed to the tenth member, quite unconfused by secondary spectrum. The method might be of advantage to anyone taking up the further study of the hydrogen spectrum The hydrogen used must be quite free from nitrogen, or the great cyanogen band at λ 388 is sure to appear and obliterate the end of the Balmer series

As is well known, the Balmer series lines in the condensed spark are greatly widened by increase of pressure It was of interest to compare the width of these lines in the spark and in the exuded luminous jet, for the pressure of the luminous gas is much reduced in passing into the latter Would the lines remain broad or not when the gas was excited in a spark of relatively high pressure, and the luminosity observed in a space of low pressure? Experiment proved that the lines become narrow. The sparking tube was covered with a metal screen, having two small holes. One of these allowed the jet to pass out as before, the other, its axis at right angles to the first, allowed a small part of the spark to be seen simultaneously with the jet Both the spark and the jet were focussed on the slit of a small spectrograph No 16 shows the result The narrow spectrum of thick lines is due to the spark The long thin lines above are from the luminous jet

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* These holes were drilled with a steel wire, fixed in a small vertical drilling machine, and fed with carborundum and oil

Stark has given reasons for believing that the broadening is due to the electric field of neighbouring atoms, and Merton and Nicholson have produced strong confirmatory evidence It is, therefore, not surprising that the removal of these to a greater mean distance, when the gas issues from the hole and expands, should cause the lines to become narrow This does not settle the question of whether the identical luminous centres in the spark survive as such after emergence into the jet or not If the experiment had given the opposite result it would have been more instructive on this point

§ 3 The Jet in Nitrogen Periodic Structure

Nitrogen has given interesting results in the same form of experiment In this case, however the simple induction coil discharge without a condenser gives much more striking effects than the condensed discharge Using the same sized silica tube as before, a long thin jet of the same reddish-orange colour as the spark itself is seen emerging from the hole and traceable as far as 5 cm from it In this experiment the pressure was 155 mm inside the tube, and 7 mm outside [*]

No 17 is a photograph of the spark and jet

Nos. 18 and 19 are photographs of the jet only, on various scales of enlargement, the spark being covered over so that the glare from it is avoided

The most striking features of this jet, both visually and photographically, are the alternations of light and comparative darkness which may be seen along its length. Five or six of these can easily be counted, but they become less distinct the farther out we go from the orifice They remain absolutely steady in position in spite of any irregularity in the working of the induction coil Their appearance is at first sight reminiscent of the striations along the positive column of a vacuum discharge, but in reality they have nothing whatever to do with this.[†]

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[*] The pressure inside was determined by a manometer attached to a side tube attached to the left-hand side of the silica sparking tube By inadvertence, this has not been represented in fig. 2

[†] If by a faulty arrangement of the experiment, any stray discharge occurs in the outer vessel, it may, indeed, issue from the same hole as the jet, but escapes to one side, out of the relatively dense stream of gas, as soon as possible, following any convenient curved path. The jet, on the other hand, takes a path confined to the axis of the hole

The periodic structure of the jet is of hydrodynamic origin It has been shown by the investigations of Mach* and Salcher and Emden† that a high pressure jet of air escaping into the atmosphere shows, when photographed by the shadow method, a periodic structure very closely analogous to the structure seen in photographs 17, 18, 19 For a full discussion reference may be made to a paper by Lord Rayleigh‡ But the general explanation of this effect is as follows —The air or nitrogen issuing from the hole expands against the lesser pressure of the surrounding air As in any other case of a vibratory system, its inertia causes it to overshoot the mark, and expansion continues beyond the point at which the pressures are equalised There is then a contraction again to something like the original density, and so on alternately Simultaneously, the mass of air under consideration is carried forward, and thus the alternating variations of density are exhibited along the length of the jet The case is analogous to that of the vibrations under surface tension of a liquid jet escaping from an elliptical orifice

It is not difficult to understand why the structure of the jet revealed in such experiments as those of Mach and Salcher and Emden should be visible in the self-luminous nitrogen jet, for variations in the density of the gas are accompanied by variations in the density of the luminous centres which it contains, with corresponding variations in the intrinsic brightness, and even apart from this the luminous condition allows the alternate swellings and contractions to be followed

According to Emden's empirical results (*loc cit*), the wave-length of the periodic structure is found to be

$$0{\cdot}88d\sqrt{\frac{p-1{\cdot}9p_1}{p_1}},$$

where d is the diameter of the nozzle, p the high pressure, and p_1 the low pressure into which the jet emerges

With the values above given, this becomes 2 mm

A measurement of photographs of the luminous jet gave the wave-length as 1 8 mm, in sufficient agreement with the value calculated from Emden's formula. It must be remembered that this formula was obtained under pressure conditions widely different Emden's jet discharged into the atmosphere at 760 mm The self-luminous jet discharged into a space where the pressure was 7 mm

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from which it issues If the silica tube is rotated about its own axis, the jet rotates with it, as if rigidly attached at right angles, and may in this way be made to impinge on the glass window of the apparatus.

* 'Wied Ann,' vol 41, p 144 (1890).

† 'Wied Ann,' vol 69, pp 264, 426 (1899)

‡ 'Phil Mag,' vol 32, p 177 (1916)

It will not fail to be noticed that the hydrogen jet (Plate 1, Nos 14 and 15) shows nothing of these periodic phenomena. The jar discharge, with its sudden explosive character, probably produces a turbulence which is fatal to the steady conditions of motion required

Nitrogen is the only gas in which striking luminous jet phenomena at relatively high pressures have been obtained, using the uncondensed discharge, thus it is the only one in which it has been possible to observe the periodic phenomena by this method *

If the gas pressure is considerably reduced by checking the supply of nitrogen, the jet remains essentially the same, but loses its periodic structure

The induction coil with break is the best source of current The 5000-volt generator hardly supplies a large enough E M F, while a high-tension transformer, with alternating supply, makes the electrodes in the confined space too hot

§ 4 *Spectrum of the Nitrogen Jet Comparison with Active Nitrogen*

So far, the nitrogen jet has been considered in relation to its hydrodynamic relations Passing to the spectroscopic side, the jet is observed to have exactly the same tint as the spark from which it originates, and its spectrum shows exactly the same bands as the spark, in the same relative intensity These are the groups known as the first, second, and third positive nitrogen groups

It is important to notice that, in spite of some superficial resemblances, the jet here described has nothing to do with active nitrogen The latter, it will be remembered, is manifested in a luminosity which lasts several seconds, or even minutes, after discharge through the gas has ceased Let us compare this with the present case. The jet issues from the side hole in the silica tube with a velocity comparable to the velocity of sound Since the distance for which it is traceable under favourable conditions is some 2 inches (5 cm), the time for which the luminosity lasts is of the order of 1/6000 of a second. Thus the duration of the light is of quite another order of magnitude from that of the active nitrogen glow

Secondly, the light is spectroscopically quite different from that of active nitrogen Let us suppose that we are dealing with gas of ordinary purity, like that used in the present experiments, not absolutely free from traces of oxygen compounds, then the following Table shows which sets of bands are present in the jet and which in active nitrogen

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* Something similar has, however, been observed in mercury vapour by Matthies, 'Verhand Deut. Phys. Gesell,' vol 12, p. 754 (1910).

Group of bands	Spectral range	In nitrogen jet.	In active nitrogen.
1st positive	Red, yellow, green	Complete set present	A small and peculiar selection alone present
2nd positive	Blue, violet, near ultra-violet	Present	Absent
β group*	Near ultra-violet	Absent	Present
3rd positive	Far ultra violet	Present	Present

It must be emphasised that those groups entered as "present" are very strong, and those entered "absent" are entirely invisible on the photographs The distinction between the two spectra is not one of degree, but of kind

It may be asked, however, what has become of the active nitrogen which one might expect to be generated in this experiment The reply is that there is very little of it, the uncondensed discharge being unfavourable for its production, and that what there is hangs around the jet in the form of a very diffuse and feebly luminous cloud, occupying a large volume, and scarcely visible in the presence of the much more luminous jet

When air is substituted for nitrogen, the jet becomes much shorter and less bright, it does not however disappear entirely, as active nitrogen does.

§ 5 *Summary*

1 The behaviour of jets of luminous gas flowing away from the region of discharge at a low gaseous pressure, has been investigated, using the principal permanent gases, also mercury vapour

In a transverse electrostatic field, the luminosity is deflected, part of it in most cases going to the positive plate, and part to the negative.

But in hydrogen when the pressure is not very low, nearly the whole of the luminosity is deflected to the positive plate, a small part remaining undeflected. As the pressure is reduced, an increasing part of the luminosity goes to the negative plate Similar results are observed in mercury vapour

2 Further observations are recorded on these jets at higher pressures, arranging a spark discharge so that the gas can flow out from it through an orifice into a sustained vacuum With hydrogen (condensed discharge) the exuded jet of luminosity, about 9 mm long, shows the Balmer series. The discharge spectrum shows widened lines These become narrow as the luminous gas emerges

3. Nitrogen in the same arrangement, with an uncondensed discharge, shows a jet with periodic swellings similar to those observed by Mach and Salcher and Emden when a jet of compressed air, examined by the shadow method,

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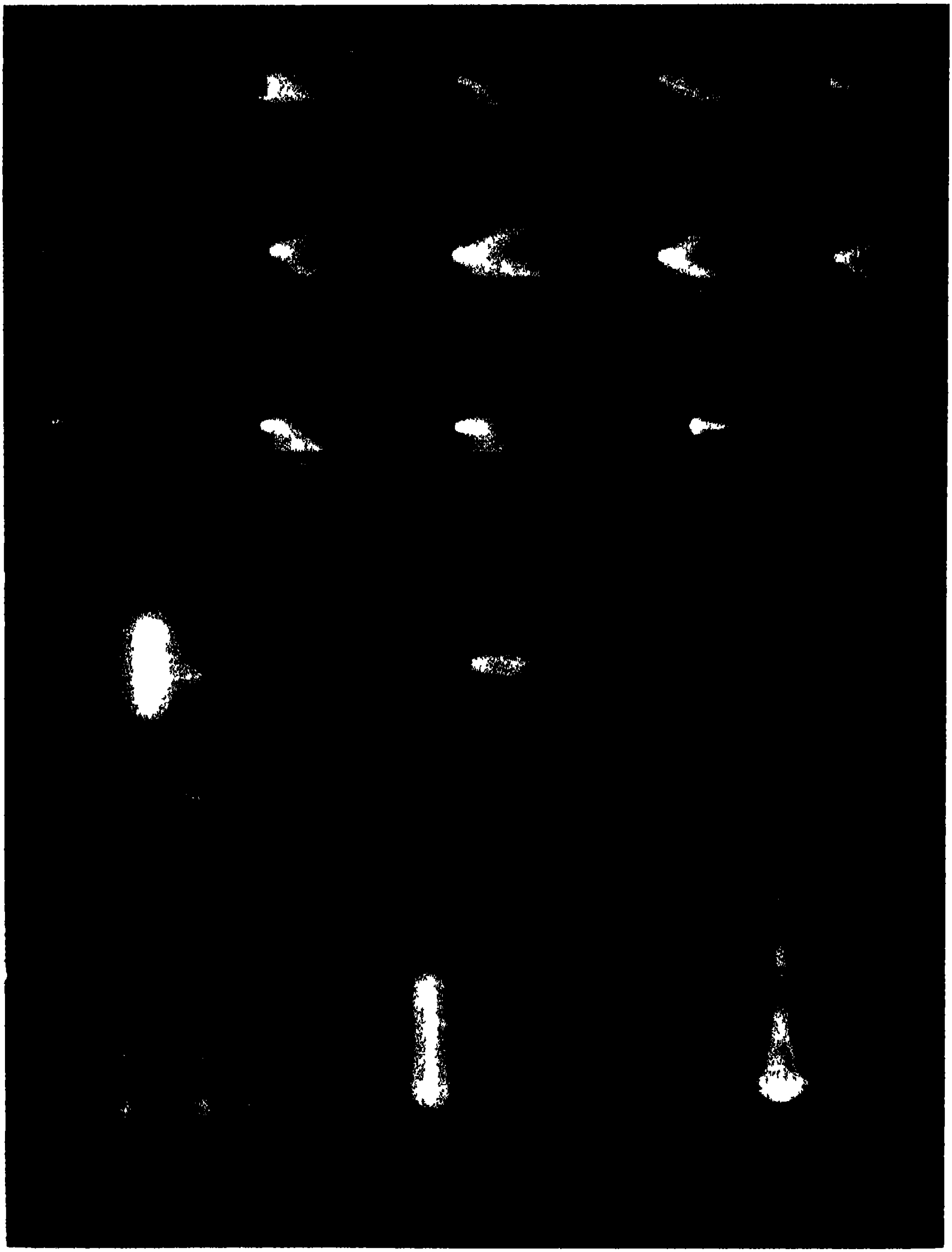
* See Fowler and Strutt, 'Roy. Soc. Proc.,' A, vol. 85, p. 377 (1911).



escapes into the open The wave-length agrees with that to be anticipated from their experiments

4 This nitrogen jet luminosity is not to be confused in any way with active nitrogen The time for which it endures is of quite a different order of magnitude, and the spectrum is essentially different

DESCRIPTION OF PLATE 1

(Photographs Nos 1 to 13 are low pressure effects Scale, slightly more than half natural size)

1 Fan-shaped jet of luminosity, hydrogen, 1 inch (25 mm) dark space No electric field

(Nos 2 to 13 show similar jets in an electric field)

2 Hydrogen, ¼ inch (6 mm) dark space Negative and neutral luminous particles

3 Hydrogen, ½ inch (13 mm) dark space Negative luminous particles only

4 Hydrogen, 1 inch (25 mm) dark space Negative luminous particles, with a few positive

5 Hydrogen, 2 inch (51 mm) dark space Negative luminous particles, with a considerable proportion of positive

6 Oxygen, ¼ inch (6 mm) dark space Negative luminous particles, somewhat more conspicuous than positive

7 Oxygen, ½ inch (13 mm) dark space Negative and positive about equally conspicuous

8 Oxygen, 1 inch (25 mm) dark space Negative and positive again appear equally

9 Carbon dioxide, 1 inch (25 mm) dark space Positive and negative appear equally

10 Nitrogen, 1 inch (25 mm) dark space Positive luminous particles rather in excess

11 Mercury vapour, ¾ inch (19 mm) dark space Negative only.

12 Mercury vapour, 1¼ inch (32 mm) dark space Negative luminous particles, with a few positive

13. Mercury, 1½ inch (38 mm) dark space Negative luminous particles, with some positive

(Photographs 14 to 19 refer to effects at higher pressures (several mm of mercury)

14 Hydrogen spark and jet One and a half times natural size

15. Hydrogen jet, same as 14, but spark screened off

16 Spectrum of hydrogen spark and jet, showing thinning of Balmer series lines when the jet emerges The lines beginning from the left are H_β, H_γ, H_δ, H_ϵ

17 Nitrogen spark and jet, showing periodic structure About one and a half times natural size

18. Nitrogen jet only, showing periodic structure About two and a half times natural size.

19 The same, enlarged to about five times natural size

Characteristic Frequency and Atomic Number

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(Communicated by Prof O W Richardson, F R S Received May 21,—Revised July-August, 1917)

Part I.—Atomic Frequency and Atomic Number

§ 1 *Characteristic Atomic Frequency*

Recent work on the specific heat of chemical elements in the solid state has led to important conclusions with regard to the values of the atomic heat at constant volume The curves showing the variation of atomic heat with temperature are all of the same form, and any given curve can be transformed into any other merely by altering the scale on which the temperature is plotted This implies that the atomic heat C_v is a function of the temperature T, dependent on a single parameter, the function being the same for different elements Thus

$$C_v = F(\Theta/T),$$

where Θ is a certain temperature characteristic of the element in question According to the quantum theory $\Theta = \beta\nu = h\nu/k$, where

ν is the characteristic atomic frequency,

h is Planck's constant, $6\ 558 \times 10^{-27}$,

k is the gas constant for a single molecule, $1\ 372 \times 10^{-16}$

The work of Moseley* has shown that the frequency of the X-radiation from an element depends upon the atomic number, that is the number which determines the position of the element in the periodic classification, and is probably equal to the number of positive unit charges in the core of the atom. Isotopic elements, which occupy the same place in the periodic table, have the same atomic number, and chemically are inseparable They must therefore have the same chemical constant or chemical affinity But the affinity may be expressed in terms of the atomic heat at constant pressure C_p. Hence C_p must be the same for isotopes But C_v is a function of the frequency It is, then, probably fair to assume with Lindemann† that the atomic frequencies are identical for isotopic elements This leads to the conclusion that the characteristic frequency is a function of the atomic number, and is independent of the atomic weight. It is the object of Part I of the present paper to show from the observed values that a simple relation holds between the atomic frequency ν and the atomic number N.

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* Moseley, 'Phil. Mag.,' vol 26, p. 1024 (1913), vol 27, p. 703 (1914).

† Lindemann, 'Nature,' vol 95, p. 7 (1915)

§ 2. *The Relation between* N *and* ν.

Determinations of the characteristic temperature, or of the atomic frequency, from the observed values of the specific heats, have been made by Nernst and Lindemann, E. H. Griffiths[*] and E. Griffiths,[†] and at very low temperatures by Keesom and Onnes.[‡] Some further results have been collected for a number of elements in a paper by C. E. Blom,[§] but to these smaller value must be attached.

Examination of the graph obtained by plotting $\log \nu$ against $\log N$ suggested that, for certain sets of elements the product $N\nu$ had a constant value. Further investigation showed that the value of the product in some cases was a simple multiple of the value in other cases. Thus, the value of $N\nu \times 10^{-12}$ for Ag is 211·0, for Fe 209·0. For Al the value is 107·5, which is nearly one-half the former number. For Pt we find 413·4, and for Ir 415·8, numbers which are nearly double those for Ag or Fe. Finally it was found that, in almost every case examined, the value of $N\nu$ could be expressed with an accuracy of 4 or 5 per cent. as a simple multiple of a certain definite frequency. This is illustrated in Table I, which contains the results for those metals for which the atomic frequency has been determined with the greatest accuracy. All metals are included for which low temperature measurements are available, with the exception of sodium. The product of the atomic number and the atomic frequency can be expressed in the form

$$N\nu = n\nu_A,$$

where n is an integer, which may be termed the *frequency number*, and ν_A is a definite frequency, for which the weighted mean value is $21{\cdot}3 \times 10^{12}\ \text{sec}^{-1}$. The variations in the values of ν_A recorded in the Table are not greater than can be accounted for by experimental errors in the determination of the frequency ν.

Commenting on the want of concordance in the values found by Nernst for the atomic heat of lead, Messrs. Griffiths remark: "Lead, from our own experience, appears to be a metal with which it is difficult to obtain concordant results."

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[*] Nernst, 'Vorträge über die Kinetische Theorie der Materie,' 1914, p. 77.

[†] E. H. Griffiths and E. Griffiths, 'Phil. Trans.,' A, vol. 214, p. 319 (1914).

[‡] Keesom and Onnes, 'K. Akad. Amsterdam Proc.,' vol. 18, p. 1247 (1916).

[§] C. E. Blom, 'Ann. d. Phys.,' vol. 42, p. 1397 (1913).

Table 1.

Element	N	$N\nu \times 10^{-12}$			
		Nernst	Griffiths	Keesom-Onnes	Mean
Al	13	5 × 21·6	5 × 21·4		5 × 21·5
Fe	26		10 × 20·9		10 × 20·9
Cu	29	9 × 21·3	9 × 21·9	9 × 21·2	9 × 21·3
Zn	30	7 × 20·6	7 × 20·6		7 × 20·6
Ag	47	10 × 21·2	10 × 21·1		10 × 21·1
Cd	48		8 × 21·1		8 × 21·1
Hg	80	8 × 20·2			8 × 20·2
Pb	82	7 × 22·2	7 × 22·5	7 × 21·6	7 × 22·0
Weighted mean		*21·1*	*21·3*	*21·4*	**21·3**

In discussing the experimental values recorded in the Table, attention must be directed to the temperature at which the observations have been made. There are several reasons why determinations at low temperatures are to be preferred. In the first place, the difference between C_p and C_v becomes negligible at such temperatures. Again "It is at very low temperatures that the assumption made by Debye, namely, that the vibrations whose frequency is greater than ν are negligible, can best be justified, since at such temperatures the slow heat vibrations would be the most important" (Griffiths). Thirdly, at such temperatures the formula of Debye simplifies to the form $C_v = \text{const}\ (T/\Theta)^3$, which is probably the correct type at low temperatures, and must be approximated to by any formula that is to give an accurate representation of the variation of specific heat with temperature.

Accordingly greater weight has been attached to the determinations of Keesom and Onnes, who determined the specific heat of copper between 14° and 90° K, and that of lead between 14° and 80° K, taking special precautions as to the temperature measurements and the purity of the material employed. It is to be noticed that the value of ν_A for lead from these observations is much nearer to the mean value than the results obtained by the earlier investigators.*

Messrs. Griffiths give two sets of values for Θ or $\beta\nu$. In the first set the values are chosen so as to give agreement with Debye's formula over the lowest portion of the temperature range considered. In the second set $\beta\nu$ is chosen so as to give coincidence at about 125° K, with the result that, in general, a slightly smaller value of ν is obtained. In Table I the values for ν are those belonging to the first set.

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* The results of Keesom and Onnes for lead have been confirmed by Eucken and Schwers in a series of experiments for which great accuracy is claimed. The latest determinations in Nernst's laboratory give for aluminium $N\nu = 5 \times 21·0 \times 10^{12}$, and for copper $9 \times 21·1 \times 10^{12}$, showing even better agreement than the earlier figures. Schwers, 'Phys. Rev.,' vol. 8, p. 117 (1916).

These experimenters also investigated ν for sodium, which appears to be somewhat exceptional in its thermal behaviour, and for that reason has not been included in the Table. At low temperatures they found $\beta\nu = 180$, whilst at 125° K the value is considerably lower, $\beta\nu = 152$. Taking the low temperature value, $\nu = 3{\cdot}77 \times 10^{12}$, and $N\nu = 2 \times 20{\cdot}71 \times 10^{12}$ a result which is in good agreement with those recorded in the Table.

Nernst has published determinations of the atomic frequency for two non-metals, carbon (in the form of diamond) and iodine, and it is interesting to find that they fall into line with the metallic elements. For C we find $N\nu = 12 \times 20 \times 10^{12}$, and for I, $N\nu = 5 \times 21{\cdot}6 \times 10^{12}$.

Similar results were obtained for nearly all the elements contained in the list given by Blom. Table II gives the figures for those elements with the exception of Li[*] and P[†]. The value of $N\nu \times 10^{-12}$ is in brackets in those cases in which the atomic frequency is stated by Blom to be less reliable. Even in these cases it will be noticed that ν_A has practically the same value as was found to hold when accurate determinations of ν were available.

Table II

Element	N	$N\nu \times 10^{-12}$	Element	N	$N\nu \times 10^{-12}$
Be	4	4 × 20·3	Ni	28	11 × 20·4
B	5	6 × 20·1	Mo	42	(12 × 21·0)
Mg	12	4 × 20·1	Pd	46	(11 × 20·9)
Si	14	9 × 20·2	Sn	50	(9 × 21·1)
K	19	(2 × 20·0)	Sb	51	12 × 21·2
Ti	22	11 × 21·0	Ir	77	20 × 20·8
Cr	24	11 × 21·4	Pt	78	(20 × 20·7)
Mn	25	8 × 21·6	Au	79	(16 × 20·7)
Fe	26	10 × 21·8	Bi	83	(12 × 22·1)
Co	27	10 × 20·8			

The weighted mean value found from the results of Table II is $\nu_A = 20{\cdot}85 \times 10^{12}$.

It will be noticed that the mean value for ν_A found from Table II, $20{\cdot}85 \times 10^{12}$, is somewhat smaller than the mean value found from Table I, $21{\cdot}3 \times 10^{12}$. This may arise from the fact that the values of ν given by Blom are derived from observations of the specific heat at higher temperatures, a procedure which, as we have noticed already, usually gives a smaller value for the frequency.

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[*] For Li Blom gives $\nu = 8{\cdot}3 \times 10^{12}$, which would make $N\nu = 24{\cdot}9 \times 10^{12}$. This suggests that $n = 1$ in this case. Determinations of the specific heat at low temperatures are much to be desired.

[†] For red phosphorus Blom's value is $\nu = 6{\cdot}3 \times 10^{12}$. This non-metallic element is exceptional; we may write $N\nu$ in the form $4\frac{1}{2} \times 21{\cdot}0 \times 10^{12}$.

§ 3. *Application of the Theory of Probability.*

In view of the unexpected character of these results it is desirable to have some check upon them in order to be reasonably sure that the agreement between the values of ν_A is not accidental. Such a check is provided by the theory of probability. The case is similar to that discussed by Strutt[*] in a paper on the tendency of the atomic weights to approximate to whole numbers. Individual values of $N\nu$ cannot deviate by more than a fixed amount $y (= \frac{1}{2}\nu_A)$ from the nearest integral multiple of ν_A. "What we require is the probability that after a given number (i) of 'trials' the sum of the results should not exceed a certain given amount r, the result of each trial lying between 0 and y, and any value between these limits being equally likely." A formula for the probability has been given by Laplace[†]

$$\frac{1}{i!}\left\{\left(\frac{r}{y}\right)^i - i\left(\frac{r}{y}-1\right)^i + \frac{i(i-1)}{1\ 2}\left(\frac{r}{y}-2\right)^i - \quad\right\}$$

The series is to be continued only so long as the quantities raised to the power i are positive.

Applying this formula to the results of Table I, taking $y = \frac{1}{2} \times 21{\cdot}3$ and for r the sum of the differences between the observed and the calculated values of $N\nu \times 10^{-12}$, the approximate value of the probability is found to be —

Results of Nernst, $\frac{1}{9}$,

Results of Griffiths, $\frac{1}{11}$,

Mean results, $\frac{1}{26}$.

That is, there is one chance in 26 that the mean values of $N\nu \times 10^{-12}$ should approximate so closely to integral multiples of the number 21·3 by accident.

The application of the formula to the results of Table II gives for the probability approximately $\frac{1}{31}$. Consequently when the results of both Tables are taken into consideration there is but a small chance that the assumed regularity may be accidental.

§ 4. *The Formula of Debye.*

In the theory of Debye[‡] the characteristic frequency is a maximum frequency which limits abruptly the range of vibrations forming the

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[*] Strutt, 'Phil Mag.,' vol. 1, p. 311 (1901).

[†] 'Théorie Analytique des Probabilités,' p. 259.

[‡] Debye, 'Ann. d. Phys.,' vol. 39, p. 789 (1912).

"spectrum" of the solid. Its value in terms of the elastic constants is given by the formula

$$\nu = \left(\frac{3N'}{VF}\right)^{1/3}$$

where N′ is the number of atoms in volume V, and

$$F = \frac{4\pi}{3}\rho^{3/2}K^{3/2}\left|\,2\left(\frac{2(1+\sigma)}{3(1-2\sigma)}\right)^{3/2}+\left(\frac{1+\sigma}{3(1-\sigma)}\right)^{3/2}\right|,$$

ρ being the density, K the compressibility, and σ Poisson's ratio. When the mass of the solid considered is equal to the atomic weight, V becomes the atomic volume, and N′ Avogadro's constant.* It should be noticed that this formula for the frequency contains no undetermined constant; all the quantities involved can be measured experimentally. Table IX of Debye's paper gives the value of the characteristic temperature found from the elastic constants for 12 metals, and these results have been employed in the calculation of ν and of $N\nu$ in the following Table (III). It will be seen that in each case the product $N\nu$ can be expressed in the form $n\nu_A$:—

Table III.—Atomic Frequency by Debye's Formula

Element	N	$\nu \times 10^{-12}$	$N\nu \times 10^{-12}$
Al	13	8·26	5 × 21·5
Fe	26	9·07	12 × 20·9
Ni	28	9·01	12 × 21·0
Cu	29	6·81	9 × 21·7
Pd	46	4·22	9 × 21·6
Ag	47	4·39	10 × 20·6
Cd	48	3·48	8 × 20·9
Sn	50	3·83	9 × 21·3
Pt	78	4·68	18 × 20·3
Au	79	3·44	13 × 20·9
Pb	82	1·49	6 × 20·4
Bi	83	2·30	9 × 21·2

Mean value of $N\nu = 21·02 \times 10^{12}$

Calculation of the probability in this case shows that there is only one chance in 45 that the product $N\nu$ should by accident approach so nearly to integral multiples of a single fundamental frequency, ν_A. Considering the difficulties in the determination of the elastic constants, and the uncertainty attaching to the temperature at which the measurements should be made, this is a striking result.

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* Debye assumes $N' = 5·68 \times 10^{23}$; Millikan's value, $N' = 6·062 \times 10^{23}$, would increase the values of ν by rather more than 2 per cent.

A comparison of the values of the frequency number, n, obtained by the use of Debye's formula, with those obtained from the specific heat, reveals the unexpected result that in several cases the integers are not the same. This difference in the values of n no doubt arises from variations in the physical conditions of the substance under examination The temperature at which the determinations are made must be taken into consideration. It is well known that in the case of certain elements various physical modifications exist consequent upon structural changes in the solid

§ 5 *Choice of the Atomic Numbers*

In the foregoing work the atomic numbers employed have been those given by Moseley It has been suggested by Rydberg that two unknown elements should be included in the Periodic Table between hydrogen and lithium, so that lithium would have an atomic number 5 instead of 3 For all the elements later in the Table, Rydberg's ordinals are greater by 2 units than Moseley's numbers It is found that the relation $N\nu = n\nu_A$ fits the results of observation more closely when Moseley's numbers are used for N. This is clearly shown by calculating the probability for the two sets of numbers Taking the observations of Nernst, the probability given by Rydberg's ordinals is 1/1·3 instead of 1/9 For the observations of Griffiths, the probability given by Rydberg's ordinals is 1/1·5 instead of 1/11. When Debye's formula for the characteristic frequency is employed, the probabilty is 1/5·4 for Rydberg's ordinals, but 1/45 for Moseley's numbers These results may be regarded as evidence strongly in favour of the atomic numbers proposed by Moseley *

§ 6 *The Physical Significance of the Relation* $N\nu = n\nu_A$

The results so far obtained may be summarised in the equation $N\nu = n\nu_A$ Thus for each element the characteristic frequency (or frequencies) may be expressed in terms of a single fundamental frequency,† ν_A, by employing two integers N and n The integer N denotes Moseley's atomic number, the physical significance of n is not as yet clear, but it may be suggested that it is related to the number of (valency) electrons which determine the crystalline structure of the solids A possible interpretation may be given to the empirical relation by the Quantum Theory Multiply each side of the equation by Planck's constant, h, which denotes the quantum of action Then

$$Nh\nu = nh\nu_A$$

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* The same conclusion was arrived at, in a different way, by van den Broek, 'Phil Mag,' vol. 28, p. 630 (1914)

† It is possible that, in some cases, instead of the value $\nu_A = 21{\cdot}3 \times 10^{12}$, it may be necessary to take a simple submultiple, such as ½, of this quantity

Considering an atom of any element, $h\nu$ represents one quantum of energy corresponding to the critical temperature Θ The left-hand side of the equation accordingly represents the amount of energy for as many quanta as there are positive charges in the core of the atom The right-hand side may be regarded as the energy corresponding to an integral multiple of a certain fundamental quantum, $h\nu_A$. It is necessary to emphasise the fact that this does not imply the actual existence of "atoms of energy" We are here concerned with the characteristic frequency, that is, with a certain limiting condition affecting each element in the solid state, and the occurrence of the quantity $h\nu_A$ implies only that there is a certain limiting amount of energy involved, which is the same for various elements

According to the theories of Debye and of Born and Kármán, the vibrations of the atoms in a solid form a continuous spectrum, limited by a definite boundary on the side of the shorter wave-lengths It is this limiting frequency which is taken as the characteristic atomic frequency It is not unreasonable to suppose that this frequency may be subject to a condition similar to that expressed by Einstein's relation, $Ve = h(\nu - \nu_0)$, which has been proved true by the experiments of Richardson and Compton, Hughes and Millikan, for the photoelectric effect It has, in fact, been proved by several investigators that the relation $Ve = h\nu$ accurately defines the boundary on the side of the shorter wave-lengths of the spectrum of X-radiation. In the equation $Nh\nu = nh\nu_A$ we may substitute Ve for $h\nu$, and V_Ae for $h\nu_A$, where V and V_A denote potentials at present undefined, and e is the charge on an electron. Then $NeV = neV_A$, or $NeV - neV_A = 0$ But Ne is E, the charge on the nucleus of the atom, and $-ne$ is the charge carried by n electrons If we may identify V with the potential of the nucleus, and V_A (equal to about $\frac{1}{12}$ volt) with the potential of a ring of electrons (or of the valency electrons), the relation expresses the fact that the energy of the atomic system is zero, or perhaps a minimum, in the condition corresponding to the limiting frequency. This involves the supposition that in the limiting condition the potential of the electrons in question assumes a constant value (or perhaps a multiple of some constant value, since n may be $p \times q$, where p and q are integers)

The relation may also be written in the form $EV = nA$, where A is constant. This may be interpreted as expressing the fact that in the limiting condition the energy of the nucleus is an integral multiple of a certain quantity of energy A.

PART II.—ELECTRONIC FREQUENCY AND ATOMIC NUMBER

§ 7. *Electronic Frequencies.*

A relation similar to that already discussed appears to hold for certain electronic frequencies. When, however, the vibration of an electron is in question it is necessary to replace the atomic constant ν_A (21.3×10^{12} sec^{-1}) by the fundamental electronic frequency, $\nu_E = 3.289 \times 10^{15}$ sec^{-1}, which is Rydberg's constant in spectral series, usually expressed as the wave number 109679.22 (Curtis). The relation then takes the form

$$N\nu = n\nu_1, \qquad \text{or} \qquad N\nu = (n+\tfrac{1}{2})\nu_E$$

In these cases, as in dealing with the characteristic atomic frequency, ν refers to some limiting frequency or to a frequency associated with a maximum value of some variable quantity.

§ 8. *The Maximum of the Photoelectric Effect.*

Pohl and Pringsheim* have determined the value of the wave-length corresponding to the maximum of the "selective" photoelectric effect with an accuracy of about 2 or 3 per cent. Their results have been employed in the construction of Table IV, which gives the values of ν and of $N\nu$ for the four alkali metals examined.

Table IV

Element	N	λ in $\mu\mu$	$\nu \times 10^{-14}$	$N\nu \times 10^{-15}$
Li	3	280	10.71	1×3.21
Na	11	340	8.82	$3 \times 3.23_5$
K	19	435	6.90	$4 \times 3.27_8$
Rb	37	480	6.25	7×3.30

The figures in the last column of the Table show that $N\nu$ may be expressed in the form $n\nu_E$, where n is a simple integer and ν_E is very nearly constant. The mean value of ν_E for these four elements is 3.255×10^{15} sec^{-1}, which is so near the Rydberg value 3.289×10^{15} sec^{-1} that there can be little doubt as to the identity of the two numbers.

A maximum photoelectric activity has been recorded for the four elements, magnesium, aluminium, calcium, and barium, which do not show a true "selective" effect. It is possible that the corresponding frequency and the atomic number are related in a similar way in these cases also, but the results are not decisive.

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* 'Die Lichtelektrischen Erscheinungen' (Vieweg, 1914).

§ 9 *The Limiting Frequency of the Photoelectric Effect*

The emission of electrons under the influence of light takes place only when the frequency of the exciting light exceeds a certain limiting frequency, ν_0 Richardson and Compton* have determined this limit both from the maximum energy of the electrons emitted, and also from their mean energy The results have been employed in the construction of Table V

Table V

Element	N	$N\nu_0 \times 10^{-15}$	
		Maximum energy	Mean energy
Na	11	2×2.83	2×2.86
Al	13	3×2.73	3×3.16
Mg	12	3×3.14	3×3.20
Zn	30	8×3.00	8×3.15
Sn	50	13×3.19	13×3.42
Bi	83	23×3.28	23×3.21
Cu	29	9×3.22	9×3.18
Pt	78	25×3.24	25×3.21

With the exception of the values for sodium and one value for aluminium, the results are in fair agreement with the relation $N\nu_0 = n\nu_E$

§ 10 *Ionisation Potentials*

The minimum potential required for the ionisation of a gas is probably connected with ν_0, the least frequency of radiation which can ionise the gas photoelectrically †

Einstein's equation is

$$eV_0 = h\nu_0,$$

where V_0 denotes the ionisation potential If then ν_0 is subject to the relation $N\nu_0 = n\nu_E$, we should expect to find $NV_0 = nV_E$, where V_E denotes a constant potential determined by $eV_E = h\nu_E$ From this relation the value of V_E is found to be 13.5 volts This is the value in Bohr's theory for the ionising potential in the case of (atomic) hydrogen. For, according to this theory, the work done in moving the electron from its orbit to a position of rest at infinity is $W = 2\pi^2me^4/h^2$ and $\nu_E = 2\pi^2me^4/h^3$ Hence W, which is eV_E, is equal to $h\nu_E$

The value of the ionisation potential for (molecular) hydrogen determined experimentally is of the right order of magnitude, but is only 11 volts instead of 13.5 volts.

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* Richardson and Compton, 'Phil Mag,' vol. 24, p 575 (1912), vol. 26, p 549 (1914)

† Einstein, 'Ann. d. Phys.,' vol 17, p 132 (1905), Richardson, 'Phil. Mag,' vol 24, p. 570 (1912), K. T. Compton, 'Phys. Rev,' vol 8, pp 386, 412 (1916)

The following Table (VI) gives the value of the ionisation potential, V_0, and of the product NV_0, in all cases where direct experimental determinations have been carried out.* The results in the last column of the Table show that, with the exception of hydrogen and oxygen, there is remarkable agreement with the relation $NV_0 = nV_E$.

Table VI

Element	N	V_0 (volts)	NV_0
Hydrogen	1	11.0	1 × 11.0
Helium	2	20.5	3 × 13.7
Nitrogen	7	7.5	4 × 13.1
Oxygen	8	9.0	5 × 14.4
Neon	10	16.0	12 × 13.3
Argon	18	12.0	16 × 13.5
Mercury	80	4.9	30 × 13.1

In the case of mercury ionisation of a second type also occurs for a potential of 10 volts,† which is almost exactly double the value recorded in the Table, so that the corresponding frequency number, *n*, would be 60.

§ 11. *Thermionic Potentials.*

Intimately connected with the potentials here discussed are the potentials observed in dealing with the emission of electrons from glowing solids and the contact potentials between different metals. In these cases the results obtained depend to such an extent on surface conditions and the presence of gaseous films, that as yet it is hardly possible to assign to the various elements reliable values that shall be characteristic of the elements themselves. The work that an electron would have to do to escape from the substance may be measured by the equivalent potential difference, ϕ. The values quoted in Table VII for ϕ, the "electron affinity" of the elements in volts, are derived from thermionic measurements,‡ and for the reason stated must be received with some reserve. It is, however, interesting to find that the values of $N\phi$ approximate fairly closely to multiples of 13.5 volts.

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[*] Franck and Hertz, 'Verh. Deutsch. Phys. Gesell.,' vol. 15, p. 34 (1913); vol. 16, pp. 457, 512 (1914); McLennan and Henderson, 'Roy. Soc. Proc.,' A, vol. 91, p. 485 (1915); Goucher, 'Phys. Rev.,' vol. 8, p. 561 (1916); Bazzoni, 'Phil. Mag.,' vol. 32, p. 566 (1916).

[†] Tate, 'Phys. Rev.,' vol. 7, p. 686 (1916).

[‡] O. W. Richardson, 'The Emission of Electricity from Hot Bodies,' pp. 69–79, 164–178 (1916); Langmuir, 'American Electrochemical Society,' pp. 341–396 (1916).

Table VII

Element	N	φ (volts)	Authority	Nφ
Carbon	6	4·14	Langmuir	2 × 12·4
		4·51	Deininger	2 × 13·5
Calcium	20	3·04	Horton	5 × 12·2
Titanium	22	2·4*	Langmuir	4 × 13·2
Iron	26	3·2*	Langmuir	6 × 13·9
Nickel	28	2·9	Schlichter	6 × 13·5
Molybdenum	42	4·31	Langmuir	14 × 13·0
Tantalum	73	4·31	Langmuir	23 × 13·7
Tungsten	74	4·52	Langmuir	25 × 13·4
Platinum	78	5·02	Deininger	30 × 13·1
		5·1	Horton	30 × 13·3
Thorium	90	3·36	Langmuir	22 × 13·7

* Preliminary measurements by Dr Dushman

§ 12 *Conclusion*

The empirical relations discussed in Part II may be summarised in the formula

$$\mathrm{N}\nu = n\nu_{\mathrm{H}}.$$

On multiplying each side of this equation by h, we obtain

$$\mathrm{N}h\nu = nh\nu_{\mathrm{H}},$$

or, by using the quantum relation $h\nu = e\mathrm{V}$,

$$\mathrm{N}e\mathrm{V} = ne\mathrm{V}_{\mathrm{H}}.$$

But Ne is equal to the charge, E, on the atomic nucleus Hence

$$\mathrm{EV} - ne\mathrm{V}_{\mathrm{H}} = 0$$

This suggests that in the limiting conditions which arise in all the physical phenomena under discussion, we have to deal with a minimum value of the energy of a system comprising the nucleus and a certain number of electrons

The Nature of the Ions produced by the Spraying of Water

By J J NOLAN, M A, D Sc, University College, Dublin

(Communicated by Prof J A McClelland, F R S Received June 16, 1917)

INTRODUCTION

In a paper on the electrification of water by splashing and spraying* the author gave a brief account of the nature of the ionisation which accompanies these processes In a later paper† these results were extended It was shown that when pure water is sprayed, ions of both signs are produced, negative being in excess These ions are found in a certain number of distinct groups, each group having a characteristic mobility There is no variation of mobility with time, that is to say, ions of any specified mobility will be found if an examination is made over as wide a range of time as is experimentally possible between the production and disappearance of the ionisation The mobilities found were —0 00038, 0 0010, 0 0043, 0 013, 0 046 0 12, 0 24, 0 53, 1 09, 1 56, 3 27 and 6 5, all being expressed as the velocity of the ion in centimetres per second in a field of 1 volt per centimetre As far as mobilities are concerned, no difference between the positive and the negative ions was found except that the fastest ion (mobility 6 5) was not found with a positive charge

In an investigation of the ionisation produced by bubbling air through mercury, Prof McClelland and P J Nolan‡ found five distinct groups of ions, the mobilities of the ions in this case depending on the time elapsing from their production and also on the degree of dryness of the air When the air was dried, or when the ions were examined as soon after formation as possible, comparatively high values of mobility were obtained With damp air the mobilities were lower, and by increasing the time interval between production and observation the ions could be brought to a final steady mobility The mobilities of the five groups in this state agreed very well with the mobilities of the five slowest ions produced from water. In a joint discussion it was suggested that these five ions are the same in each case, but that when produced by the breaking up of mercury, they take some little time to add on enough water to attain to their final steady state We imagined them therefore as water aggregates of five different sizes In the present paper a more definite theory will be submitted as to the constitution of these ions, a theory which in the light of the fuller account which can now be given of the more mobile ions can be made to cover all the ions observed

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* 'Roy Soc Proc,' A, vol 90 (1914)

† 'Royal Irish Academy Proc,' A, vol 33, p 9 (1916)

‡ *Ibid*, p 24 (1916)

PART I —DETERMINATION OF MOBILITIES OF VERY RAPID IONS

In the previous experiments the method of observation employed was to draw the ionised air at a steady rate through a tube having a coaxial cylindrical electrode suitably insulated. The outer tube was charged to various voltages, and the current to the electrode measured When the current-voltage graph was plotted it consisted of a number of straight lines, the intersections of the straight lines showing the voltages at which the various groups of ions were saturated The determination of mobilities then reduces itself to the accurate drawing of the current-voltage graphs If the observations of current are unsteady, or if the number of ions of any class is small, so that the bend due to the saturation of that class is not well marked, the possibility of an accurate determination of mobility is accordingly reduced Great attention was given, therefore, in the previous work to the accurate drawing of these graphs In the result it was felt that the mobilities of the different groups were fairly well established—apart from a few cases in which there appeared to be a real variation It was felt, however, that much more information could be obtained about the more mobile ions, if a method which would more effectively separate out the different classes of ions could be employed—and especially if such a method could be devised to examine the ions as soon as possible after their production, when their numbers had not been reduced by recombination or diffusion The method described below was found to be suitable, and it was immediately found that ions of much higher mobility than those previously observed were present Ions which had been obscured in the other observations were readily separated out A full account can now be given of the more mobile ions

Experimental Method

The apparatus used is shown in fig 1 A strong wooden box of rectangular shape was made measuring internally 100 cm long, 25 3 cm wide and 14 cm deep. To the top and bottom were fitted metal plates The bottom plate was 50 cm. long and its end was 20 cm. from the open end of the box This plate was supported by two metal rods passing through ebonite plugs in the bottom of the box. The plate came within 0 4 cm. of the bottom Strips of ebonite of this thickness were placed transversely under the plate so that its surface was quite level. Exactly above this plate were two plates C and D, each 25 cm. long, separated by an interval of about 0·2 cm. These

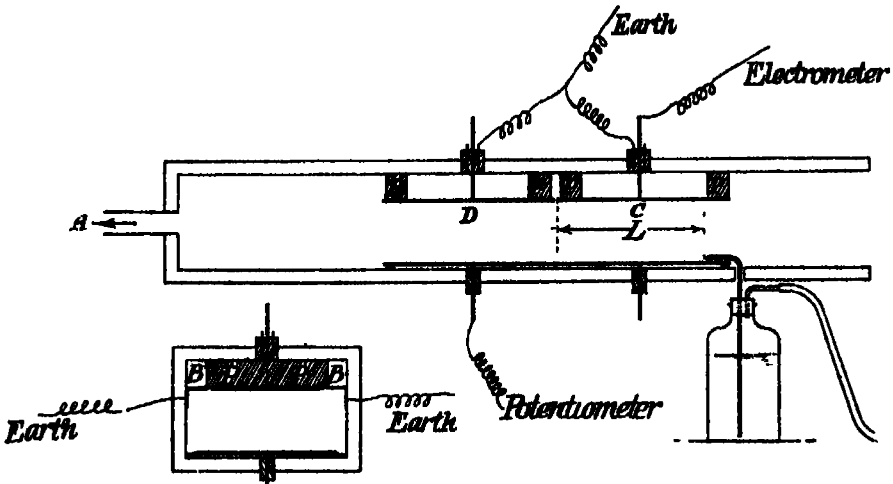

FIG. 1

plates were supported by metal rods passing through ebonite plugs fitted with guard-rings connected to earth. Blocks of paraffin (P, P) served to keep the plates in position. The lower plate was 24 cm. wide, giving a little clearance on each side. The upper plates were 15 cm. wide. Running along on each side of the upper plates were strips of metal 5 cm. wide (shown in end view of apparatus). These were fastened on to wooden blocks (B, B) and were connected to earth. There was thus a gap of 0·15 cm. at each side between the upper plates and the earthed strips. The spraying apparatus consisted of an ordinary metal scent-spray connected to a large glass bottle containing distilled water. It was driven by air at a constant pressure of 27 cm. of mercury. The nozzle of the sprayer passed through the bottom of the box and projected a distance of 5 cm. over the lower plate. The distance between the nozzle and the plate was approximately 0·2 cm. Connection was made with a gasometer at A, so that a strong current of air could be drawn through the apparatus at a uniform rate. The lower plate was connected to a potentiometer by means of which it could be charged to any desired voltage. In practically all the experiments the plate C was connected to the electrometer and the plate D to earth. In a few experiments C and D were both connected to the electrometer, but the other arrangement was generally more suitable. The apparatus was tilted slightly so that the water sprayed along the lower plate drained away into a vessel provided for it.

If a certain number of ions of the same mobility, u, are produced at the tip of the sprayer, they will all move along the same path and arrive at the end of the plate C, having traversed the horizontal distance L at a voltage V, when

$$uV = Qa/Lb,$$

Q being the volume of air passing between the plates per second, a being the distance between the plates, and b the inside width of the apparatus If a current-voltage graph is plotted the following result should be obtained From zero up to a certain voltage the plate C should receive no charge, at a certain voltage the fastest group of ions should arrive together at the plate and the graph go up steeply It should then become level again, the current being constant while the voltage is rising until the next group of ions is received The graph will then resemble a series of steps, each step representing the collection of a group of ions of a certain mobility The mobilities are readily calculated from the voltage at which the step occurs by the formula given above It is obvious that if this can be realised in practice, the method is much easier and more accurate than that previously employed for the determination of mobilities It has the special advantage also that it separates out ions of mobilities lying close together and makes it possible to obtain a good value of mobility for each

When the method was put into practice the step-like nature of the current-

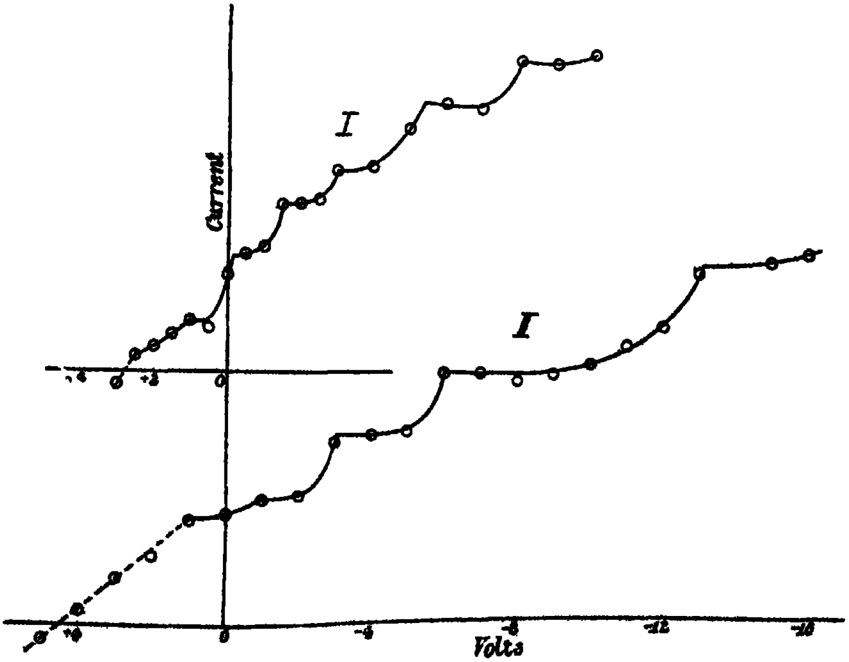

Fig 2

voltage graphs was at once apparent But an unexpected complication had arisen which will be clear from an inspection of fig 2 These two graphs deal with negative ions The steps are very well developed, but in each case for zero voltage a very considerable negative current is recorded Positive voltages, in one case three, and in the other case over four volts, have to be applied in order to bring the negative current down to zero Independently of the applied voltage there seems to be a negative field in operation of the order of three or four volts A little consideration makes the origin of this field clear The jet from the sprayer contains (see previous paper*) minute water drops with a high positive charge and large numbers of positive and negative ions with mobilities of all values from very fast to very slow The negative ions are in excess, the difference corresponding to the positive charge on the water When this jet is spread out over the lower plate a certain fraction of the water-drops are caught by the plate and give up their charge to it A commensurate number of ions will not be discharged, because, whereas the water drops may be carried against the plate by the velocity of their projection from the sprayer, the ions will not retain such a velocity, but will take up almost at once the stream-like motion of the air in the apparatus The result is that an excess of negative electricity is formed close to the lower plate In this way a field could easily be set up of the order of magnitude observed The distance between the plates is approximately 10 cm The positive voltage necessary to secure a zero reading of current varied between zero and 6 volts, but was generally about two These variations occurred in such a way as to lend support to the explanation just given Slight variations in the position of the spraying nozzle would mean that more or less of the water-drops would be discharged against the lower plate It was found that the field varied accordingly Again, if the velocity of the air in the apparatus is increased, it is to be expected that the negative excess present at any time would be diminished and that the field would be correspondingly smaller It is found that the field does vary in this way with the air-velocity.

It is very important to be clear as to how far this phenomenon affects the determination of mobilities In practically all the experiments dealt with here the rate of drawing air through the apparatus was such that V*u* (saturation voltage × mobility) had the value 47·0 The voltage corresponding to the determination of the mobility of the ordinary small ion would be between 25 and 30 As the value of the extra voltage was generally two, and sometimes less, it was easy to choose conditions so that the effect is negligible for the small ion and for any ion of lower mobility. It is in connection with the fast ions, and especially with the very fast ones, that the effect is most important. The question is this. Can we treat this field as being equivalent to a uniform steady field, which is simply to be added on to the applied field if the latter is negative, or subtracted from it if it is positive? It is easy to verify that, for any fixed position of the spraying nozzle and for a uniform air-blast, the voltage necessary to give zero current is quite constant, so that the field is steady in that respect at least. It will not be affected either by the different motion of the ions under different applied fields, because the small voltages used in dealing with the very mobile ions leave the motion of the great majority of the ions absolutely unaffected. The voltage to be added or subtracted will not therefore be different for different values of the applied field. As to whether the field is sufficiently uniform to be treated simply in an additive way, we can look to experimental results for guidance. Observations of the current-voltage curve have been obtained under various conditions, the "correcting voltage" or positive voltage necessary to give zero current varying between zero and 4.5. The observations when the correcting voltage was zero are especially valuable. The Table given below contains the results of these observations; it shows the voltage at which steps occurred on the current-voltage curve—these voltages having been corrected by the addition of the correcting voltage. These observations deal with negative ions.

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* 'Roy. Soc Proc,' A, vol. 90 (1914).

Table I

Correcting voltage	Voltages at which steps occur (corrected)					
0.0	1.7	—	—	5.75	—	9.5
2.8	1.8	2.9	4.3	5.8	8.2	10.8
3.75	1.75	3.35	—	—	—	—
4.5	—	3.5	—	5.5	7.5	10.5

This Table deals with some of the steps corresponding to the very rapid ions. It is clear that the corrected voltages agree very well among themselves, and that they agree very well with the voltages observed in the case when no correction had to be applied. In every series of observations, consequently, whatever part of the current-voltage curve is being examined, it is necessary to take a few observations with low positive voltages, in order to determine the point at which the graph crosses the line of zero current. It is much easier to proceed in this way than to endeavour to arrange the apparatus so that the correcting voltage is always zero. Efforts were directed to keeping the necessary correcting voltage as low as possible, and to determining its value accurately for each experiment.

An interesting example of this determination is given (III, fig. 3). As the curve is drawn, the voltage corresponding to zero current is +3·75

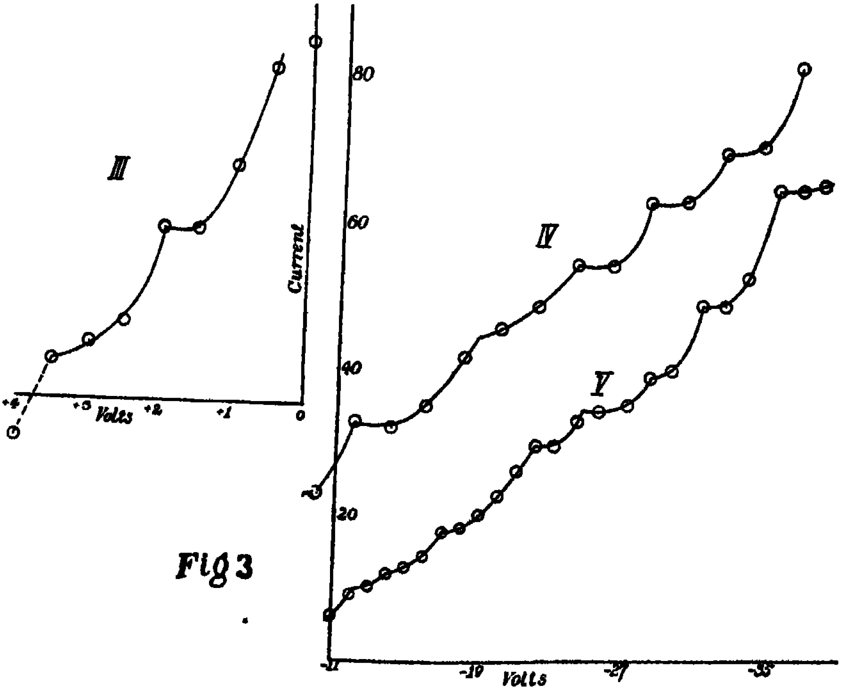

Fig 3

Taking this as the correct zero of voltage, we see that the first well marked step is at 1·75 volts. But, before that, there seems to be an abrupt rise between zero and 0·25 volt. Under the conditions of the experiment, V*u* being 47·0, the mobility of the ion corresponding to this step would be 47/0·25 = 188 cm./sec., or some higher value. The same indications are present in curve I (fig. 2), but the curve just mentioned is the best example, as the readings of current were big and steady. We can only attribute this effect to the presence of free electrons. There are no corresponding indications on the positive curves. The point is worthy of fuller examination.

Two examples are given (IV and V, fig. 3) of the graphs obtained with higher values of voltage. Both refer to negative ions. The voltages have been corrected.

The five curves given were obtained under the same conditions as regards air-velocity, etc.; the value of V*u* is consequently the same for all, and they are therefore strictly comparable with one another. A consideration of the curves will illustrate some points in connection with these experiments. Thus, if we compare I and II, we find that they agree in giving steps of current at (corrected) voltages of approximately 6, 8, and 11 They also agree in giving a step at a voltage of about 3, but curve I shows steps at 1 8 and 4 3 volts, which do not appear in curve II An explanation can be found in the fact that sufficient observations at various voltages were not taken in the second instance, and that the steps in the curve were thus obscured In the case of curve I, observations were taken at intervals of half a volt over this part of the curve, whereas the intervals in curve II are double this value

If we compare curves IV and V, we find that they agree in giving steps at 12, 24, 28 and 31 approximately, but that between 12 and 24 they are in disagreement, curve V showing steps which do not appear on curve IV One of these steps—that at 14 volts—is very faintly marked, in fact, there is hardly any justification for indicating a step at this point except that it shows up more strongly in other curves And again both of these curves are in disagreement with curve II, which gives no indication of a step at 12 volts Part of this disagreement may be attributed as before to the fact that if observations are not taken close enough together steps may be obscured But in addition there appears to be over part of the curve, say from 10 to 24 volts, some unsteadiness or variation which prevents a consistent agreement Most of the steps on the current-voltage curves are found easily and are always present In this particular region some have been established fairly well, but others must be regarded as in some degree open to doubt. The latter are specially noted later

In the actual determinations of mobility, shorter curves than those illustrated were generally used, the observations being taken close together round about the regions in which steps were indicated in preliminary surveys

Statement of Results

The following Tables give the values of mobility of the ions deduced from the best observations. Only ions of mobility greater than unity have been included, those of lower mobility have not yet been so fully examined by this method.

Comparing the positive and negative ions we find that the fastest positive and the second fastest negative are practically the same After that there is disagreement, but the six slowest in each case are practically identical

Table IIA.—Negative Ions.

Mobility	Remarks
27·0	
15·2	
11·4	Observed three times. Not well marked.
8·34	
6·06	
(4·42)	It is not quite certain whether these mobilities represent different ions or different observations of the same ion. Two observations of 4·42.
3·83	
2·69	
2·28	Two observations.
1·94	
1·70	
1·49	
1·34	
(1·15)	Sometimes absent.
1·06	

Table IIB.—Positive Ions.

Mobility	Remarks
15·5	One observation.
4·86	
3·43	
2·76	
2·42	One observation.
1·98	
1·72	
1·56	
1·37	
(1·18)	Not well marked.
1·05	

Possible Sources of Error.

Some of the mobilities are much higher than would be anticipated. It would be useful at this stage to see if this could be due to any error in the method of working. In estimating mobilities by the formula $uV = Qa/Lb$ certain assumptions have been made.

1. It was assumed that all the ions are produced at the tip of the sprayer. It is possible that the ions are not all produced at one point, or that, if they are, that point is not the tip of the sprayer. But the fact that the steps in the current-voltage graph are so steep and sharply marked is a pretty clear indication that the origin of the ions is sharply localised, at least within the limits of experimental observation. It might be said, however, that the point at which the ions are produced is not the tip of the sprayer but some point 2, 3 or 4 cm. in front of it. This is very unlikely, and if it were true it would mean that our estimate of L is too high and that the true values of mobility are higher than those given above.

2 The quantity Q was estimated by measuring the total quantity of air passing between the upper and the lower plates per second But the origin of the ions was some distance (0 2 cm) above the lower plate Therefore the quantity Q is not the total volume of air, but is less than it by an amount equal to the quantity of air flowing between the lower plate and a plane drawn parallel to it through the tip of the sprayer But it is easily seen that if the air velocity at the plates is zero, and if the distribution of velocities between the plates is approximately parabolic, the error involved in this estimation is altogether negligible, when the distance between the plates is as great as 10 cm

3 The disturbing effects of the rush of air from the spray were neglected But if this rush of air had any effect on the ions it would mean that the ions were carried along with a certain velocity parallel to the bottom plate instead of partaking of the undisturbed stream-line motion of the air This would mean that a higher voltage than is allowed for in our calculation would be necessary to bring them over to the upper plate, and that consequently the values for mobility are too low It is very unlikely, however, that there is any effect of this kind When an ion is produced at the tip of the sprayer it comes under the influence of the field and at once moves up out of the rush from the sprayer and takes up the velocity of the layers of air into which it successively moves

4 The principal assumption is in connection with the "correcting voltages" This has been discussed already, but perhaps two points may be emphasised once more The first is that our values for the mobilities from, say, mobility = 2 downwards, are practically unaffected by any uncertainty that might be felt in this connection The second point is that the values deduced from the experiment in which no correcting voltage had to be applied do not disagree in any way with those deduced from other experiments

The facility with which mobilities can be determined by this method is clear from an inspection of the graphs There are a few other points about the method which are worthy of note

Each group of ions travels along a path of its own through air free from any other ions. Thus there is no loss by recombination, and no complication introduced by encounters with other ions. Again, if any member of the group changes in mobility, say by an increase of mass, it drops out of the group and follows a different path for the rest of the way The unchanged ions arrive at the upper plate together, causing an abrupt upward step in the curve. The ions which have increased in mass arrive at various higher voltages, and their effect on the curve is to give a slight upward slope to the approximately horizontal part between steps. This effect is very marked in the part of the positive curves corresponding to high mobilities. The conclusion is that the very small positive ions are varying in a more rapid way than has been noticed in the case of the negative ions. The very well-marked steps in the negative curves show that in this case there is little variation of mobility during the time of experiment. The values of mobility deduced from the curves are therefore the true mobilities of ions observed at their formation and undergoing no change during the process of observation.

A general study of the curves shows that it is the fastest ions and then the slower ones from 1·94 to 1·06 which give the sharpest steps. The conclusion is that these ions have a fair degree of stability and a definite period of existence. The ions, on the other hand, of mobilities between 4·4 and 2·28 do not give indications that are so satisfactory or consistent. It is probable then that these ions are of a more fugitive character.

Before we endeavour to arrive at an idea of the nature of these rapid ions, it is convenient to consider the ions of low mobility which are formed at the same time. From a preliminary study of the slower ions we can derive some suggestions as to the magnitudes involved in the case of the ions which have just been described.

PART II.—THE NATURE OF THE LARGER IONS PRODUCED FROM WATER.

In the introduction to this paper an account has been given of the variety of slowly moving ions produced when distilled water is sprayed. The five slowest ions have mobilities respectively, 0·00038, 0·0010, 0·0043, 0·013 and 0·046. It will hardly be disputed that they are composed of water. The circumstances of their production, the way in which their growth depends on time and on the moisture of the air when, as in the experiments of McClelland and P. J. Nolan, they are not produced directly from water, all lend support to this idea, which from any point of view seems to be the only feasible one. Assuming, then, that they consist of water, there are two ways in which we can calculate their size, (1) by applying Stokes' Law, and (2) by using the theoretical formulæ deduced for the mobility of an ion which is great in comparison with the molecules of the gas which contain it.

Application of Stokes' Law.

The modified form of Stokes' Law which follows was deduced by Millikan* from observations on the fall of minute oil-drops in air:

$$v = \frac{2}{9}\frac{ga^2(\sigma-\rho)}{\eta}\left\{1+\frac{l}{a}(0{\cdot}874+0{\cdot}32e^{-1{\cdot}54a/l})\right\},$$

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* Millikan, 'Phys. Rev.,' vol. 1 (1913).

where l is the mean free path of the gas molecules. Substituting for g the acceleration Xe/m due to an electric field of intensity X, e being the charge on the ion, and assuming for the present that the charge is the electronic charge $4·77 \times 10^{-10}$ electrostatic units, we can calculate the radius of these five ions. For η the value 0·000182 is taken, and for l the value $9·27 \times 10^{-6}$ cm. The values deduced are tabulated below.

Table III

Mobility	Calculated radius	Mass	Ratio of masses
	cm	grm	
0·00038	$4·08 \times 10^{-6}$	2850×10^{-17}	1770
0·0010	2·41	587	365
0·0043	1·12	58·7	36·5
0·013	0·637	10·8	6·7
0·046	0·337	1·61	1

The masses of the ions calculated in this way are related to one another in a very definite way. If we exclude the second ion, there is an almost uniform ratio between the masses. The value of this ratio is about 6·5. The special reason for excluding the second ion will be dealt with at a later stage.

The validity of Millikan's formula for drops of very small radius might be questioned. We can show, however, that the results deduced are in close agreement with those deduced from theoretical formulæ for the mobility of ions.

Application of Mobility Formulæ

Sir J. J. Thomson,* on the assumption that the effects of collisions between ions and molecules are the same as those between two hard elastic spheres, gives the formula

$$\text{Mobility} = \frac{e}{N\pi\sigma^2\sqrt{M}\quad\Omega}\left\{\frac{M_1+M_2}{M_1M_2}\right\}^{\frac{1}{2}}\frac{5}{4}\left\{\frac{M_2+3M_1}{M_2+5M_1}\right\},$$

where N is the number of molecules of the gas per unit volume, M the mass of a hydrogen molecule, Ω its average velocity at the temperature of the gas, M_1 the mass of the ion, M_2 that of a molecule, σ the sum of the radii of an ion and a molecule, and e the charge on an ion. If the mass of the ion is large compared with that of the gas molecule this becomes

$$\text{Mobility} = \frac{e}{N\pi\sigma^2\sqrt{M}\quad\Omega}\,\frac{3}{4}\sqrt{\frac{1}{M_2}}$$

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* Sir J. J. Thomson, 'Phys. Soc. Proc.,' Part I, vol. 27 (1914).

On the same supposition with regard to elastic collisions, Townsend,[*] following Langevin's[†] investigation of the coefficient of interdiffusion of two gases, gives for the velocity U of an ion in the direction of an electric force X the formula

$$U = \frac{3eX}{8\sigma^2 N} \sqrt{\left(\frac{h(M_1+M_2)}{\pi M_1 M_2}\right)},$$

where $2h = N/\Pi$, Π being the pressure of a gas containing N molecules per cubic centimetre When the mass of the ion is large compared with that of the gas molecule, this becomes identical with the corresponding formula of Sir J J Thomson

Using this formula, then, we can find the value of σ corresponding to the various ions. Assuming once more that the charge is the unit electronic charge, we find for the radius of the ion of mobility 0 00038, the value $4\,16 \times 10^{-6}$ cm. as compared with the value $4\,08 \times 10^{-6}$ cm deduced from Millikan s formula The agreement between the two values is remarkably good

But a serious difficulty arises at this stage If the large ion has the size which these two very different methods of calculation agree in ascribing to it, it should be very easily visible in the ultramicroscope But de Broglie[‡] states that the large ion produced from flames cannot be seen in the ultramicroscope, and suggests that it consists of some sort of loose grouping too diffuse to give an ultramicroscopic image We have considered each ion so far as if it could be represented as a spherical globule of water We must recognise that this is a very definite assumption and that the alternative possibility must be considered, namely, that some at least of the ions consist of loose aggregates of some smaller drops which have not coalesced

This possibility can be considered more effectively at a later stage, for even if we at once accept de Broglie's experiment as decisive and regard the largest ion as a grouping of something smaller, we have as yet no clue to what that smaller globule may be, nor have we as yet any evidence as to how many of the ions are built up in this loose manner When we arrive at the stage at which the ions are compact spheres of different sizes the formulæ we have used will then be valid Meanwhile we will continue the calculations based upon the original assumption which regards all the ions as consisting of compact aggregates of water

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[*] Townsend, 'Electricity in Gases,' p 81 (1915)

[†] Langevin, 'Ann. de Chim et de Phys.' (8), vol 5, p. 245 (1905)

[‡] De Broglie, 'Comptes Rendus,' vol. 148, p. 1317 (1909)

Sizes of Ions Treated as Drops Calculated from Mobility Formula

The theoretical formula for mobility may be put in the form

$$\text{Mobility} = \text{constant}/a^2,$$

where a is the radius of the ion. In the case of the first five ions the average ratio of mobility of each ion to the next slowest is 3·3. Therefore the radii of the ions are in the ratio 1 : $\sqrt{3\cdot3}$ and their masses are in the ratio 1 : $(3\cdot3)^{3/2}$, which is approximately 1 : 6. The values deduced from Millikan's formula have already suggested that the masses of the ions are related to one another in a ratio almost identical with this. Assuming then that the ions are built up in this way, each ion having a mass six times that of the next smallest, we can calculate the mobilities of the ions starting from the slowest. The mobility of each ion is got from the next slowest by multiplying by $6^{2/3}$, which is equal to 3·302. The following Table shows the calculated mobilities compared with those experimentally found. The agreement is very good except in the case of the second ion.

Table IV

Observed mobility	Calculated mobility	Mass
		grm
0·00038	(0·00038)	$\frac{4}{3}\pi(4\cdot16\times10^{-8})^3 = 3\cdot02\times10^{-16} = m$
0·0010	0·00129	$m/6$
0·0043	0·0043	$m/6^2$
0·013	0·014	$m/6^3$
0·046	0·047	$m/6^4$

The ions next in order had the mobilities 0·12, 0·24, 0·53, and 1·09. The ratio of mobilities is now different. The value of 0·12/0·046 is about 2·6, and the subsequent ratios have values approximately 2. If the ratio of the masses of the ions was 3, the ratio of the mobilities would be $3^{2/3} = 2\cdot08$. This agrees very well with the ratio found for the faster ions, but the value 2·6 found for one ratio is in disagreement. The latter value suggests a mass ratio of 4, which would give a ratio of mobilities of the value of $4^{2/3} = 2\cdot52$. It is possible that the ratio of the masses of the ions is uniformly 3 and that the disagreement is in part due to experimental error and in part to the imperfection of our theory. For the moment, however, we will assume that the slowest ion has a mass one-fourth that of the last ion in the Table above and that the succeeding ions have masses in the ratio of 3. Calculating the values of the mobilities on these assumptions, we find that they correspond very closely with the experimental values.

Table IVA

Observed mobility	Calculated mobility	Mass
0 12	0 12	$m/6^4 \times 4$
0 24	0 24	$m/6^4 \times 4 \times 3$
0 53	0 51	$m/6^4 \times 4 \times 3^2$
1·09	1 06	$m/6^4 \times 4 \times 3^3$

Possibility of Multiple Charges

These calculations rest on the assumption that the charge on each of these ions is the unit electronic charge Kennedy* has shown, however, that the large ions produced from flames may have various charges, e, $2e$, etc, up to at least $6e$ We are obliged, therefore, to consider the possibility that here also multiple charges are present In that case we should expect to find a mixture of ions of charges e, $2e$, and $3e$, etc, with certain charges predominating Kennedy's results support this view In the case of ions as large as those with which we are dealing, there is no theoretical justification whatever for the idea that the size could depend *ceteris paribus* on the charge. Therefore, on any theory of mobility, we should expect to find a 1 2 3 4 .. relationship between the observed mobilities of the ions But no such relationship is apparent from an inspection of the mobilities obtained. It is most unlikely, therefore, that the majority of the ions have other than single charges There remains the case of the second ion. A great many observations of this ion gave the value 0 00088, other readings varied from that up to 0 00131 The last value would agree with that calculated, 0 00129 (see Table IV), and with the value observed by McClelland and P J Nolan, 0 0013 It was noted also in the previous work that the determination of the mobility of the final ion was difficult The observations varied from 0 00031 to 0 00044, and on some occasions it was difficult to decide whether this ion was present at all Again, very good observations were obtained of ions of mobility 0 00065, which could not be made to fit into any group It is clear, then, that in the case of the two slowest ions, we have a variety of values which might be accounted for by the existence of multiple charges

On the whole, however, the assumption that the charge on the ions is the electronic charge is justified

We arrive, then, at the idea that when pure water is broken up, water aggregates of certain definite sizes are produced These have at least a certain degree of stability, and, on our present hypothesis, their masses are related to one another, first in a ratio of 6, and then for the faster ions in a ratio of 3 But intervening there is a step from the ion of mobility 0 047 to the ion of mobility 0 12, where we have supposed a mass-relationship of 4 It is possible, of course, that the true ratio of the masses of the ions is 3 . 1, in harmony with the other lighter ions, in that case the calculated mobilities of the faster ions would have to be lowered about 20 per cent This would involve a disagreement of about that magnitude between the calculated and the observed values Such a disagreement would not be very serious, seeing that any error in calculation due to the inadequacy of the theory is cumulative in the method which we have employed.

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* Kennedy, 'Roy Irish Academy Proc.,' A, vol. 33, p 58 (1916)

The ion of mobility 0 047 seems to occupy a special position. There are reasons for believing that it is a particularly stable aggregate Taking the radius of the largest ion (mobility 0 00038) as $4{\cdot}16 \times 10^{-6}$ cm, the radius of this ion will be

$$4\,16 \times 10^{-6}/6^{4/3} = 3\,8 \times 10^{-7} \text{ cm approximately}$$

Now, it has been shown by Sir J J Thomson* and Langevin,† that drops of a certain radius should have a very high degree of stability The value of the radius of such a drop is calculated as 5×10^{-7} cm This is in rough correspondence with the value that has been found above, and it is possible, therefore, that the ion of mobility 0 047 is the true persistent ion deduced by Sir J J Thomson and Langevin rather than the ordinary "large ion," the radius of which we have shown to be $4\,16 \times 10^{-6}$ cm or larger

The Larger Ions Considered as Loose Groupings

In discussing the larger ions as loose assemblages of some smaller unit rather than as compact masses, we must make some assumption as to the nature of the unit This unit we may assume to be the largest possible single water-globule, the true "persistent ion" of Langevin and Sir J. J Thomson, which we have just suggested is identical with the ion of mobility 0 047 The next largest ion, instead of being formed from six of the latter, coalescing into a larger water-sphere, as we supposed before, is, on the present view, formed from a loose group of three. The viscous resistance to motion offered by this group of three will probably be something greater than three times the resistance of each of the drops from which it is formed, corresponding with the observed ratio of the mobilities of 3 3. Hence we might regard the ions of mobility 0 013, 0·0043, 0 0013, and 0·00038 as loose groups of 3, 9, 27, and 81 water-globules of the stable size. The ions with mobilities intermediate between these which have been observed by McClelland and P. J Nolan in dried air would be built up in the same way from a smaller unit. This is only a slightly more definite statement of the theory put forward by Prof McClelland and P J Nolan, and the present author in the previous paper already referred to.

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* 'Conduction of Electricity through Gases,' p. 149 (1903).

† See Chauveau, 'Le Radium,' March–April, 1912.

The masses and dimensions calculated above for the faster ions would remain unaltered in the new arrangement The new scheme has this advantage, that it is easier to see why all the ions have, in general, unit charge Each ion is formed from three smaller ones, which will in general be two positives and a negative, or two negatives and a positive We can also easily provide for the formation of ions carrying a number of charges, the mobility of which will not be sensibly different from that of the ion with unit charge For supposing that two, three, four or any higher number of the latter ions, all of the same sign, come together to form a loose grouping, the resistance of the system will be increased in approximately the same proportion as the charge, so that the mobility will be very much the same as that of the singly charged ion.

There are thus two possible views of the larger ions, that which regards them as single water-globules and that which looks on them as groupings of a smaller drop, which is the real stable water drop There is no doubt that the ultra-microscopic experiments of de Broglie on flame gases give a very strong support to the second view The greater regularity in the process of building up of ions which it permits and the ease with which it explains multiple charges are also strongly in its favour Further experiments on the large ions should put the matter beyond doubt. Meanwhile our calculations for the smaller ions are still valid

Before considering the very rapid ions it is of interest to calculate the size of the smallest of the ions which we are considering at present This ion, of observed mobility 1·09, has a mass (Table IV) $= m/6^4 \times 4 \times 3^3$, where $m = 3{\cdot}02 \times 10^{-16}$ grm, calculating on the basis of Sir J. J Thomson's formula. Thus the mass of the ion is $2{\cdot}14 \times 10^{-21}$ grm The mass of a molecule of water is 3×10^{-23} grm., therefore this ion contains $214/3 = 71$ molecules of water If we take the value deduced from Millikan's formula we find that the corresponding result is 66 molecules of water.

PART III —THE NATURE OF THE SMALLER IONS PRODUCED FROM WATER.

When we consider the values obtained for the mobilities of the smaller ions we are confronted with mobilities which are unexpectedly high. There are two ways in which we might try to make these mobilities fit in with normal values: (1) We might suppose, at least in the case of the negative ion, that what we are dealing with is an electron which has become attached to a molecule or group of molecules for a certain portion of its path, or (2) we might suppose that there are present ions with a variety of multiple charges. On the first hypothesis, it is very difficult to imagine how a variety of high mobilities sharply differentiated from one another could be produced. But, apart from that, the fact that we find a positive ion with mobility 15·5 shows that we have to deal with molecules. The question of multiple charges is not so easily decided. If there were molecules present with one, two, three charges, etc., we would expect to find a 1 : 2 : 3 : 4 relation between the mobilities. An inspection of the table of mobilities shows that it is easy enough to pick out here and there mobilities which are roughly in the ratio 1 : 2 or 1 : 3 but we can find no sustained 1 : 2 : 3 : 4 relation. The first three well-established mobilities, 27, 15·2 and 8·34, may suggest 3 : 2 : 1. But the numerical agreement is not sufficiently good. Even if we admitted that these ions had three, two and one charges respectively, we have to explain why the doubly charged positive ion (mobility 15·5) occurs while there is no trace of this ion with a single positive charge.

Any such allocation of multiple charges, moreover, does not make much change in the difficulty as regards high values of mobility. For example, even if the 3 : 2 : 1 ratio of charges mentioned above could be successfully established, it would leave us with a singly charged ion of the unusually high mobility of 8·34.

Calculation of the Mobility of an Ion

The very high values of mobility obtained in the present work seem to support the view that the collisions between ions and molecules may be regarded as taking place between hard elastic spheres. The other hypotheses as to ionic collision made familiar by the work of Wellisch* and Sutherland† would yield much lower values.

On the elastic collision hypothesis Sir J. J. Thomson,‡ finds for the mobility of an ion the equations

$$R = \frac{e}{N\pi\sigma^2\sqrt{M}\;\Omega}\left(\frac{M_1+M_2}{M_1M_2}\right)^{\frac{1}{2}}\frac{5}{4}\left(\frac{M_2+3M_1}{M_2+5M_1}\right)$$

and
$$R = \frac{e}{N\pi\sigma^2\sqrt{M}\;\Omega}\left(\frac{M_1+M_2}{M_1M_2}\right)^{\frac{1}{2}}\frac{5M_2(M_1+3M_2)}{15M_2^2+10M_1M_2-M_1^2},§$$

according as M_1 is greater or less than M_2. N is the number of molecules of the gas per c c , M the mass of a hydrogen molecule, Ω its average velocity at the temperature of the gas, M_1 the mass of the ion and M_2 that of the molecule We have already used the formula for the case when M_1 is large compared with M_2 We can use these formulæ for calculating the mobilities of various possible aggregates of molecules, but two difficulties arise The diameter of the sphere of influence of a molecule is not known accurately, and even if we could assign a value to σ in the case where the ion is a single molecule, we can only endeavour to approximate to it when the ion is a cluster of several molecules

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* Wellisch, 'Phil. Trans.,' A, vol. 209, p. 272 (1909).

† Sutherland, 'Phil. Mag.,' Sept., 1909.

‡ 'Phys. Soc. Proc.,' Part I, vol. 27 (1914).

§ There is a misprint in this formula as published.

As the agreement between Sutherland's values of σ for the ordinary gases and those deduced from the limiting density is very good, it was decided to rely upon these two methods of calculation The values $2 9 \times 10^{-8}$ cm. for air and $3 45 \times 10^{-8}$ cm for water have therefore been used in deducing the mobilities of various possible ions In calculating the mobility of a composite ion formed by n molecules of a gas, the diameter of the molecules being s_1, in a gas of which the molecules had diameter s_2, the formula $\sigma = \frac{1}{2}(s_2 + n^{\frac{1}{3}}s_1)$ has been employed.

As to the composition of the ions, two possibilities present themselves—they may be aggregates of air-molecules (making no distinction for the moment between oxygen and nitrogen) or they may be aggregates of water The balance of probabilities is altogether in favour of the latter view The slowest of these fast ions (mobility 1 06) is identical with the ion which was considered to consist of a cluster of about 70 water-molecules. From that ion up to the fastest there are a variety of ions increasing in mobility and therefore decreasing in size There is no indication of any transition where the ions cease to be of water and begin to be composed of air There is no special reason in these experiments why a greater number of groupings of air-molecules should be formed; whereas there is very good reason to expect all possible aggregations of water Further reasons in support of this view will be brought forward at a later stage in this paper

Assuming then that we are concerned altogether with ions composed of water, we can employ the formulæ given by Sir J J. Thomson to calculate the mobility of ions composed of 1, 2, 3, and various higher numbers of water-molecules In the following Table mobilities calculated for various groupings of water-molecules are given together with the mobilities observed, negative and positive

Table V.

Number of H_2O molecules	Calculated mobility	Observed mobility	
		Negative	Positive
1	12·3	15·2	15·5
2	7·40	8·34	—
3	5·66	6·06	—
4	4·70	—	4·86
6	3·67	3·63	—
7	3·36	—	3·43
12	2·49	2·69	2·76
(15)	2·17	(2·28)	(2·42)
18	1·97	1·94	1·93
24	1·72	1·70	1·72
30	1·53	1·49	1·56
36	1·31	1·34	1·37
48	1·14	(1·15)	(1·18)
54	1·04	1·06	1·05

Comparison of Calculated with Experimental Values of Mobility.

The mobility calculated for an ion consisting of one water-molecule with unit charge (12·3) is in fair agreement with the values 15·2 and 15·5 found for negative and positive ions respectively. A water-molecule with two charges would have a mobility 24·6, according to our calculations. This corresponds fairly well with the fastest ion observed (mobility 27). No place has been found in the above Table for the ion sometimes observed, the mobility of which was 11·4. If the ion consisting of three water-molecules had a double charge, its mobility should be 11·3. This is the most probable explanation of this ion. The values for mobility of the negative ions fit in fairly well with the values calculated for ions consisting of 2, 3, 6 and 12 water-molecules, with a tendency for the observed values to be higher than the calculated ones; the agreement for 18, 24, 30, 36, 48 and 54 is very good, both for positive and negative. Of course, in the last two cases, a fairly big change could be made in the number of water-molecules without very seriously affecting the mobility; but the way in which the observed values agree with those calculated for even increments of 6 water-molecules from 18 to 36 is very striking. With regard to the ions of 12 and 15 water-molecules, it is possible that in this case we are really only concerned with one ion consisting of 12 rather than 15 molecules. There are only two negative observations giving the value 2·28, and one positive giving the value 2·42. Perhaps the disturbing element is one of the slower ions with a double charge. If we eliminate the ion of 15 molecules, then the negative ions go fairly evenly 1, 2, 3, 6, 12, 18, 24, 30, 36, 48 and 54 In the case of the positive ions nothing was found corresponding to the ions of 2 and 3 water-molecules The next highest mobilities after that which we suppose to be due to one molecule do not agree with any values found in the case of negative ions. These mobilities (4·86 and 3·43) are deduced from good experiments, and no doubt is felt about their approximate accuracy As will be seen from the Table, they fit in very well with the calculated mobilities for ions consisting of 4 and 7 water-molecules respectively This suggests that these ions are formed by the union of a water-molecule which has lost an electron, to groups of 3 and 6 water-molecules It is possible that all the positive ions are formed in this way, that is, by the union of a positively charged molecule with a stable or semi-stable group of water-molecules which is already in existence When the number of molecules becomes considerable, the resulting ion has practically the same mobility as the negative ion, which is formed from a similar group by the adding on of an electron

All the ions observed have now been accounted for with the exception of the negative ion 4·42 There are only two observations of this ion, so that the exact value of its mobility is not very well established The best explanation then that can be given is that it consists of 5 water-molecules, such an ion would have a mobility about 4·15, as compared with the observed value of 4·42

It is worthy of note that it is just those ions which show up most consistently and easily on the experimental curves which fall most readily into a regular classification These are the very fast ions and the ions with mobilities from 1·97 to 1·04 inclusive We cannot so easily classify the others, partly because of an uncertainty as to their accurate mobility values, and partly, no doubt, because some of them represent aggregates of a small degree of stability which are not easily made to show a relationship with what we may regard as the more definite or stable forms

On the supposition then that the ions are composed of various numbers of water-molecules we have been able to show that if the ions are built up in a certain regular way the observed mobilities agree fairly well with a theoretical estimate from the elastic collision point of view The agreement is better the greater the number of molecules in the ion, and for the last six ions the agreement is extremely good. Then the last ion is found to consist of 54 water-molecules (assuming a regular step of 6 or a multiple of 6), agreeing as well as could be expected with our previous estimate of 66 to 71.

Possibility that the Ions are Groups of Air-Molecules

It might be argued, however, that a set of ions could be built up from air-molecules which would fit the observed values as well as our theoretical water-ions But the previous work on the large ions has led to the idea that the slowest of the present ions (mobility 1 06) is altogether water We would have to suppose, then, that as the ions became smaller a certain proportion of air-molecules entered into their composition, and that finally at a certain small size they were altogether composed of air-molecules But the very good agreement between the calculated and theoretical values for the last six ions suggests that they at least are altogether water There is nothing to suggest a change in composition Assuming that aggregations of air-molecules are possible, we can calculate the mobilities which ions composed of various small numbers of air-molecules should have Using Sir J J Thomson's formula as before, we find that ions containing 1, 2, 3, 4, 5 and 6 air-molecules would have mobilities 12, 8 05, 6 4, 5 47, 4 85 and 4 43. The first of these numbers (12) is practically the same as the mobility calculated for a water-molecule, and indeed the two following, 8 05 and 6 4, are not far removed from the corresponding values for water, 7 4 and 5 66 If we consider the ratios of the mobilities of the first three ions we find that the observed numbers are as 100 55 40 The values calculated for water are as 100 60 46, while those for air are as 100 67 . 53 The observed numbers, therefore, agree better with the supposition that the ions are composed of water than with the idea that they are composed of air The explanation of the two positive ions which do not correspond to any negative ions fits in very well on the supposition that the ions are of water If the ions are composed of air-molecules it is hard to see why there should be any difference between positive and negative, because the ion will then consist of a cluster of molecules owing whatever stability it has to its charge The number of molecules in such an ion would depend only on the charge. We cannot imagine a cluster of gaseous molecules becoming an ion by picking up an electron or a positive molecule, because there is no evidence that such clusters exist in a gas like oxygen or nitrogen In fact the only evidence of the existence of such uncharged clusters is in the case of moist gases, where everything points to these clusters being composed of water. We cannot, of course, exclude the possibility that the first ion (mobility 15 2) may be a molecule of oxygen or nitrogen, because the observed value would agree as well with that assumption as with the supposition that it is a molecule of water.

Identity of Some of the Water Ions with Ordinary Gaseous Ions

If we consider the four ions, the observed mobilities of which are (for negative ions) 1·94, 1·70, 1·49 and 1·34, we find indications of a possible connection with the mobilities of ordinary ions produced by X-rays or other ionising agents in air under different conditions. For example, in saturated air the ordinary values[*] for mobility of negative and positive ions are respectively 1·51 and 1·37, corresponding approximately to the third and fourth of the ions above. For dry air the values given by different observers vary a great deal, and there is good reason for believing that this should be attributed to the degree of drying arrived at in various cases. For example, Wellisch[†] gives the values 1·78 and 1·54 for negative and positive. Langevin's[‡] values are 1·70 and 1·40. In more recent work, where a very high degree of dryness was attained, Wellisch[§] finds the values 1·93 and 1·23, corresponding to some extent with Zeleny's[||] values 1·87 and 1·36. All these values can be grouped fairly well into four classes corresponding to the four values quoted above, the negative ion having values approximating to 1·94, 1·70, or 1·49, according to the dryness of the air, and the positive ion having values approximating to 1·49 or 1·34 in the same way, with a tendency towards the last value occurring more frequently. The small ion in air, therefore, must at the very least resemble closely in mass and size the water-ions which we have been observing. The way in which its mobility decreases in damp air resembles exactly the way in which the water-ion adds on further groups of water-molecules. The view is put forward in this paper that the ordinary small ion is composed of water, the negative ion being an aggregate of 18, 24, or 30 molecules of water, according to the condition of the air as regards humidity, and the positive ion consisting of 30 or 36 water-molecules, more often the latter number. Of course we cannot insist on the exact accuracy of the number of water-molecules assigned, but it is felt that they are fairly close to the true values. This theory of the nature of the small ions applies to any of the permanent gases as well as to air. The nature of the small ion in a readily condensable gas is probably different. The justification of the theory that the small ion is composed of water is found in the ease with which it explains several of the well-known properties of the gaseous ion.

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[*] Zeleny, 'Phil Trans.,' 1900.

[†] Wellisch, 'Phil Trans.,' 1909.

[‡] Langevin, 'Annales de Chimie et de Physique,' 1903.

[§] Wellisch, 'American Journal of Science,' May, 1915.

[||] Zeleny, *loc. cit.*

The Mobility of the Ion in Different Gases

In a number of the permanent gases the mobility of the ion is inversely proportional to the square root of the density. This result follows immediately from our theory, for, the mass of the ion being large compared with that of the gas-molecule, the formula of Sir J J Thomson, which we have been using, becomes

$$R = \text{constant}/\sqrt{M_2},$$

M_2 being the mass of the gas-molecule

We can, moreover, use the full formula to calculate the mobilities of ions having the composition that we assume in different gases For example, taking Sutherland's value for the molecular diameter of hydrogen, we can calculate the mobilities of ions composed of 18, 24, 30, and 36 water-molecules in hydrogen

Table VI —Hydrogen

Composition of ion	Calculated mobility	Observed values				
		Moist gas (Zeleny) *	Dry gas			
			Zeleny *	Chattock †	Franck and Pohl ‡	Haines §
18 H_2O mol	7·93	—	−7·95	−7·48	−7·68	−7·9
24 "	6·81	—	+6·70	—	—	—
30 "	6·00	−5·6	—	—	+6·02	—
36 "	5·44	+5·8	—	+5·4	—	+5·4

The agreement between mobilities calculated by this method and the values observed is extremely good Again, as in the case of air, we find the negative charge associated with 18 water-molecules, and the positive charge associated with 36, when the drying is specially good. Under other conditions of moisture the other two sizes tend to appear

Space will not permit of the extension of these calculations to the other gases. It may be said, however, that the agreement is good in the case of the more permanent gases, but not so good in the case of gases which are easily condensable.

The further advantages of the theory of the small ion just put forward may be summarised rapidly as follows:—

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* Zeleny, *loc. cit.*

† Chattock, 'Phil Mag,' 1899

‡ Franck and Pohl, 'Verh. d. Deut Phys. Ges.,' 1907

§ Haines, 'Phil. Mag.,' April, 1916.

(1) It permits of ready explanation of the results of Blanc* and Wellisch† for the mobilities of ions in gaseous mixtures

(2) It can be harmonised with Phillips'‡ results on the variation of mobility with temperature

(3) The sizes attributed to the ions are in fair agreement with the sizes deduced from the coefficient of diffusion

(4) The abnormally high mobilities observed by Haines§ in very pure hydrogen are readily explained if we assume that the "impurity" necessary for the formation of ions is water, and that the ions which successively appear are composed of 1, 6, 18, and 36 water-molecules respectively

CONCLUSION

It is no part of the theory given in this paper to insist on the absolute values (18, 36, etc) of the numbers of molecules supposed to constitute each ion These are the numbers which fit in best with our present knowledge of molecular sizes With fuller information, the ions might be better represented in terms of some other unit rather than 6 || But enough evidence has been given to show that, in the case of ions produced by spraying water, there is a regular building up of aggregates of water, and that certain of these aggregates, which seem to be specially stable, are at the very least similar, and are probably identical with the small ions observed in various gases

It is clear that there are many ways in which the ideas put forward in this paper can be experimentally tested Further work should render it possible to decide whether the larger ions are to be regarded as loose groupings or compact aggregates. It should be possible also to obtain more information on the more important question of the nature of the ordinary small ion, and to test the theory put forward in this paper

The author is indebted to Prof McClelland, to whose suggestion these researches are due, for his interest and advice.

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* Blanc, 'Journ de Phys.,' vol 7, p. 838 (1908)

† Wellisch, 'Roy Soc Proc.,' vol 82, p 500 (1909).

‡ Phillips, 'Roy Soc. Proc.,' vol 78 (1907)

§ Haines, 'Phil. Mag.,' April, 1916

|| Perhaps in this connection the structure of crystallised water is not without significance As is well known, snow crystals are almost always flat and hexagonal in character, taking various forms such as that of a six-pointed star, but showing evidence of having been formed around a single nucleus and having characteristic angles of 60° and 120°.

Phenomena Connected with Turbulence in the Lower Atmosphere.

By G I. Taylor, M.A.

(Communicated by Sir Napier Shaw, F R S —Received October 16, 1917)

The object of the present paper is to bring together some of the meteorological phenomena which depend on the turbulence of the lower atmosphere, to show how they depend on one another, and to demonstrate some numerical relationships which exist between them

The transference of heat, water vapour and momentum by means of eddies has been discussed by the present writer in a previous paper [*] It was shown there that the effect of turbulent motion on the atmosphere is to endow it with a power of transmitting heat in much the same way as a solid possessing a large coefficient of conductivity

The temperature, however, which enters into the equations of conductivity in the atmosphere is potential temperature,[†] instead of being the actual temperature, as it is in the case of the equations for flow of heat in a conducting solid Thus, if the turbulent air is at a uniform temperature, heat will be transmitted downwards because, under those conditions, the potential temperature increases with height

The power possessed by the atmosphere in virtue of its turbulence of transmitting heat and momentum may be represented by the symbol K, where K is proportional to the velocity and to the scale of the turbulence As a rule the atmosphere is stratified in the sense that temperature and velocity vary much more rapidly in a vertical than in any horizontal direction In this case the rate at which heat flows into unit volume of air is $K\rho\sigma\,\delta^2\theta/\delta z^2$, where θ is the potential temperature, ρ the density and σ the specific heat of air, and z is height measured from the ground The rate at which momentum parallel to the horizontal axis x is communicated to unit volume is $K\rho\,\delta^2 u/\delta z^2$, where u is the mean component of velocity parallel to the axis x, and the quantity K should, according to this theory, be the same in the two cases Roughly K may be taken as equal to $\frac{1}{2}wd$, where w represents the mean vertical component of velocity due to the turbulence, and d represents roughly the mean vertical distance through which any portion of the atmosphere is raised or lowered while it forms part of an eddy till the time when it breaks off from it, and mixes with the surroundings. This may be taken to be roughly equal to the diameter of a circular eddy

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[*] "Eddy Motion in the Atmosphere," 'Phil. Trans.,' A, vol. 215, p. 1, 1915

[†] The potential temperature of the air at any height is the temperature to which it would be reduced by expanding or compressing it adiabatically to a standard pressure.

In the paper already referred to, values of K have been found for the turbulence of air blown over the sea, from the measurements of the temperature over the Great Banks of Newfoundland A few simultaneous measurements of wind velocity over the same area tend to show that the value of this K involved in the equations which represent the transference of momentum is of the same order of magnitude as the K involved in the heat transference equations The average value obtained for K by these measurements was 3×10^3 in C G S units.

On the other hand, observations taken by means of pilot balloons over Salisbury Plain, of the K involved in the momentum transference equations indicate that the value of K on land is much greater The mean value was of the order 5×10^4 In 1914 I did not know of the existence of any observations which would enable me to calculate the value of the K involved in the equations for heat transference through the atmosphere over the land, but I have since found, in the 'Annales du Bureau Central Météorologique de France,' for 1894, a series of temperature measurements made at various heights on the Eiffel Tower, which supply the data necessary for calculating this quantity It will be seen later that these measurements indicate that the value of K over the land is of the order 10×10^4, but that it varies considerably with the time of year, and also, to a certain extent, with the height above the ground.

Temperature Observations on the Eiffel Tower.

During the five years 1890–94 hourly observations of temperature were taken at three different heights on the Eiffel Tower, at 123, 197, and 302 metres above ground. The results have been given by M Angot, who discussed the observations, in the form of the mean temperature of the air for each hour in the day for each month in the year Each figure given by M Angot is, therefore, the mean of about 150 observations.

On examining the rise and fall in temperature during the day at the various stations on the Eiffel Tower, it will be found that the temperature may be represented very approximately by a simple harmonic function of the time That is to say, the curve which represents the variation in temperature during the day is approximately a sine curve A specimen curve (fig 1) has been picked out at random from the observations, and the true sine curve which most nearly represents the real curve has been drawn in as a dotted line beside it It will be seen that no serious error will result from taking the sine curve instead of the true curve for the purposes of calculation, and it will be found that this introduces considerable simplification into the calculations.

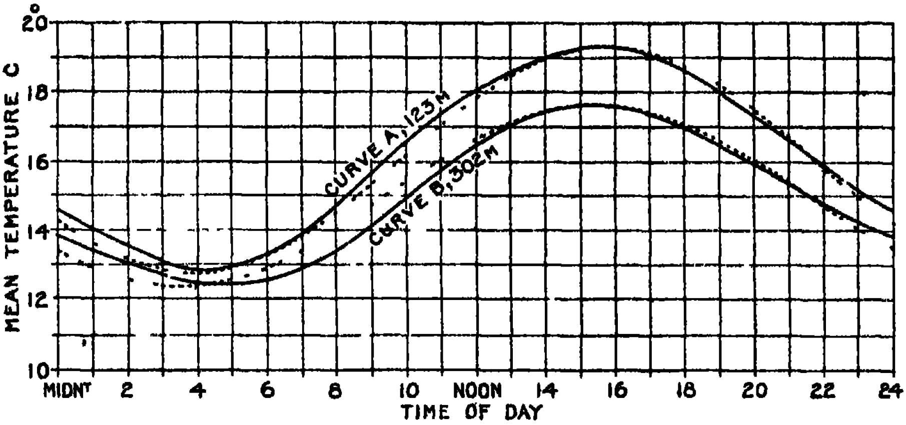

Fig 1

Table I—Amplitude in Centigrade Degrees of the Daily Variation of Temperature at Various Heights

Month	Terrace of the Bureau Météorologique	Eiffel Tower		
	18 metres	123 metres	197 metres	302 metres
	°	°	°	°
January	2 99	2 46	1 93	1 31
February	4 52	3 92	3 22	2 30
March	6 46	5 67	4 90	3 90
April	8 45	7 15	6 39	5 12
May	7 76	6 40	5 63	4 82
June	7 76	6 48	5 86	5 21
July	7 57	6 22	5 59	4 98
August	8 06	6 94	5 94	5 12
September	7·95	6 48	5 47	4 34
October	5 61	4 53	3 66	2 78
November	3 24	2 79	2 32	1 55
December	2 74	2 33	2 04	1 40

In Table I are shown the amplitudes of the daily variation in temperature for the various months of the year at various heights above the ground. As might be expected, the daily range in temperature decreases with the height above the ground, and the rate at which this range decreases evidently depends on the amount of turbulence in the atmosphere These observations of daily range will now be used to calculate the value of K for each month in the year. It will be found that the calculations are very greatly simplified if it is assumed that K is constant at all heights This assumption will therefore be made in the first place, and afterwards the effect of varying K will be considered.

If z represents the height above the ground, the equation for convection of heat by means of turbulence is $\frac{\delta}{\delta z}(K\rho\sigma\,\delta\theta/\delta z) = \rho\sigma\,\delta\theta/\delta t$, where $\delta\theta/\delta t$ represents the rate of increase in temperature with time

If K be independent of z this becomes

$$K\,\partial^2\theta/\partial z^2 = \partial\theta/\partial t. \tag{1}$$

A solution of (1) is

$$\theta = Ae^{-bz}(2\pi t/T - bz), \tag{2}$$

where

$$b^2 = \pi/TK. \tag{3}$$

If the daily variation of temperature at the height $z = 0$ is represented by $\theta = A\sin(2\pi t/T)$, where $T = 24$ hours or 86,400 sec, the daily range is 2A

From (2) it will be seen that the ratio of the daily ranges, R_1 and R_2, at two heights z_1 and z_2, is

$$R_1/R_2 = (2Ae^{-bz_1})/(2Ae^{-bz_2}) = e^{-b(z_1-z_2)}$$

Hence

$$b(z_2-z_1) = \log_e R_1 - \log_e R_2.$$

Hence, if the ratio of the daily ranges at two heights which differ by an amount h is known, the quantity b may be found from the equation

$$b = (\log_e R_1 - \log_e R_2)/h \tag{4}$$

The value of K may then be found from equation (3)

In Table I are given the observed daily ranges at three stations on the Eiffel Tower at heights of 123, 197, and 302 metres above the ground, and also the daily range at a station on the terrace of the Bureau Météorologique at a height of 18 metres above the ground If K were independent of z, it could be found by taking the ratio of the daily ranges at any two heights, and the same result would be obtained whichever pair of ranges were taken. If, therefore, all the values of K obtained by applying equations (3) and (4) to various pairs of stations at different heights are the same, we are justified in assuming that this is the true value of K* On the other hand, if it be found that the values of K found in this way diminish as the mean height of the air between the two stations increases, it seems justifiable to assume that K diminishes with height, while, if the values of K obtained in this way are found to increase, the real value of K increases

This method of calculation will not give the true values of K, unless K is found to be constant or nearly constant at all heights, but it may be taken as giving, qualitatively, certain general results concerning the changes in the amount of turbulence at various heights above the ground during the course of the year In Table II are given the results of applying equations (3) and (4) to the daily ranges in temperature given in Table I, in order to find the mean value of K in different months of the year

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* Strictly speaking, this is only true if the stations extend up to such a height that the daily variation in temperature is small compared with the daily variation near the ground, but no considerable error is likely to arise owing to the fact that this condition is not fulfilled.

Table II —Mean Values of K between Various Heights from 18 to 302 Metres above the Ground

Month	1 18 to 302 metres	2 123 to 302 metres	3 197 to 302 metres	4 18 to 123 metres
January	4 3 × 10^4	2 9 × 10^4	2 7 × 10^4	11 × 10^4
February	6 4 × 10^4	4 1 × 10^4	1 6 × 10^4	20 × 10^4
March	10 5 × 10^4	8 3 × 10^4	7·7 × 10^4	24 × 10^4
April	10 2 × 10^4	10 5 × 10^4	8 2 × 10^4	14 × 10^4
May	12 9 × 10^4	14 4 × 10^4	16 7 × 10^4	11 × 10^4
June	18 3 × 10^4	24 4 × 10^4	28·8 × 10^4	12 × 10^4
July	16 7 × 10^4	23 4 × 10^4	30 1 × 10^4	13 × 10^4
August	14 6 × 10^4	13 1 × 10^4	19 6 × 10^4	18 × 10^4
September	8 0 × 10^4	7 2 × 10^4	7 5 × 10^4	10 × 10^4
October	5 9 × 10^4	4 9 × 10^4	5 3 × 10^4	9 × 10^4
November	5 4 × 10^4	3 2 × 10^4	2 5 × 10^4	18 × 10^4
December	6 5 × 10^4	4 4 × 10^4	2 8 × 10^4	15 × 10^4
Mean	10·0 × 10^4			

The most noticeable feature of the Table is the way in which the turbulence appears to decrease with height in the winter and to increase in the summer In June and July, for instance, the mean values of K in the whole height of the tower, from 18 to 302 metres, are 18 and 17 × 10^4, while the mean values from 197 to 302 metres are 29 and 30 × 10^4 respectively In the winter months—November, December, January, and February —the mean values of K from 18 to 302 metres are 5 4, 6 5, 4 3, and 6 4 × 10^4, while the mean values from 197 to 302 metres are 2 5, 2·8, 2 7, and 1 6 × 10^4 respectively

The explanation of this diminution of turbulence at the top of the Eiffel Tower in the winter must be looked for in the temperature gradient. The mean temperature gradient up to 300 metres is considerably less than the adiabatic gradient in the winter, and the number of occasions when the adiabatic gradient is reached is comparatively small. A gradient less than the adiabatic has a tendency to prevent the spontaneous formation of turbulence, and to suppress it when formed by any outside agencies, such as obstacles on the ground

In the summer the mean temperature gradient in the first 300 metres is much more nearly adiabatic, and the number of occasions when it reaches the adiabatic gradient is large. A gradient equal to the adiabatic gradient has a tendency to encourage the spontaneous formation of turbulence Under these circumstances, an increase in the value of K with height is to be expected, because K is roughly proportional to the vertical component of turbulent velocity and to the diameters of the eddies It is to be expected that the eddies will increase in size as the height above the ground increases, because they have more room to grow There will, therefore, be a tendency for K to increase with height

Looking at the first and third columns in Table II, it will be seen that the mean values of K between 18 and 302 metres and between 197 and 302 metres both have a maximum in the summer and a minimum in winter, but the variation is much greater in the latter case than in the former It is a matter of some interest, therefore, to find out whether the value of K near the ground shows any marked monthly variation In column 4, Table II, is given the result of applying equations (3) and (4) to the daily ranges of temperature at 18 and 123 metres, but it must be remembered that the errors due to the method used in deducing K are greater near the ground, where the most rapid variations are likely to occur, than they are higher up Accuracy in the figures of column 4 is therefore not to be expected.

It will be seen that the values of K near the ground appear to vary in a haphazard manner, but that they show no marked monthly variation of the type exhibited by the figures in all the other columns. It seems, therefore, that the effect of the mean temperature gradient, which reduces the turbulence at 300 metres in the winter and allows it to increase in the summer, does not have any marked effect on the mean amount of turbulence near the ground This must be governed almost entirely by wind velocity, which shows no marked monthly variation, and by the nature of the ground

On the other hand, it will be seen later that the daily variation in temperature gradient near the ground has a very marked effect on the turbulence near the ground

Identity of K *found from Temperature Measurements with* K *found from Wind Measurements.*

On looking at column 1, Table II, it will be seen that the mean value of K for the turbulence over Paris, as calculated from the Eiffel Tower temperature measurements, is about 10×10^4. It was pointed out on p 138 that the mean value of K for the turbulence over Salisbury Plain, as calculated from wind-velocity measurements, is about 5×10^4. It is to be expected, from the nature of the ground, that the turbulence over Paris would be greater than the turbulence over Salisbury Plain.

Quite recently, since finishing the work described above, I have seen a paper by Dr. F Akerblom,* in which he uses the change in wind direction between the top and the bottom of the Eiffel Tower to find the coefficient of viscosity of the atmosphere due to turbulence This quantity, which is, of course, equal to K/ρ, was found to be about 85 C G S units in the winter and 115 in the summer Åkerblom finds, therefore, that the mean value of K in the height of the Eiffel Tower is greater in the summer than in the winter, but the difference is considerably less than that indicated in the first column of Table II The mean value obtained for K/ρ is given by Dr Åkerblom as 95 C G S units Taking ρ as 0 00125, the value of K over Paris, calculated from wind-velocity measurements, is therefore 95/0·00125, or $7\ 6 \times 10^4$ C G S units

The agreement between this and the value of 10×10^4 obtained above from temperature measurement is quite as good as could possibly be expected, when it is remembered that neither Akerblom's nor the present writer's equations are rigidly applicable under the conditions which actually occur in the atmosphere It is sufficiently good at any rate, to provide a satisfactory confirmation of the theory that momentum and heat are transmitted by the same agency, and that the behaviour of the lower atmosphere in regard to heat transmission can be calculated from observations of the retardation of the lower layers of the earth's atmosphere by the friction of the ground.

Daily Variation in Wind Velocity

We have seen how a study of the change in wind velocity with height and of the diurnal variation in temperature leads to a knowledge of the amount of turbulence in the atmosphere near the ground We have seen, also, how much the turbulence is reduced in the winter by the smaller average temperature gradient which is characteristic of that part of the year. So far, however, no mention has been made of what is by far the most noticeable feature of the meteorological effects of the turbulence of the lower atmosphere, namely, the daily variation of wind velocity The remainder of the paper will be devoted to discussing this variation It will be shown that the knowledge which had been gained concerning turbulence is sufficient to explain all the facts which recent observations have brought to light concerning the daily variation in wind velocity at various heights above the ground and at various seasons of the year.

It is well known that the wind near the ground is usually less strong at night than it is in the day time. Observations taken on mountains and on high buildings show that at comparatively small heights above the earth this variation in wind velocity is much less than it is quite close to the ground The Eiffel Tower observations show that at a height of 1000 feet the daily variation in wind velocity is reversed, so that it has a maximum at night and a minimum during the day

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* F. Åkerblom, 'Nov Act Soc. Scient.,' Upsala, 1908

Quite recently the diurnal variation in wind velocity has been made the subject of an elaborate series of observations by Dr Hellmann * A brief description of his results will form the best introduction to the more theoretical discussion which will follow later

In a piece of flat meadow-land Dr Hellmann set up three anemometers at heights of 2, 16, and 32 metres above the ground, and the results of one year's observations are dealt with in his paper

During periods when strong winds were blowing, all three anemometers showed a maximum velocity in the middle of the day and a minimum during the night. The daily variation in light winds was of a different character At 2 metres there was a maximum in the day and a minimum at night At 16 and 32 metres, however, there were maxima in the middle of the day, and also in the middle of the night, with minima during the morning and afternoon At 16 metres these two maxima were about equal, while at 32 metres the night maximum was greater than the day maximum

On examining the seasonal effect, it was found that there was a distinct tendency in summer for the day maximum at 16 metres to be greater than the night maximum, while in the winter the two maxima were about equal

These results are shown graphically by means of curves in figs 2 and 3 It will be seen that in light winds the height at which the day and night maxima are equal is about 16 metres in winter and between 16 and 32 metres in summer At greater heights the wind is greatest in the middle of the night, while at smaller heights it is greatest in the middle of the day This change in the type of the daily variation in wind velocity will be referred to as a "reversal"

Dr Hellmann's results are confirmed by his analysis of the daily variation in wind at the observatory at Potsdam, where the anemometer is placed at a height of 41 metres above the ground In fig 4 is shown the variation in wind velocity for light and for strong winds in the summer and in the winter It will be seen that for strong winds the maximum occurs in the middle of the day, at all times of the year In the case of light winds there

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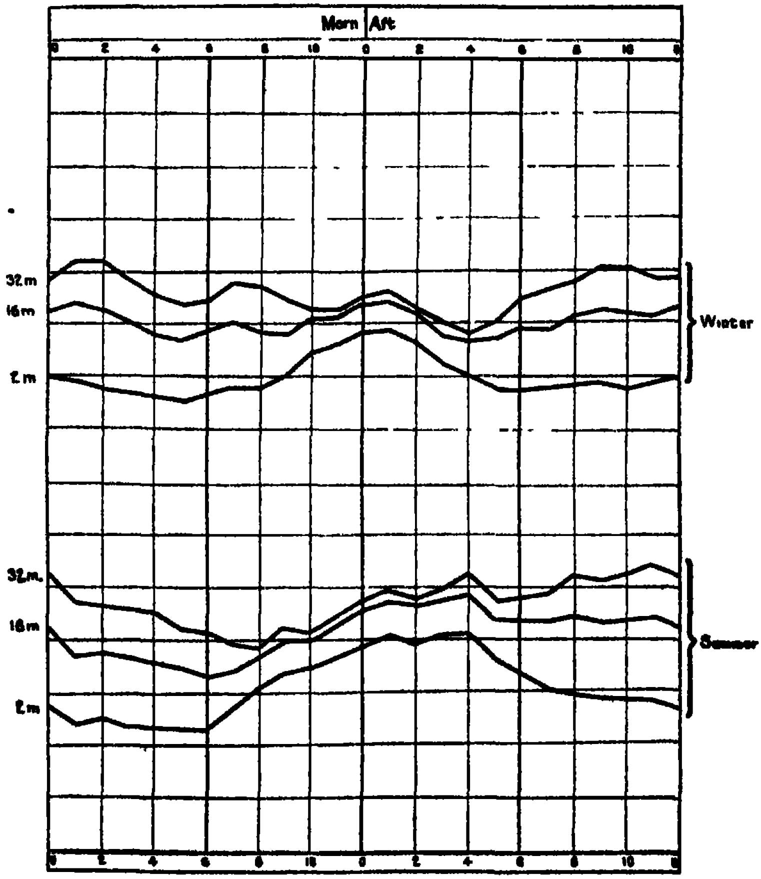
* "Über die Bewegung der Luft in den untersten Schichten der Atmosphäre," 'Meteorologische Zeitschrift,' January, 1915

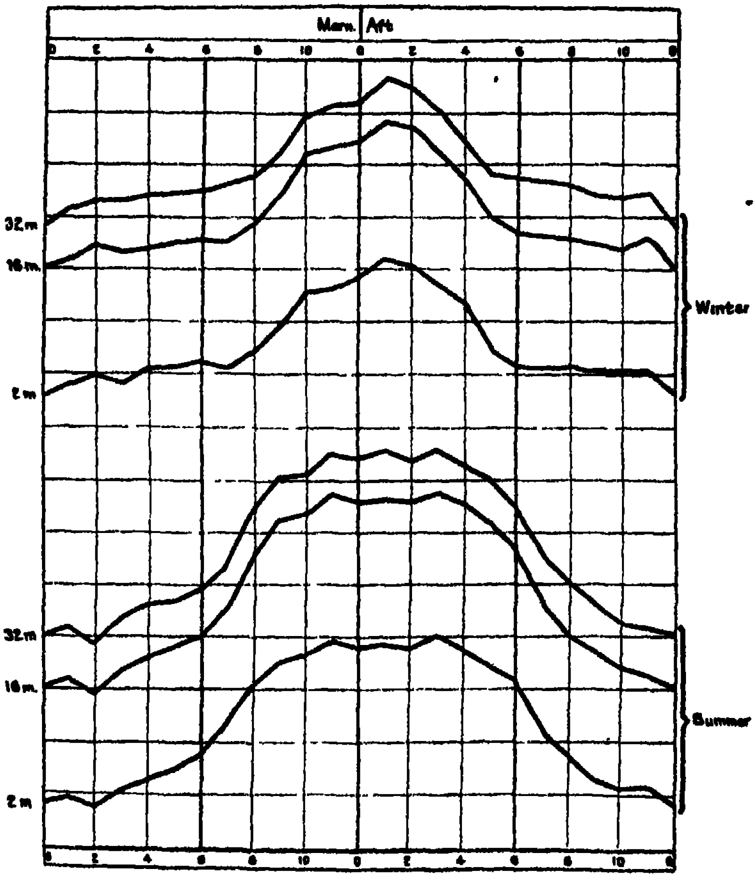
is a minimum in the middle of the day in the winter, while in the summer there is a small maximum in the middle of the day, and a much larger maximum in the middle of the night It appears, therefore, that the height at which the reversal in type of daily variation occurs is greater than 41 metres for strong winds, but that for light winds it is less than 41 metres. During light winds in winter the height of the reversal is so much less than 41 metres that the day maximum has altogether disappeared at that height In the summer, however, the height of the reversal is evidently much nearer to 41 metres, for the day maximum is still quite well developed at that height These results agree well with Dr. Hellmann's observations.



Daily Variation of Wind Velocity with Light Winds

FIG 2.

To account for the facts brought to light by these observations, Dr Hellmann mentions the theory of Espy and Koppen, according to which both types of daily variation are the results of ascending and descending currents produced by the heating of the earth in the day time The descending currents carry the upper wind down to the ground, thus increasing the surface wind, while the ascending currents carry the more stagnant air at the surface up into the higher layers, an action which reduces the velocity in those regions

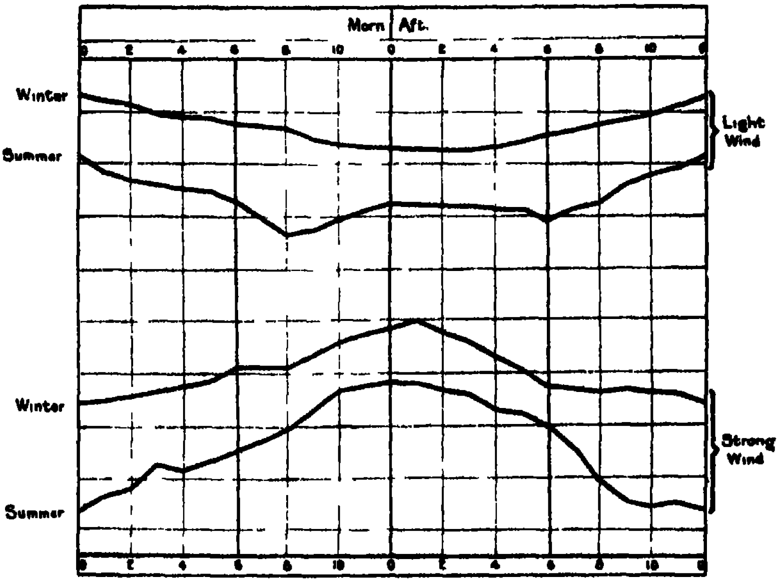


Daily Variation of Wind Velocity with Strong Winds

FIG 3

This theory appears to contain some elements of truth, but in several ways it is very unsatisfactory. In the first place it is purely qualitative Presumably, the height at which the reversal takes place should, according to this theory, be some definite fraction of the height to which the vertical currents produced by the heating of the ground extend In order, therefore, that the theory may account for Dr. Hellmann's observations, it is necessary to assume that the vertical currents, due to heating of the ground, extend to a much greater height in strong winds than they do in light winds. This is contrary to the commonly accepted idea of vertical heat currents, and is almost certainly untrue.

On examining the Espy-Koppen theory closely, it will be found that it not only fails to give quantitative expression to the phenomena of daily variation in wind velocity, but that it involves implicitly several special assumptions regarding the nature of the vertical currents which convey horizontal momentum from the upper layers to the lower layers of the atmosphere, and from these to the ground. Moreover, there is no other evidence that these special assumptions are true. We shall now enquire into the nature of these special assumptions.



Daily Variation of Wind Velocity in Potsdam with Light and Strong Winds

FIG. 4.

The vertical currents in the atmosphere promote an interchange of momentum between the different layers. The action of an increase in the amount of vertical currents in the lower atmosphere is to increase the amount of rapidly moving air brought down from the upper layers to the layers close to the ground, but at the same time it increases the amount of slowly moving or stationary air brought up from the surface of the ground. The velocity of the lowest stratum of air is determined by the balancing of these two effects, and in order that an increase in the velocity of the wind close to the ground may result from an increase in the amount of the vertical currents, it is necessary to assume that the structure of the vertical currents is such that the increase in velocity due to the extra influx of momentum from above is greater than the decrease due to the extra outflow of momentum into the ground It has been shown by the present writer in a previous paper* that, with a given amount of eddy motion and a given pressure gradient, there is a certain distribution of wind velocity near the ground, which exists as a steady state when the system has been established long enough for the effect of the initial conditions to have died away In the Espy-Koppen theory no account is taken of the steady state In the present paper, on the other hand, it will be assumed that the distribution of wind velocity in the lower atmosphere at any time is the distribution which would result from the gradient velocity and turbulence which exist at that time if the steady state had been reached This is equivalent to assuming that the lag of the variation in wind velocity behind the variation in turbulence which gives rise to it is small That this is sufficiently near the truth for our purpose is shown by the fact that the maxima in wind velocity occur about the middle of the day, at a time when the turbulence might be expected to be at its maximum

It will be shown that a daily variation in turbulence, by an amount which might have been expected from our previous knowledge of the subject, is sufficient to explain qualitatively—and also, in a rough way, quantitatively—all the known characters of the daily variation in wind velocity

We have seen that the effect of turbulence on the distribution of velocity in the lower atmosphere depends on a certain quantity K, which can be regarded as a number which expresses the amount of turbulence It has been shown also that the mean distribution of wind velocity in the lower layer of the atmosphere is very nearly the same as that in an ideal atmosphere in which K is constant at all heights We shall therefore find how the wind velocity at various heights would vary in the ideal case when its pressure gradient remains constant and K is constant at all heights but has a daily variation in magnitude.

For this purpose it is necessary to take account of the fact that the frictional force of the wind over the ground is proportional to the square of the wind velocity † The magnitude of the frictional force has been calculated for Salisbury Plain. It was shown that over the grassy land of which Salisbury Plain is composed, the frictional force, F, is 0·0023 ρQ_0^2, where Q_s is the wind velocity close to the ground, and ρ is the density of the air. Using this value for F it may be shown that

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* "Eddy Motion in the Atmosphere," 'Phil. Trans,' A, vol 215 (1915)

† 'Roy Soc. Proc.,' A, vol 92, p 196 (1916)

$$\frac{F}{\rho} = 0.0023\, Q_s^2 = 2KQ_G \sin\alpha/B, \qquad (5)$$

where α is the angle between the wind at the ground level and the gradient direction, Q_G is the gradient velocity, K is the coefficient of eddy diffusion already referred to, and B is equal to $\sqrt{(\omega \sin\lambda/K)}$,* where ω is the angular velocity of rotation of the earth, which is numerically equal to 0.000073, and λ is the latitude of the place in question, which in the case of Salisbury Plain is 50° N, so that $\sin\lambda = 0.77$. Equation (5) may be transformed into the form

$$\frac{1}{BQ_G} = \frac{1}{2\sin\alpha}\left(\frac{Q_s}{Q_G}\right)^2\left(\frac{0.0023}{0.000073 \times 0.77}\right), \qquad \text{but}\dagger\ Q_s/Q_G = \cos\alpha - \sin\alpha$$

Hence
$$\frac{1}{BQ_G} = \frac{20.4}{\sin\alpha}(\cos\alpha - \sin\alpha)^2$$

The values of $1/BQ_G$ may therefore be tabulated for a series of values of α. These are given in Table III.

Since B depends only on K, it might be simpler to consider the variations in wind velocity at various heights when B rather than K undergoes a daily variation; but since the numerical value of K has been measured in several cases already, it is convenient to have a Table showing the relationship between K and α. The quantity K/Q_G^2 is a function of α, and its values are given in the last column of Table III.

Table III may now be used in conjunction with the equations given on pp. 15 and 16 of "Eddy Motion in the Atmosphere," to find the wind velocity at any height for a given gradient velocity and a given value of K. The numerical work is laborious, but the results can be shown simply by means of a series of curves. These curves are shown in fig. 5. The abscissæ represent the wind velocity as fractions of the gradient, while the ordinates represent the quantity z/Q_G, where z represents the height above the ground. If the curves be regarded as giving the variation in wind velocity with height, the scale of the ordinates will depend on the gradient velocity. Two scales of height in metres have been drawn on the right-hand side of the figure, corresponding with the two gradient velocities of 4·6 and 15·6 metres per second, which correspond with the classes "light" and "strong" winds for which mean values of K have been calculated.‡

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* "Eddy Motion in the Atmosphere," *loc. cit.*, p. 15.

† *Ibid.*, p. 16.

‡ 'Roy. Soc. Proc.,' A, vol. 92, p. 196 (1916).

Each of the curves in fig 5 represents the distribution of wind velocity at various heights with a given value of K and Q_G. When K undergoes a daily variation, the daily variation in wind velocity is given by the variation in the abscissæ of the different curves for a fixed ordinate

Table III

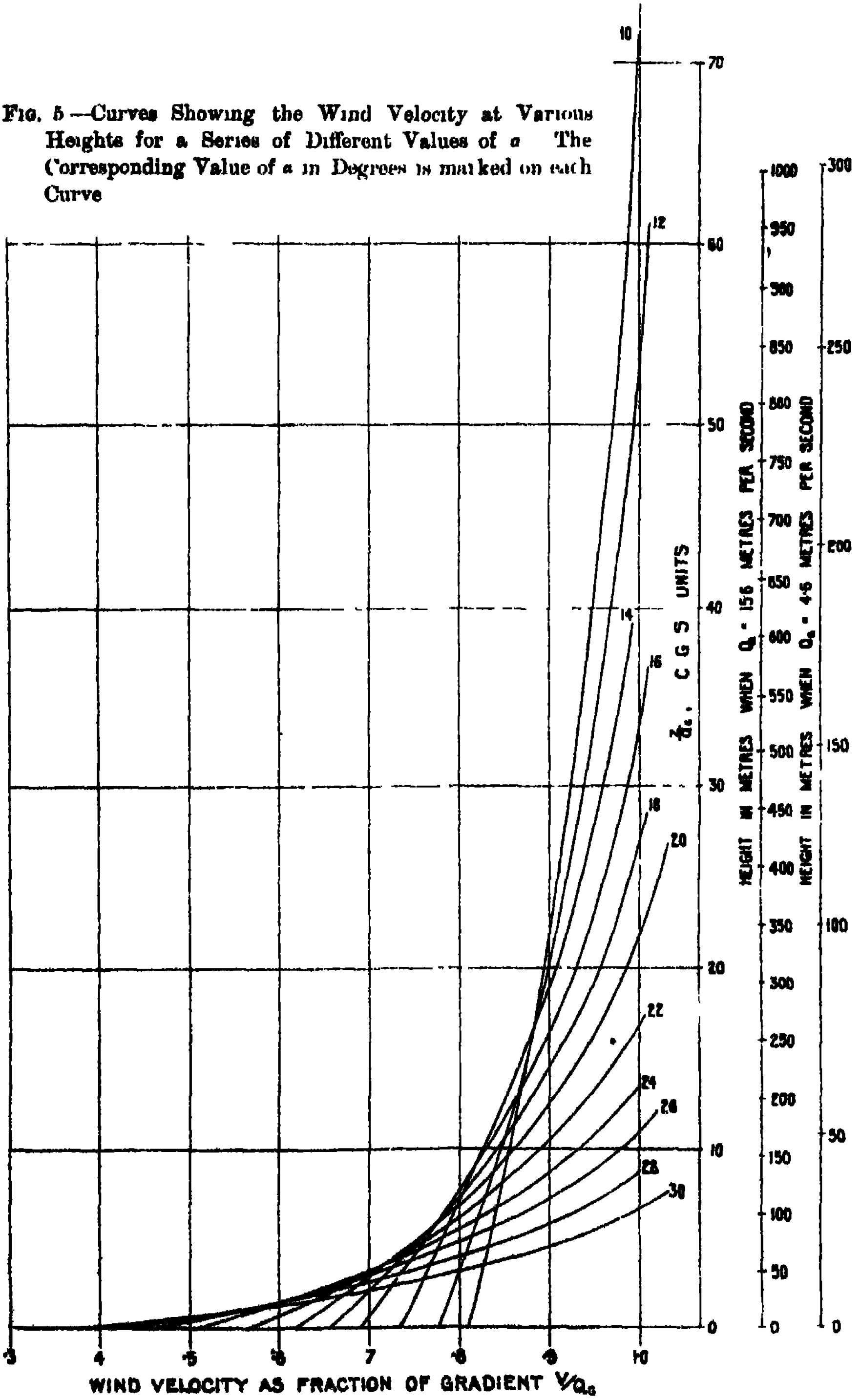
α	$1/DQ_G$	K/Q_G^2
Degrees	C G S units	C G S units
4	252	3·54
6	155	1·35
8	106	0·685
10	77·5	0·338
12	58·5	0·192
14	44·8	0·116
16	34·9	0·069
18	27·3	0·042
20	21·9	0·027
22	16·7	0·0156
24	12·9	0·0094
26	9·9	0·0055
28	7·4	0·0031
30	5·5	0·0017
32	3·7	0·00085
34	2·6	0·00038
36	1·7	0·00016

In order to simplify the application of the curves in fig 5 another set of curves, shown in fig. 6, has been constructed from them These represent the variation in wind velocity at a given height when α varies owing to the variations in K Each curve represents the variation in value of V/Q_G for a given value of z/Q_G when α varies from 6° to 30°.

An inspection of the curves of fig 6 shows that the daily variation in K will account for the character of the daily variation in wind at different heights Suppose, for instance, that the value of K varies in such a way that $\alpha = 10°$ at midday and 30° at midnight. At heights above that at which $z/Q_G = 15$ the wind will be at a maximum at midnight and at a minimum at midday At heights for which z/Q_G is equal to or less than 1 there is a maximum at midday and a minimum at midnight At all intermediate heights there are two maxima, one at midday and the other at midnight. When $z/Q_G = 4$ the maxima at midday and at midnight are about equal for the particular range of K we have chosen

The several characteristics of the daily variation in wind velocity which this theory indicates are shown in fig 7. In this figure the value of K has been assumed to vary in a continuous manner so that it is a maximum at midday and a minimum at midnight Each curve represents the variation

Fig. 5.—Curves Showing the Wind Velocity at Various Heights for a Series of Different Values of α. The Corresponding Value of α in Degrees is marked on each Curve.



in wind velocity at a given height, represented, on an arbitrary scale, by the figure attached to the curve

It will be seen that this agrees remarkably well with the variation which was actually observed by Dr Hellmann. In the case of light winds in winter, for instance, the wind velocity at 2 metres was a maximum at midday and remained practically constant from 6 P M to 6 A M The wind at 16 metres had equal maxima at 1 A M. and 1 P.M The wind at 32 metres had a maximum at about 1 A M and a smaller maximum about 1 P M It also had a maximum at about 7·A M, but this appears to be an irregularity due to the fact that Dr Hellmann's observations only extend over one year The curves showing the daily variation of wind velocity at Potsdam, which are based on five years' observations, certainly do not possess the corresponding maximum.

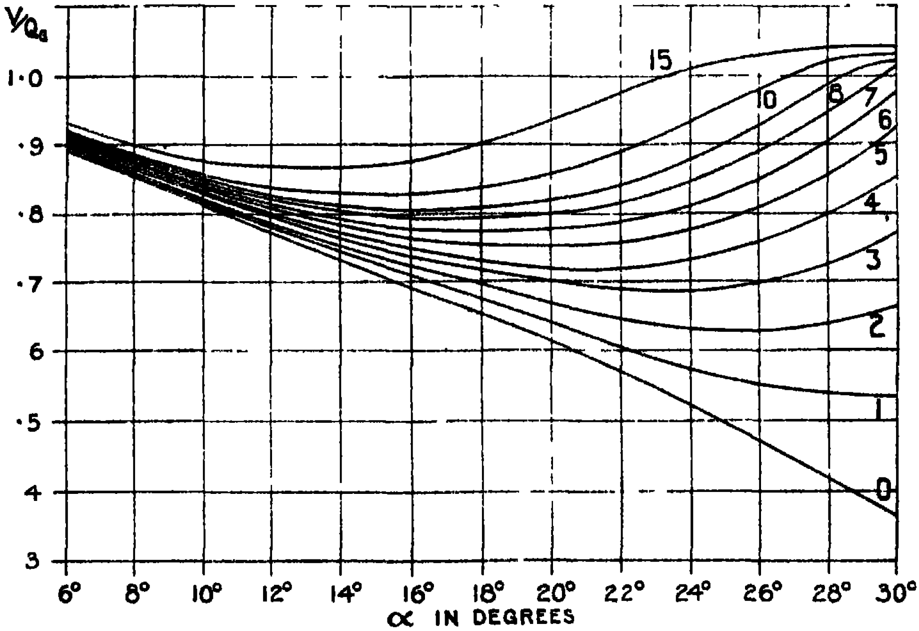


Fig 6.—Curves Showing the Variation in Wind Velocity at Various Heights above the Ground when the Gradient Velocity remains Fixed and α Varies owing to Variations in K Each Curve Represents the Variations at a Fixed Height, and the value of z/Q_G which corresponds with that is marked on the Curve in Question

It is interesting to see how far the information which we have obtained in various ways concerning the value of K may be used to set numerical limits to the height to which the ground type of daily variation in wind velocity is likely to extend.

It has been shown* that the mean value of K in the summer in the height of the Eiffel Tower is about 15×10^4, while in the winter it is about 5×10^4 These include strong as well as light winds, and night as well as day. Since the value of K must be very low on all clear nights it seems probable that one might expect the mean value of K at midday to be at least as great as 30×10^4 It will be higher for strong winds and lower for light winds, and in the absence of any further data we might guess the values 40×10^4 for strong winds and 20×10^4 for light winds For the winter the values of K may be taken as one-third of the summer values, because the mean value of K in winter is one-third of the mean value in summer. Hence they may be taken 13×10^4 and 7×10^4.

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* See Table II

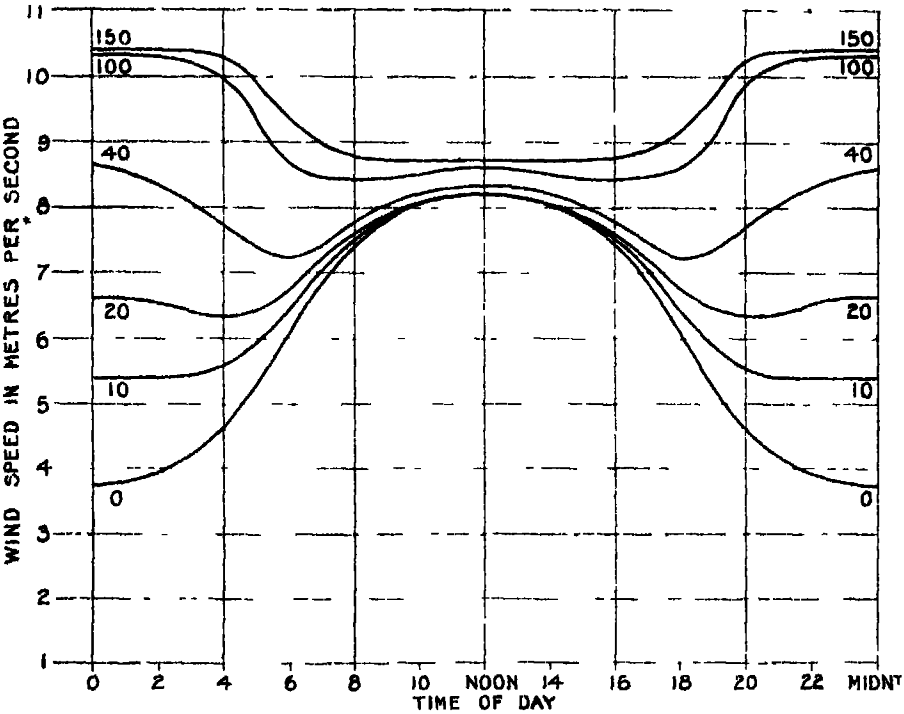

FIG 7

The value of K at night is governed by the fact that the temperature gradient is less than the adiabatic gradient, in fact it is frequently reversed near the ground Under these conditions the stability of the air tends to prevent the formation of turbulence So far the value of K has not been determined on land under circumstances in which the temperature gradient is known. At sea, however, the value of K, under circumstances when there was an inversion of temperature, was found to be of the order 3×10^3 for moderate winds and 10^3 for light winds. In strong winds at sea, therefore, we might expect K to be of the order 6×10^3 when there is an inversion of temperature gradient On land there is likely to be a still greater difference between the values of K for light and for strong winds at night because the air temperature at the ground does not go down so much when the wind is strong as when it is light, whereas the temperature of the air near the sea is practically the same as that of the sea itself, whatever the wind velocity may be

Taking the values of K given above and taking the mean gradient velocities in light and strong winds as 4 6 and 13·6 metres per second respectively, it will be found from Table III that the corresponding values of α are those set out in Table IV

From the curves in fig 6, the value of z/Q_G at which the maxima at midday and midnight are equal may be found In the case of strong winds in summer, for instance, the variation of α from 12° to 28° gives equal maxima of wind velocity, equal to 0 83 Q_G when $z/Q_G = 4\frac{1}{2}$. This corresponds with a height of 60 metres At heights lower than this the greatest wind will be at midday, while for greater heights the maximum will be at midnight

Table IV

		α at midday	α at midnight	z/Q_G	Height at which maxima at midday and midnight are equal
		°	°		Metres
Strong winds	Summer	12	28	4	60
	Winter	17		3	50
Light winds	Summer	7	26	7	30
	Winter	10		5¼	25

The heights at which the reversal in the character of the daily range occurs are given in the last column. It appears, therefore, that for light winds the reversal might be expected at heights between 25 and 30 metres. In strong winds, however, the type of daily variation in which the wind velocity is a maximum at midday would extend up to 50 or 60 metres. If, as was anticipated above, the value of K at midnight is considerably greater in strong winds than we have taken it to be, then the height of the reversal of the type of daily range would be much higher than 50 or 60 metres.

On referring to the curves of figs 2, 3, and 4, it will be seen that these theoretical conclusions agree with Dr. Hellmann's observations. He found the reversal to occur in light winds at about 16 metres in the winter and about 32 metres in the summer In the case of strong winds there was no reversal in the first 32 metres.

In the case of Potsdam observations, it was found that the height of the reversal was less than 41 metres for light winds and greater for strong winds In the case of light winds in summer, there is a maximum at midday at Potsdam, but it is smaller than the maximum which occurs at midnight

It appears, as a result of these calculations, that a variation in K by an amount which fits in well with all the other known data concerning the turbulent motion of the air near the ground is sufficient to explain, both qualitatively and quantitatively, all the facts concerning the daily variation of wind velocity at different heights above the ground which are brought to light by Dr Hellmann's observations

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The Discharge of Gases under High Pressures

By L Hartshorn, B Sc, A R C S

(Communicated by the Hon R J. Strutt, F R S Received September 28, 1917)

Introduction.

This work was undertaken, following up a suggestion made by Lord Rayleigh.[*] De St Venant and Wantzel in 1839 stated, as a result of their experiments, that when gas was discharged through an orifice from a vessel in which the pressure was p_0, into one in which the pressure was p_1, then the rate of discharge was sensibly constant from $p_1 = 0$ upwards to $p_1 = 0\cdot3p_0$ or $0\cdot4p_0$, but then, as p_1 further increases, the discharge falls off, slowly at first, afterwards with great rapidity.

In 1885 Osborne Reynolds quoted some experiments which seemed to show that the flow remained constant from $p_1 = 0\cdot50p_0$ or $0\cdot53p_0$ He explained this by pointing out that the maximum "reduced velocity" occurs when the actual velocity coincides with that of sound under the conditions then prevailing, as then the effect of a further reduction of pressure in the recipient vessel cannot be propagated backwards against the stream. If $\gamma = 1\cdot408$ this argument suggests that for a nozzle ending abruptly at the narrowest part, the discharge reaches a maximum when the pressure in the recipient vessel falls to 0·527p_0 and then remains constant. Lord Rayleigh pointed out the importance of examining with more accuracy than has hitherto been obtained the extent to which this constancy of flow for different values of p_1 is true.

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[*] 'Phil. Mag,' vol 32, p. 178 (1916)

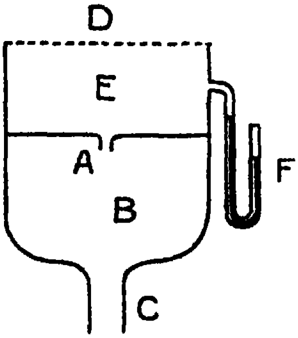

Fig 1

He suggested using for this purpose the arrangement indicated diagrammatically in fig 1. A is the nozzle, B the recipient vessel, which is partially exhausted. The passage D from the external air to the discharging vessel E is somewhat choked and thus the reading of the pressure gauge F is a measure of the discharge. The discharge is absorbed by a powerful air-pump, communicating with B by the passage C. If the flow remain constant in spite of variations in the pressure in B, then the reading of F will remain constant.

Details of Apparatus

The details of the actual apparatus used are shown in fig 2. The nozzle A was fitted into a brass tube a little more than a centimetre in diameter. The receiving vessel B and the discharging vessel E were made out of glass tubing 4 cm in diameter. The brass tube holding the nozzle was fitted into these by means of rubber corks. The pressure in B was measured by means of the straight manometer tube H and the barometer tube K. The difference

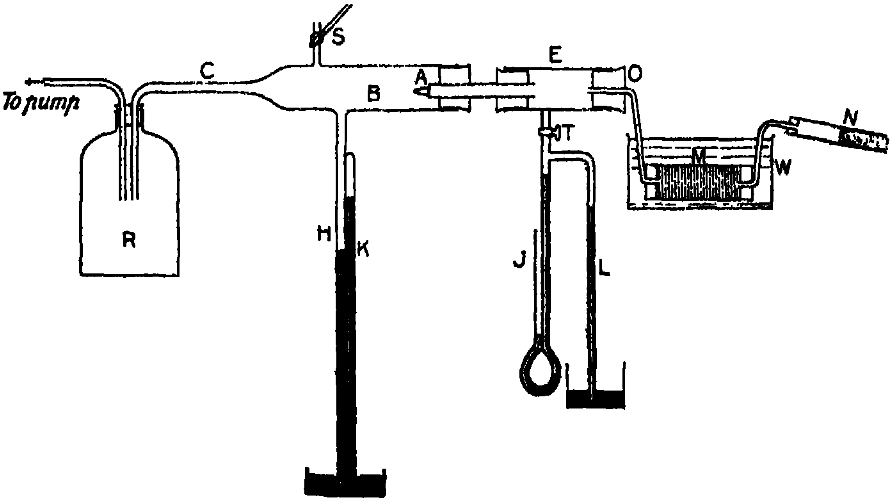

Fig. 2

between the pressure in E and the atmospheric pressure was measured by means of the oil-gauge J, which could be cut off by means of the tap T The small variations in this pressure were measured by observing the meniscus of the oil in the tube L The lower end of this tube dipped into a wide vessel of oil, so that for small variations the whole change was shown by the change in level of oil in the tube These changes were measured by means of a microscope having a scale in the eyepiece

M is a cylinder of thin brass, 23 cm long and of 5 cm diameter It was tightly packed with discs of copper gauze, which completely filled the tube It was placed in a large bath of water W at the temperature of the laboratory, and was connected to the entrance of the discharging vessel by glass tubing fitting into rubber corks. The other end of this cylinder communicated with the glass tube N, which contained a plug of cotton-wool Thus the air passed first up the tube N, then through the gauze in M into the discharging vessel E It then passed through the nozzle into the receiving vessel B, and then to the pump through the large bottle R which served to steady the pressure in B The choking at the entrance to E was supplied partly by the gauze in M and partly by the cotton wool in N The latter also served to filter out the dust The pump used was a Lennox vacuum pump driven by a 2 H P motor

Preliminary Experiments

In the first experiments M and N were not used and the choking was supplied by a piece of capillary tubing fitting into the cork O The nozzle in these experiments consisted of a thin brass plate with a small hole in it This plate was soldered on to the end of the brass tube Under these conditions accurate measurement was found to be impossible owing to the fact that the level of the oil in L was not steady even when the pressure in B was constant The changes were of two kinds · (1) a slow regular fall of the meniscus, (2) rapid and irregular fluctuations of the level The slow change was found to be due to the cooling effect at the nozzle. When the apparatus was left running for some time, this slow change gradually got slower and slower, and finally the mean position of the meniscus became constant, although the irregular fluctuations still persisted When the brass nozzle was replaced by one drawn down from silica tubing, which has a very small coefficient of expansion, this slow change did not occur The irregular fluctuations were, however, just as bad as they were with the brass nozzle It was found that slightly warming the air entering E caused quite a considerable change of level in L, and this led to the suspicion that the irregular fluctuations were due to the non-uniformity of the air entering the apparatus.

The most probable source of any such temperature variation was the pump, which, although water-cooled, became warm after working for some time. Accordingly steps were taken to draw the air from the part of the room most remote from the pump This was the purpose of tube N, which was of considerable length A great improvement in the steadiness of the oil-gauge reading was obtained in this way Additional improvement was obtained by the use of the cylinder of gauze M immersed in a large bath of water at the temperature of the laboratory This served to mix the air thoroughly and to keep it at the room temperature It should be mentioned that the oil fumes from the pump discharge were led into a fume cupboard, as if by any chance they contaminated the air drawn through the apparatus, the result was condensation of oil in the nozzle The air used was not specially dried

Method of Performing an Experiment

Owing to the difficulty of making silica nozzles to an exactly predetermined shape, it was decided to use invar The sample obtained had a coefficient of expansion larger than that of silica, so that the slow change in the oil-gauge reading, due to alteration in the size of the nozzle with temperature, was quite appreciable Further there were also slight changes due to expansion or contraction of the manometer oil with slow changes in the room temperature Any error due to this was avoided by proceeding as follows The pump was set working so as to exhaust B as far as possible The amount of choking was then adjusted, by varying the amount of cotton wool in N, until the reading of L reached a convenient value The pressure in B was then slightly increased by opening the tap S, through which B communicated with the atmosphere .When the pressure in B and E had become steady, the tap S was closed, thus restoring the original low pressure in B The level of the oil in L was continually watched by means of the microscope and any change noted This was now repeated for a still higher pressure in B, and so on In each case the pressure in B was restored to its original value, so that in each case the total change was measured, and as the change only occupied a few seconds, there could be no error due to slow change of level in L with temperature Readings were taken until the change in level was too large to be measured on the microscope scale

The Nozzles

The nozzles were all made out of a rod of invar, 8 mm in diameter They were soldered into a brass collar, which in turn was soldered into a length of brass tube Thus it was quite easy to change from one nozzle to another.

The various shapes used are shown in fig 3. No. 1 consists of a parallel hole of 0 8 mm. diameter and 7 mm long No 2 is a parallel hole of about the same diameter, but only about 1 mm long No 3 is really a hole in a thin plate, the edges being almost sharp The remainder are

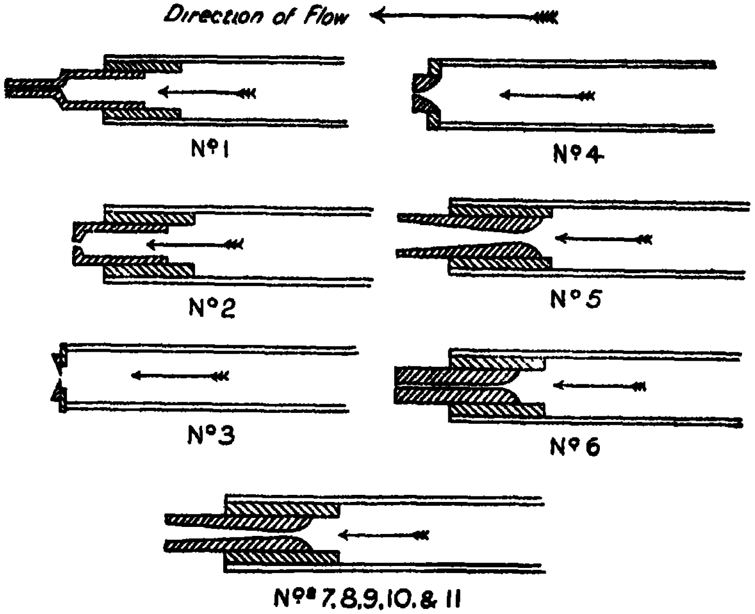

Fig 3

all opened out similarly at the entrance end No 4 is cut off at the narrowest part To make this a parallel hole was first bored, then one end opened out, and the other end ground down until the hole just began to get larger, the measurement of its diameter being made with the microscope Nos 5, 7, 8, 9, 10 and 11 end in divergent cones of angles 16 6°, 7·8°, 4·6°, 4 4°, 2 5° and 4 4° respectively. The length of the conical portion is about 2 cm in each case No 6 ends in a parallel hole

All the nozzles were tested to make sure of their being air-tight at the joints by closing the hole with the finger and blowing into the nozzle with the bellows, the end of the nozzle being under water

The minimum diameter was in each case measured by means of the cathetometer microscope, using the eyepiece scale. In the case of nozzles ending in divergent cones the diameter at the base of the cone was measured in the same way, and the distance between these two diameters was obtained by measuring the distance through which it was necessary to advance the microscope in order to change the focus from one to the other The angle of the cone was deduced from these measurements The values are shown in Table I

Table I

Nozzle	Diameter at narrowest part	Angle of cone
	mm	°
1	0 8	—
2	0 8	—
3	0 5	—
4	0 95	—
5	1 1	16 6
6	1 0	0
7	1 2	7 8
8	1 0	4 4
9	1 0	2 5
10	1 0	0·7
11	2 4	4 4

It has been mentioned that the chief source of trouble in the experiment was the unsteadiness of the reading of the oil-gauge, the irregular variations in particular Further, we have seen that these were largely due to variations in the temperature of the air flowing through the nozzle They were not entirely due to this, however, for the readings were much steadier with some nozzles than with others, other conditions being exactly the same In no case were the readings perfectly steady Nozzle No 3 gave the best results, the variations in this case being very minute. No 4 gave the least steady readings The others were about equally good

The scale in the eyepiece of the microscope had 50 divisions, and 30 of these corresponded to 0 1 inch (or 11 8 to 1 cm) The pressure registered by the oil-gauge was about 17 inches (or 43 cm) of oil in each case, so that one division on the microscope scale corresponded to about 0 02 per cent. of the whole pressure Owing to the irregular variations mentioned above, it was not always possible to read the variations correct to less than one division on the microscope scale, though in nearly all cases it was possible to detect a total variation of half a division, and in many cases of less than this

Results.

The results are shown in Table II, where p_0 is the pressure in the discharging vessel It is equal to the atmospheric pressure diminished by the pressure indicated by the oil-gauge It is considered constant, as the variations in the oil-gauge reading are much too small to need taking account of in obtaining the ratio p_1/p_0, p_1 is the pressure in the receiving vessel, and is given by the difference of the readings of the mercury columns H and K (fig 2) These readings were only taken correct to the nearest millimetre, further accuracy being unnecessary owing to the errors involved in making the other measurements The readings on the microscope scale, shown in column 7, give the variations of the pressure p_2 indicated by the oil-gauge, and thus show the variations in the discharge of gas

Table II

Nozzle	Barometer reading	Oil gauge reading	p_0	p_1	p_1/p_0	Microscope reading	Corrected microscope reading	p_2 (max = 5000)
	cm Hg		cm Hg	cm Hg		scale divisions	scale divisions	scale divisions
1	75·3	18·85	72·8	5·1	0·071	0	0	5000
		ins of oil		15·0	0·208	0	0	5000
				19·1	0·264	0·3	0·3	4999·7
		3·0		23·5	0·325	1	1	4999
		cm Hg		25·2	0·348	1·7	1·5	4998·5
				27·3	0·378	3·5	3	4997
		5650		30·1	0·417	10	9	4991
		mic div		32·4	0·448	25	22	4978
		of oil		33·8	0·468	38	34	4966
2	76·3	17·7	73·5	5·6	0·076	0	0	5000
		ins of oil		12·6	0·171	0	0	5000
				15·6	0·212	0·3	0·3	4999·7
		2·8		19·0	0·258	2	2	4998
		cm Hg		22·0	0·300	4	4	4996
				24·9	0·339	8	8	4992
		5310		28·0	0·381	16	15	4985
		mic div		30·1	0·410	26	25	4975
		of oil		31·1	0·423	35	33	4967
				32·3	0·440	48	45	4955
3	76·2	17·4	73·4	3·0	0·041	0	0	5000
		ins of oil		12·9	0·176	0	0	5000
				15·4	0·210	0·2	0·2	4999·8
		2·8		17·6	0·240	0·5	0·5	4999·5
		cm Hg		20·0	0·272	1	1	4999
				22·3	0·304	2·5	2·5	4997·5
		5220		25·7	0·350	6	6	4994
		mic div.		28·1	0·383	11	11	4989
		of oil		30·8	0·420	23	22	4978
				32·3	0·440	34	33	4967
				33·6	0·458	48	46	4954
4	76·7	17·7	73·9	9·0	0·123	0	0	5000
		ins of oil		13·0	0·176	0	0	5000
				17·4	0·236	1	1	4999
		2·8		19·5	0·264	2	2	4998
		cm Hg		22·5	0·304	4	4	4996
				25·5	0·345	7	7	4993
		5310		28·5	0·386	13	12	4988
		mic div.		30·5	0·413	22	21	4979
		of oil		32·7	0·443	37	35	4965
				33·7	0·465	47	44	4956
5	76·4	17·7	73·6	11·2	0·152	0	0	5000
		ins of oil		28·2	0·386	0	0	5000
				30·9	0·420	0·3	0·3	4999·7
		2·8		33·3	0·452	1	1	4999
		cm Hg		36·0	0·490	4	4	4996
				39·1	0·531	10	9	4991
		5310		41·6	0·565	23	22	4978
		mic. div.		43·1	0·586	33	31	4969
		of oil		44·0	0·598	43	40	4960

Table II—*continued.*

Nozzle	Barometer reading	Oil gauge reading	p_0	p_1	p_1/p_0	Microscope reading	Corrected microscope reading	p_2 (max = 5000)
	cm Hg		cm Hg	cm Hg		scale divisions	scale divisions.	scale divisions
6	77·0	17·1 ins of oil 2·8 cm Hg 5130 mic div of oil	74·2	8·5	0·115	0	0	5000
				13·3	0·179	0	0	5000
				16·2	0·218	0·5	0·5	4999·5
				19·1	0·258	1·5	1·5	4998·5
				21·7	0·292	3·5	3·5	4996·5
				24·4	0·329	7	7	4993
				27·4	0·369	14	14	4986
				30·6	0·413	28	27	4973
				32·6	0·440	43	42	4958
7	75·5	17·35 ins of oil 2·8 cm Hg 5200 mic div of oil	72·7	12·9	0·178	0	0	5000
				37·3	0·514	0	0	5000
				40·3	0·555	0·5	0·5	4999·5
				43·3	0·596	1·5	1·5	4998·5
				46·3	0·637	4	4	4996
				48·9	0·673	8	8	4992
				51·2	0·705	15	14	4986
				53·6	0·737	27	26	4974
				55·4	0·762	46	44	4956
8	76·0	17·1 ins of oil 2·8 cm Hg 5130 mic div of oil	73·2	10·2	0·139	0	0	5000
				56·1	0·766	0	0	5000
				57·4	0·785	0·7	0·7	4999·3
				58·4	0·797	2·5	2·5	4997·5
				58·8	0·804	3·4	3·4	4996·6
				59·2	0·809	5	5	4995
				59·8	0·818	9	9	4991
				60·2	0·822	12	12	4988
				61·3	0·838	30	29	4971
				62·0	0·847	46	45	4955
9	76·9	17·1 ins of oil 2·8 cm Hg 5130 mic div of oil	74·1	9·7	0·131	0	0	5000
				55·6	0·751	0	0	5000
				57·2	0·772	0·5	0·5	4999·5
				58·6	0·791	0·8	0·8	4999·2
				59·7	0·805	1	1	4999
				60·4	0·815	1·5	1·5	4998·5
				61·1	0·825	3	3	4997
				61·7	0·833	8	8	4992
				62·2	0·840	12·5	12·5	4987·5
				62·7	0·846	32	31	4969
				63·1	0·852	45	44	4956
10	76·8	17·0 ins of oil 2·8 cm Hg 5100 mic div of oil	74·0	9·5	0·128	0	0	5000
				28·7	0·388	0	0	5000
				30·5	0·412	0·8	0·8	4999·2
				32·0	0·433	1·7	1·7	4998·3
				33·6	0·454	4	4	4996
				35·1	0·475	7	7	4993
				36·7	0·496	11·5	11·5	4988·5
				38·6	0·522	22	22	4978
				40·5	0·547	39	38	4962
				41·5	0·561	51	50	4950

Table II—*continued*

Nozzle	Barometer reading	Oil gauge reading	p_0	p_1	p_1/p_0	Microscope reading	Corrected microscope reading	p_2 (max = 5000)
	cm Hg		cm Hg	cm Hg		scale divisions	scale divisions	scale divisions
11	76·3	17·6 ins of oil 2·8 cm Hg 5280 mic. div of oil	78·5	49·2	0·670	0	0	5000
				58·5	0·795	0	0	5000
				59·6	0·811	0·5	0·5	4999·5
				60·8	0·827	1	1	4999
				61·5	0·837	3	3	4997
				62·2	0·846	7	7	4993
				62·8	0·855	14	13	4987
				63·1	0·859	18	17	4983
				63·7	0·866	38	36	4964
				64·0	0·870	50	47	4953

In order to allow the results for various nozzles to be conveniently compared, the values of p_2 are given corresponding to the various values of p_1 expressed as a fraction of p_0, the maximum value of p_0 in each case being reckoned as 5000. Since the actual value of p_0 was not exactly 5000 microscope divisions, the microscope readings were corrected to make them correspond to this value.

Curves were plotted showing the relation between p_2 and p_1 for the various nozzles. They are shown in figs 4 and 5.

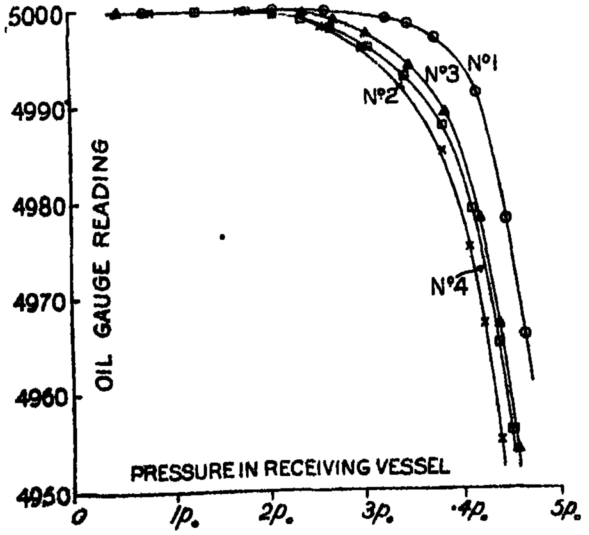

Fig. 4.

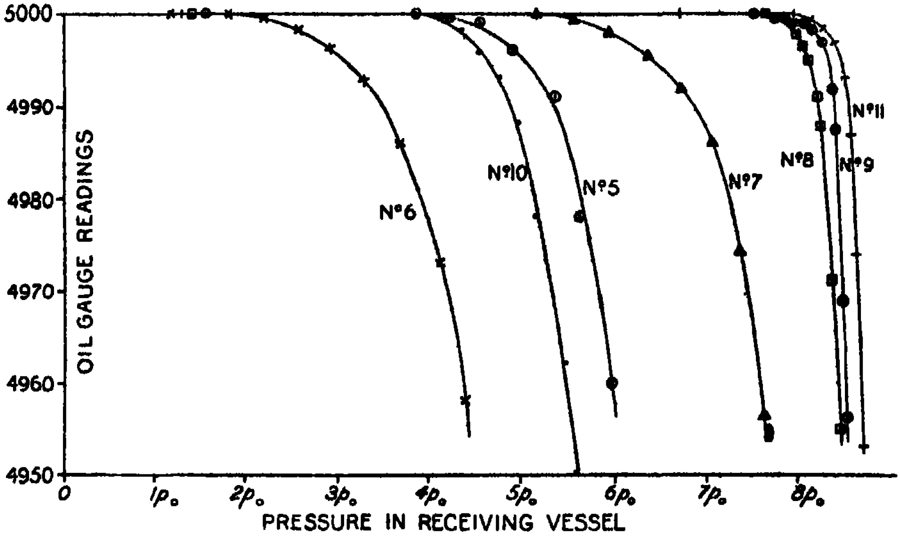

Fig 5

Discussion of Results.

It will be seen from the curves that the fact that the discharge is independent of the pressure in the receiving vessel, when this is less than a certain fraction of that in the discharging vessel, is amply verified by the experiments. In each case the flow was constant, probably to within 0·01 per cent, for a considerable range of p_1, the pressure in the receiver.

The experiments also show that we cannot accept the theoretical value of this critical pressure below which the flow from a convergent nozzle remains constant, viz., 0·527 p_0 as applying to all nozzles; indeed, we cannot accept any particular value as applying even approximately to all nozzles. In some cases the value is as low as 0·2 p_0, and in others it is as great as 0·8 p_0. The value 0·2 p_0 holds very approximately, however, for all the nozzles tried except the divergent ones. Those ending in divergent cones all have higher values, and the value increases as the angle of the cone decreases from large values down to about 3°. In this case the critical pressure is about 0·8 p_0. As the angle of the cone is further decreased the value of the critical pressure seems to decrease. For an angle of 0·7° it is about 0·4 p_0, and when the angle becomes zero, *i.e.*, for a parallel hole, we obtain the value 0·2 p_0.

It seemed possible that these results might be of practical importance in connection with turbine nozzles. Accordingly, in order to ascertain whether the phenomena were very considerably modified in nozzles of larger diameter, No 11 was made with a diameter as large as the pump would allow The flow of air was then so large that the cylinder of gauze M produced too much choking to give a convenient reading of the oil-gauge, and it had to be removed, so that the choking was then supplied by the tube connecting E and N (fig 2) and a very small piece of cotton-wool in N Even under these conditions the readings of the oil-gauge were fairly steady with this nozzle The first reading of p_1 was of course not very low in this case, but it is seen to lie well on the horizontal portion of the curve The result for the nozzle of larger diameter is seen to be practically the same as that for a smaller one of the same shape

In conclusion I wish to thank Prof the Hon R J Strutt, F R S, for advice on many points, and for the keen interest he has taken in the work throughout its progress

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On the Relation between Barometric Pressure and the Water-level in a Well at Kew Observatory, Richmond

By E G. Bilham, A R C Sc, D I C, B.Sc, F R Met Soc

(Communicated by Sir Napier Shaw, F R S Received October 13, 1917)

I *Introductory*

The investigation which forms the subject of the present communication must be regarded as forming part of a general inquiry into the mode of action of the well at Kew Observatory, Richmond The well was sunk during the course of some extensive alterations to the building about four years ago, and the water-level has been continuously recorded since the end of July, 1914 The mean level for each day expressed in centimetres above Ordnance Datum has been published in the 'Geophysical Journal,' and the extreme values for each month have usually been given in the Monthly Weather Report. After being in action for two years, it was thought desirable to undertake a somewhat detailed examination of the records which had accumulated The most interesting feature of the inquiry proved to be the periodic variations of level, which were to be attributed to the tidal oscillations of the neighbouring River Thames The most noteworthy response of the well was found in the case of the lunar fortnightly oscillation, which could, indeed, be plainly seen by merely plotting successive daily mean levels on a fairly open scale

The solar and lunar diurnal, semi-diurnal, ter-diurnal, . , oscillations were sought for by forming solar and lunar diurnal inequalities at two-hour intervals and analysing the resultant variations During the course of this work, it soon became evident that the extent of the solar diurnal variation was very much greater than one had any right to expect, and one inferred that the solar tides were not alone concerned in its production The effect of barometric pressure suggested itself as a likely factor, and an investigation was accordingly taken in hand

Full details of the method of registration and of standardising the traces, as well as a description of the site of the well, will appear in the paper embodying the results of the work on the tidal phenomena Some of the results are quoted in this paper where necessary With these exceptions, the inquiry relating to the effect of barometric pressure forms a more or less complete study in itself, and, from this point of view, the results would seem to be of sufficient interest to justify their being brought before the Society

The well referred to penetrates to a depth of 11 feet below the basement of the Observatory, the bottom being 150 cm above Ordnance Datum The soil consists of sand and coarse gravel resting upon London Clay, at a depth of about 9 or 10 feet The well is about 300 yards from the Thames

II *The Relation between Changes of Pressure and Water-level*

The direct detection of a relation between atmospheric pressure and water-level is a matter of no little difficulty, on account of the multiplicity of factors which have been found to control the latter Beside the regular tidal effects, the well shows the usual type of seasonal variation, with a maximum late in winter and a minimum in the late autumn During the winter the changes of level may be very rapid and irregular, particularly at times of flood The well also appears to be highly sensitive to rainfall. Any connection between the height of the barometer and the height of the water in the well is therefore almost certain to be masked in a short series of observations by the large deviations due to those causes After consideration, the following method was adopted The change of level between 4 h. and 7 h G M T each day was measured on the trace in millimetres The figure given in the Daily Weather Report for the barometric change in three hours ending at 7 h was then written down in an adjacent column. This was done for each day of the two years, August, 1914, to July, 1916, with only such omissions as were necessitated by loss of record. For brevity, the changes of level and of pressure during the three hours ending at 7 h. will be denoted by δU and δP respectively, δu and δp will denote corresponding small changes of level and pressure in general.

In the Daily Weather Report the value of δP takes the form of an integer—the unit being the half-millibar—to which a positive or negative sign is prefixed, to indicate a rising or falling barometer.* The same convention was adopted in tabulating the value of δU. On account of the diurnal variation of pressure, the mean value of δP is about +0·3 mb, but the passage of depressions and anticyclones introduces sufficient variation to produce a fairly normal distribution, in which values up to ±4·5 mb occur with sufficient frequency. Retaining the unit of the half-millibar, we may, then, place a given value of δU in one of 19 classes, according as the value of δP given in the Daily Weather Report is −9, −8, ..., 0, ..., +8, or +9. A few values of δP of higher magnitude occurred during the period under examination, but they have not been employed in the subsequent work. The mean value of δU for each class, together with the frequency, for the whole period, is given in Table I.

Table I

δP (unit 0·5 mb)	−9	−8	−7	−6	−5	−4	−3	−2	−1	0
δU in millimetres	+2·00	+4·75	+9·40	+3·43	+1·80	+2·00	+0·46	+0·56	−0·43	−1·20
No of observations	2	4	2	8	5	21	31	33	56	122

δP (unit 0·5 mb)	+1	+2	+3	+4	+5	+6	+7	+8	+9	Mean +0·720
δU in millimetres	−1·24	−2·05	−1·94	−1·93	−0·81	−3·75	−3·71	−0·80	−14·25	−1·09
No of observations	126	125	66	31	14	11	7	2	2	668

There are some irregularities, but the tendency for negative values of δP to be associated with positive departures from the mean value of δU is manifest. The values of δP and δU in the Table give a correlation coefficient of −0·88, which is high enough to place the reality of the connection beyond reasonable question. The corresponding regression equation is

$$\delta u = -0.533\,\delta p$$

or, converting the pressure units to millibars,

$$\delta u = -1.07\,\delta p,$$

whence it appears that on the average an increase of pressure of 1 mb, spread over a period of three hours, is accompanied by a depression of the water-level amounting to rather more than a millimetre.

The preliminary work having been successful, in so far as it indicated a definite connection between changes of pressure and water-level, it was thought desirable to extend the inquiry with the object of determining

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* More accurately, a negative sign indicates that the barometer was higher at 4 h. than at 7 h., and *vice versâ*.

(1) Whether the value of the ratio $\delta u/\delta p$ depended upon the sign of δp, or upon the value of dp/dt, in other words, whether the relation between δU and δP were linear.

(2) Whether the value of $\delta u/\delta p$ were subject to seasonal variation,

(3) Whether there were any lag in the response of the water-level to changes of pressure

III *Form of the Relation between δu and δp*

The figures given in Table I are too irregular to permit of any conclusions being drawn regarding the linearity or otherwise of the relation between δu and δp In order to obtain a smoother curve the fifteen classes included between $\delta P = -7$ and $\delta P = +7$ were reduced to five by combining successive groups of three The results obtained in this way are given in Table II

Table II

<table>
<tr><td>Mean δP</td><td colspan="2">−5·80</td><td colspan="2">−2·86</td><td colspan="2">+0·23</td><td colspan="2">+2·58</td><td colspan="2">+5·78</td></tr>
<tr><td>Mean δU</td><td colspan="2">+3·69</td><td colspan="2">+0·88</td><td colspan="2">−1·07</td><td colspan="2">−2·00</td><td colspan="2">−2·46</td></tr>
<tr><td>δu/δp (in mm /mb)</td><td></td><td colspan="2">−1·91</td><td colspan="2">−1·26</td><td colspan="2">−0·79</td><td colspan="2">−0·29</td><td></td></tr>
</table>

Plotting δU against δP (fig 1) it is seen that the slope of the curve decreases uniformly in going from negative to positive values of δP Before accepting this conclusion as representing a genuine phenomenon there are

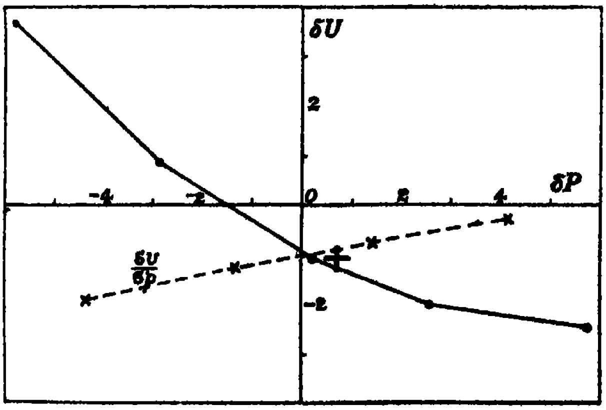

Fig 1 —The mean values of δU and δP are shown by the large cross.

certain points to be considered In the first place, the observations are not really homogeneous. The more rapid changes of barometric pressure are almost entirely confined to the winter months Consequently, values of δP of ±4 and upwards are very much more frequent during the winter than during the rest of the year We cannot, therefore, regard the first and last entries in Table II as fairly representing general conditions throughout the year. Again, it will be observed that the curve does not pass through the point representing the mean values of the two quantities This means that a small depression of water-level occurs without change of pressure, which seems to indicate that "accidental" features have not been completely eliminated

For the purpose of investigating the existence of a seasonal variation of $\delta u/\delta p$ the data were subdivided into three groups in a manner to be explained presently From what has been said above, it appears essential to treat the data according to a seasonal classification before further progress can be made towards interpreting the results so far obtained I will therefore proceed at once to the consideration of our second object of inquiry, the results of which may be expected to throw some light on the question as to whether the relation between δu and δp is really linear or not

IV *Seasonal Variation*

It has been previously remarked that the water-level at Kew shows a pronounced seasonal variation, the highest levels normally occurring in winter, followed by a steady drop to a minimum late in the autumn In the work on the tidal phenomena it was found that the sensitiveness of the water-level to all kinds of variation was intimately related to the height of the water For example, the ranges of the solar and lunar diurnal variations obtained from monthly mean inequalities were both found to increase with rise of the mean level, the correlation coefficients being about + 9 in each case. It was therefore thought best to take the water-level as a basis of classification rather than to employ the ordinary seasonal subdivisions of the year. The following three groups of months were formed —

Group 1 —High-Level Months

1915	January	(381)
1915	February	(370)
1915	March	(331)
1915	December	(355)
1916	January	(352)
1916	February	(330)
1916	March	(400)
1916	April	(364)

Mean level = 360 cm above M.S L
= 190 cm. below ground level.

Group II.—Intermediate Months.

1915	April	(273)
1915	May	(252)
1916	May	(286)
1916	June	(245)

Mean level = 264 cm above M S L
= 286 cm below ground level.

Group III—Low-level Months

1914	August	(191)
1914	September	(183)
1914	October	(176)
1914	November	(175)
1914	December	(237)
1915	June	(237)
1915	July	(214)
1915	August	(217)
1915	September	(205)
1915	October	(208)
1915	November	(225)
1916	July	(223)

Mean level = 208 cm above M S L
= 342 cm below ground level.

The figures in parenthesis give the mean heights of the water in centimetres above mean sea level. The above classification, although quite arbitrary, was found to be very satisfactory in practice. There were many advantages and apparently no serious objections attending the use of the same groups in the present work, and they were accordingly adopted

The method followed has been similar to that used in Table II, with the exception that only three classes have been retained, comprising values of δP -4, -3, -2, -1, 0, $+1$, $+2$, $+3$ and $+4$ The observations in each class are usually sufficiently numerous to give fairly reliable values, and the range of pressure is quite large enough, in view of the ultimate object of the inquiry, to determine what portion of the solar diurnal variation is to be attributed to the effect of barometric pressure

In order to get some confirmation of the figures by comparing results obtained under similar but independent conditions the low-level and high-level months have each been considered in two groups. The complete results are given in Table III and shown graphically in fig. 2.

Table III

Period	Mean Level	δP			δU			$\delta u/\delta p$	$\delta u/\delta p$ (corrected)
	cm	½ mb	½ mb	½ mb	mm	mm	mm	mm /mb	mm /mb
High-level months, 1915	359	−3·01	−0·10	+2·78	+2·46	−3·68	−2·01	−1·57	−1·63
High-level months, 1916	361	−2·91	+0·11	+2·57	+0·10	−2·89	−6·27	−2·31	−2·39
All high-level months	360	−3·00	+0·01	+2·65	+1·31	−3·24	−4·31	−2·01	−2·08
Intermediate months	264	−2·40	+0·23	+2·46	−0·55	−1·47	−2·78	−0·64	−0·66
Low level months, 1914	192	−2·78	+0·42	+2·67	+0·89	+0·39	−0·27	−0·43	−0·44
Low-level months, 1915	218	−2·69	+0·24	+2·53	+0·49	−0·35	−0·80	−0·49	−0·51
All low-level months	• 208	−2·79	+0·31	+2·57	+0·67	−0·15	−0·65	−0·49	−0·51
Mean of all results	268	−2·86	+0·23	+2·58	+0·88	−1·07	−2·00	−1·07	−1·11

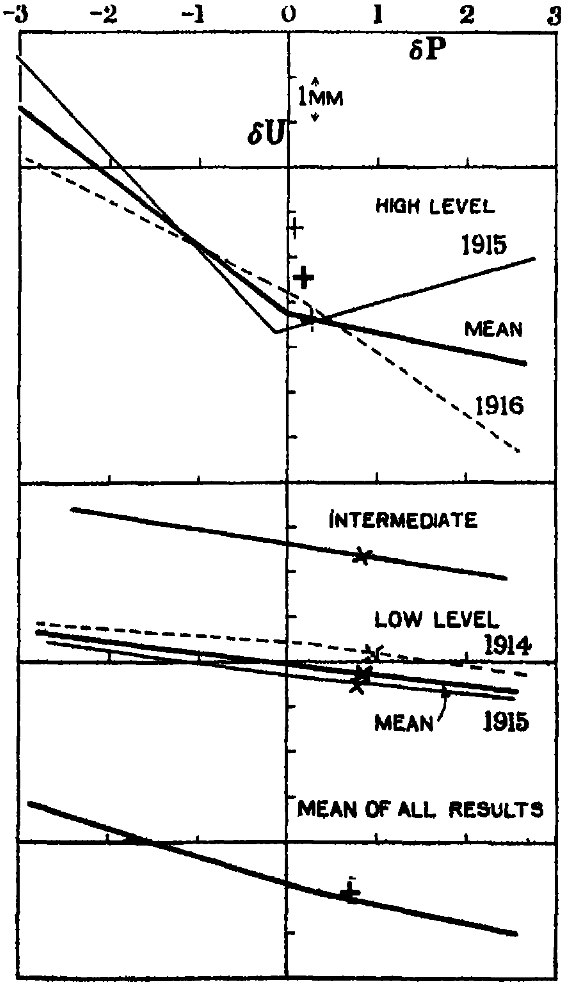

FIG 2 —Mean values are shown by crosses.

It will be seen that with one noteworthy exception (high level, 1915) the three points lie fairly on a straight line passing through the point corresponding to the mean values. It seems fair to conclude that the exceptional character of the 1915 high-level curve must be entirely due to accidental circumstances. It must be remembered that changes of level are much more rapid and irregular during the high-level months than during other periods, and consequently a greater number of observations are necessary to reduce errors due to "accidental" movements (including in the latter term all movements which are not caused by changes of pressure) to a sufficient extent. It appears, then, that under similar conditions the effect of an increase of pressure of about 1·5 mb., spread over three hours, is equal and opposite to that produced by an equal decrease of pressure. Within those limits it would seem justifiable to generalise that conclusion. From three points only it is of course not possible to obtain the true course of the curve on which they lie, but we may fairly assume that a fourth point is given by the mean values of δU and δP. The results indicate that the four points lie on a straight line. We may therefore write in general

$$\delta u = a \cdot \delta p,$$

at any rate so long as dp/dt does not exceed 0·5 mb. per hour. The factor a is always negative but varies with the amount of water in the well.

The values of the factor a, computed where necessary by the method of least squares, are given in the Table. The nature of the correction which has been applied in order to obtain the results in the last column will be explained in the following section. It will be seen that the sensitiveness of the water-level to changes of barometric pressure is conspicuously dependent upon the amount of water in the well, high levels being associated with high sensitiveness. This relation is similar to that observed in the case of lunar tides of short period and also in the case of rainfall. It appears, then, that the height of the water in the well may be regarded as an index of sensitiveness in general. The only exception which I have observed occurs in the case of the lunar fortnightly tides, the effect of which upon the well does not appear to vary much with the level.

Although it is convenient to state the general conclusion in the above form, there is good reason for believing that the connection is purely an indirect one, due to the fact that those conditions of the soil which are necessary for high sensitiveness occur only at times of high level. The condition of the soil, indeed, appears to be the chief if not the only factor which controls the sensitiveness. In the absence of accurate measurement a good idea of the sensitiveness is conveyed to the eye by the amount of incidental movement, or "embroidery" as it is usually termed, shown by the trace Now when the water is rising rapidly from a low level to a higher one, which is frequently the case late in autumn, the appearance of the trace in all the cases I have examined is very different from that occurring during a fall of level through the same range In the latter instance the sensitiveness appears to be much greater. I am unable to say whether the response to each factor is equally affected, but in the case of barometric pressure at any rate the sensitiveness characteristic of "high level" does not appear to manifest itself until the water has ceased rising for the time being It is clear, then, that the sensitiveness is decided by the condition of the soil rather than by the absolute level The difference between the high level results for 1915 and 1916 may, therefore, to some extent be real

Reference may be made here to fig. 4, which represents the relation between level and sensitiveness so far as I have been able to ascertain it I have made no attempt to represent the curve mathematically because the number of points is too small to define its exact shape with certainty It would appear to be either hyperbolic or exponential in form On the first supposition the sensitiveness is inversely proportional to the depth of the water-line below some fixed level. The extrapolation of the curve as it stands would result in the sensitiveness becoming infinite for a level of about 430 cm or about 1 2 m below the surface This is, of course, out of the question, but there can hardly be any doubt that the value of $-\delta u/\delta p$ increases very rapidly as the water-lèvel approaches the surface

V *Accuracy of the Results*

For the purpose of ascertaining the main features of the phenomena under investigation, the data employed would seem to have been quite satisfactory It will, however, ultimately be necessary to know the value of $\delta u/\delta p$ in the different groups with some accuracy It is desirable therefore to form some idea of the probable errors involved

Consider first the barometric data The "barometric tendency" is determined at Kew by means of a Dines float barograph. The change of pressure in three hours is estimated with the help of horizontal lines, ruled by hand, whose distance apart is intended to represent 5 mb of pressure (10 on the scale used in the Daily Weather Report). As the estimate has to be made while the chart is on the drum and under conditions of illumination which leave something to be desired, there is obviously scope for errors which may be important in the present connection In order to form an idea of the accuracy obtained, I have compared the mean value of the barometric change from 4 h. to 7 h in each month obtained from the "barometric tendency" (δP_1) with that obtained from the tabulated hourly values derived from the Kew photo-barograph (δP_2) The result of this comparison is given in Table IV

Table IV

Month	δP_1 from D W R	δP_2 from hourly values	$\delta P_1 - \delta P_2$	Month	δP_1 from D W R	δP_2 from hourly values	$\delta P_1 - \delta P_2$
1914	mb	mb	mb	1915	mb	mb	mb
August	0 408	0 381	0·022	August	0 580	0 464	0 116
September	0 733	0 640	0·093	September	0 416	0 380	0 036
October	0 548	0 319	0 229	October	0 242	0 203	0 039
November	0 217	0·217	0	November	0 350	0 384	−0 034
December	0 420	0 520	−0 100	December	−0 194	−0 226	0 032
1915				1916			
January	0 194	0 110	0 084	January	−0 016	−0·093	0·077
February	0 161	0 157	0 004	February	−0 017	−0 038	0 021
March	0 355	0 322	0·033	March	0 530	0 465	0 065
April	0 717	0 504	0 213	April	0 467	0 336	0 131
May	0 436	0 465	−0 029	May	0 323	0 358	−0·035
June	0 316	0 360	−0 044	June	0 350	0 293	0 057
July	0 532	0 555	−0 023	July	0 435	0·455	−0 020
Means	0 419	0 379	0·040	Means	0 289	0 249	0·040

For the two years the mean values are $\delta P_1 = 0\ 354$ mb, $\delta P_2 = 0\ 314$ mb We have now to decide whether this difference is due to some systematic error of observation The root mean square value of $\delta P_1 - \delta P_2$ for individual months is 0 086 mb If this is due to casual error the standard value of $\delta P_1 - \delta P_2$ for 24 months is $0\ 086/\sqrt{24}$ or 0 0175 mb, and the probable error two-thirds of this, or 0·012 mb The observed value (0 040) is more than three times this amount, which seems to suggest the probability of a systematic error in δP_1.

If we take 0 1 mb as the standard error of an individual hourly reading of barometric pressure, the standard error of a monthly mean for a given hour is $0\ 1/\sqrt{30}$ or 0 018 mb., and the standard error of δP_2 in a given month $0\ 018 \times \sqrt{2}$ or 0 025 mb The standard error of δP_1 is a good deal larger than this The fact that the estimate is only made to the nearest half-millibar introduces a standard error of 0 25 mb in an individual reading which would give rise to a standard error of 0·065 mb in δP_2. The standard error of $\delta P_1 - \delta P_2$ is therefore $\sqrt{[(0\ 025)^2 + (0\ 065)^2]}$ or 0·070 mb. We have probably underestimated δP_1, since we have assumed that the estimate of the tendency is correct to the nearest half-millibar The agreement with the observed value 0·086 mb is therefore satisfactory.

One opportunity for the introduction of a systematic error immediately suggests itself, namely, the ruling of the charts The accepted magnification-factor of the instrument is 2 10, with a probable error of about 1 per cent. The distance apart of the 5 mb rulings should then be 2 1 × 5 × 750/1000 or 7 875 mm The actual spacing was measured on two samples of 20 charts, selected at random from the curves for the year 1915 Both gave the same mean value, viz, 7 62 mm, the probable error being 0 03 mm It appears then that the spacing was on the average 3 3 per cent too narrow If we make a correction on this account the mean value of δP_1 becomes 0 342, giving $\delta P_1 - \delta P_2 = 0\,028$ mb If the same correction is applied to individual values of δP_1 it is found that the difference $\delta P_1 - \delta P_2$ is positive in 15 and negative in 9 cases out of the 24 There is no clear indication therefore that the remaining errors are anything but casual A correction of +3 3 per cent has accordingly been applied to all the results in Table III

From the foregoing discussion it might appear that it would have been desirable to use more precise barometric data - hourly tabulations for example The relative uncertainty in the mean value of δU is, however, probably greater in general than that in δP, so that there was nothing to be gained by obtaining the latter quantity with all possible precision The error in δU does not arise so much from errors of measurement as from fluctuations of water-level, due to causes other than changes of pressure, and is not easily estimated

The best way of appraising the results is to compare values of $\delta u/\delta p$, obtained from different groups of months belonging to the same category In the high-level class there is some discordance between the values for 1915 and 1916, although the mean levels were very nearly the same. It would appear that the uncertainty in the final value −2 08 may amount to 20 per cent or more* The difference in the case of the two low-level groups is in the direction to be expected from the difference of level The figure −1 11 for the whole period rests upon about 600 observations. δP appears to be uncertain to the extent of about 8 per cent over the whole period The uncertainty in δU is probably of about the same order of magnitude, so that we can hardly suppose that the probable error is less than about 0 1 mm /mb

VI. *Rapidity of Response to Pressure Changes*

In the preceding sections we have compared the changes of water-level which occur simultaneously with given changes of pressure, and derived certain conclusions from the results. It is obvious, however, that those conclusions may be very largely in error unless it can be shown that the response of the well is not associated with lag

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* This is the uncertainty involved in assigning a certain value of $\delta u/\delta p$ to the level of 360 cm. It necessarily includes the effect upon the average value of real differences of sensitiveness at the same level.

During high-level periods there is no difficulty in identifying movements of water-level which have taken place as the result of well-marked changes of barometric pressure The passage of a depression, for example, is shown by a peak on the water-level trace Two examples of such movements are shown in fig 3 In the diagram the time scales of the barograph and water-level recorder have been brought into equality, and the barograph trace has been inverted The former instrument actually records on a daily chart, and the latter on a weekly, one hour being represented by 0 6 inch and 0 09 inch respectively Time marks are made twice daily on the water-level chart, so that it is generally possible to obtain the time at any point on the trace to within about ten minutes.

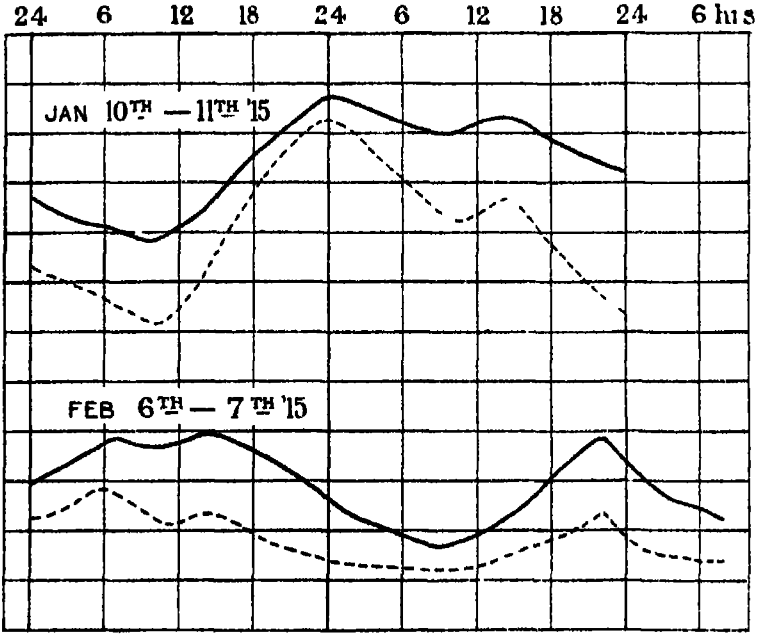

Fig 3 —Broken line—water-level trace, full line—barograph trace inverted Vertical divisions each represent 20 mm in water-level and 12 7 mb of pressure

Judging from the two examples figured, it would appear that the pressure changes are very faithfully reproduced by the water-level recorder simultaneously with their occurrence If there is any lag it must be a matter of minutes only. In order to verify this conclusion I have measured the times of occurrence of corresponding movements shown on the two traces in all cases that were suitable for the purpose during the year 1915. The chief qualifications for suitability were that the movement should be sufficiently outstanding to prevent any mistake in identifying it on the water-level trace, and that the feature to be measured—the barometric minimum during a depression for example—should be sufficiently abrupt to permit of accurate measurement of the time of its occurrence

The results are given in Table V, and it will be seen that they show no systematic indication of lag in the water-level Where times differ appreciably the difference is as often in one direction as in the other, and was undoubtedly due to some cause for uncertainty in the measurement In most cases, however, the times agree within the limits of the errors of measurement

Table V —Times of Occurrence of Barometric Maxima, etc, and of Corresponding Water-Level Movements during 1915

Date		Character of movement shown on barograph *	Time of occurrence, G M T				T_2-T_1	Remarks
			Barograph (T_1)		Water level (T_2)			
			h	m	h	m	min	
January	10	Maximum	9	40	10	10	30	See fig 3
,,	10	Minimum	24	0	24	0	0	,, ,,
,,	11	Maximum	9	40(?)	10	0	20(?)	,, ,,
,,	11	Minimum	14	40	14	20	−20	,, ,,
,,	16	Minimum	2	20	2	30	4	
February	6	Minimum	13	50	14	0	10	,, ,,
,,	7	Maximum	8	50	8	50	0	,, ,,
,,	7	Minimum	22	10	22	15	5	,, ,,
,,	9	Minimum	7	15	7	10	−5	
,,	13	Minimum	20	0	20	0	0	
March	6	Minimum	15	50	14	40	−80	Minimum not well marked Fairly steady from 14 30 to 15 50
,,	18	Sudden rise	19	4	19	0	−4	Squall
,,	19	Minimum	3	0	2	50	−10	
April	7	Minimum	2	45	2	0	−45	Times agree if steady fall of water level is allowed for
,,	8	Sudden rise	19	8	19	10	2	Squall
December	25	Minimum	12	15	13	0	45	Pressure nearly constant from 12 to 14 hours
,,	26	Maximum	20	5	20	15	10	

* The movement is inverted on the water-level trace

It may be noted in passing that the data from which fig 3 was drawn may be used to obtain values of $\delta u/\delta p$ Thus —

January 10, 1915 —Mean level = 403 cm , pressure range = 18 4 mb , water-level range = 80 6 mm , whence

$$\delta u/\delta p = -4\ 38 \text{ mm}/\text{mb}.$$

February 7, 1915.—Mean level = 344 cm., pressure range = 13·7 mb; water-level range = 23·0 mm., whence

$$\delta u/\delta p = -1\cdot68 \text{ mm./mb.}$$

These values would seem to be quite reliable and I have plotted them together with the data of Table III in order to obtain a curve, giving the variation of sensitiveness with water-level (fig. 4). The rapid increase of $\delta u/\delta p$ with rise of level is very striking.

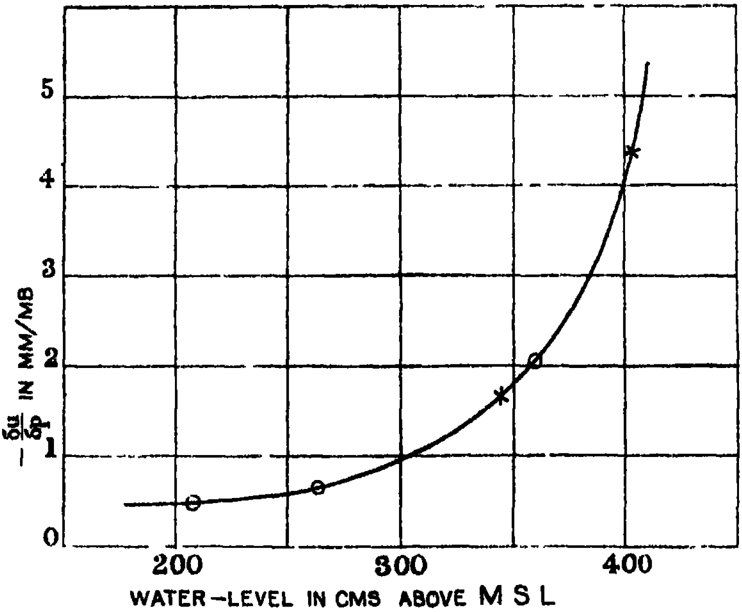

Fig. 4.

Returning to the question of lag, a further piece of evidence may be quoted in support of the conclusion arrived at. During times of high wind a broadening of the barograph trace occurs as a result of the phenomenon known as "pumping." The water-level traces are affected in a very similar manner. Now, if there were any lag in the response of the well, it is difficult to see how the rapid changes of pressure to which the movements are due could be registered. We may surmise that any sluggishness in the response to ordinary changes of pressure would be accompanied by insensitiveness to small and rapid movements. The records show, however, that the latter are recorded in a similar manner and apparently on the same scale as the former.

VII *Summary of Results and Conclusion.*

We may summarise the results of the inquiry as follows :—

(1) At all seasons of the year the water-level is sensitive to changes of pressure, a rise of the barometer being associated with a fall of water-level and *vice versâ*

(2) Within certain limits the change of level is proportional to the change of pressure producing it

(3) The magnitude of the change of level produced by a given change of pressure increases rapidly as the subsoil water-level rises

(4) There is no appreciable lag in the response of the water-level to changes of pressure

These conclusions must be understood to refer to the level of the water in the well, because we have no justification for assuming that the effect of a change of pressure upon the free surface of water is the same as its effect upon the water held in the interstices of the soil In fact, it is possible that the phenomena may depend upon a differential action of this nature.

The fact that barometric pressure is a factor of importance in relation to the movements of subsoil water appears to have been first recognised by Mr Baldwin Latham [*] His observations related chiefly to the Bourne flow in the Croydon district, and included soundings of deep wells So far as can be ascertained from the brief notices of his two papers read before the British Association, his investigation of the response of the wells to pressure changes led to conclusions very similar to those enunciated above. We may suppose, then, that the phenomenon is a general one—probably with limitations imposed by the geological formation—exhibited by both deep and shallow wells. A further result obtained by Latham was that the rate at which percolation occurred increased with diminution of pressure This conclusion he confirmed by direct observation with a percolation gauge, and attributed the phenomenon to the action of dissolved air and other gases in the water He supposed that a diminution of pressure would result in the liberation of a certain amount of the gases, and that the consequent increased aëration of the soil would facilitate percolation. An increase of pressure would reduce percolation by causing an absorption of gases in the soil by the water.

The chief difficulty in applying Latham's theory to the Kew well appears to me to lie in the fact that there is no lag in the response of the well to changes of pressure. All the evidence which has accrued from the investigation of the movements of subsoil water at Kew indicates that percolation can only operate slowly in bringing about changes of level. Even during times of flood, when conditions are most favourable for rapid penetration, the water-level in the well continues to rise for hours after the surface water has begun to subside It is therefore difficult to believe that any effect depending upon some change in the rate of percolation could follow so promptly upon the initiation of that change as we have found to be the case in the phenomenon under notice There appears, therefore, to be a strong probability that the effects of pressure upon well-level are not secondary effects due to changes in the rate of percolation

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[*] "On the Influence of Barometric Pressure on the Discharge of Water from Springs," 'B.A. Report,' 1881, p. 614 ; 1883, p. 495.

I have already indicated an alternative direction in which to look for an explanation of the observed results, namely, a differential action of pressure upon the free surface of water in the well and upon the subsoil water The well behaves very much like the open arm of a manometer connected to a reservoir of air If we suppose that the air contained in the interstices of the soil immediately above the water-line, or in cavities, is cut off from external pressure changes, the general character of the movements would seem to be sufficiently well accounted for Any change of barometric pressure would result in a readjustment of the water-level in the soil and in the well, until uniformity of pressure at any given level below the water line was re established If this view is correct, it follows that the water-level in the well does not in general represent accurately the true subsoil water-level The two levels would only coincide at some particular height of the barometer It must also be supposed that any volumes of air concerned in the action are very perfectly cut off from external pressure changes, otherwise the well would behave like a micro-barograph, responding only to sudden changes of pressure The results, however, point strongly to the conclusion that the rate at which the pressure is changing is of no consequence

It would appear at first sight that this postulate is in opposition to the well-known phenomenon of the emission and absorption of gases by the soil in response to barometric changes In deep mines this effect is of so great importance that special precautions may be necessary to prevent explosions under certain barometric conditions. In the case we are dealing with, however, the local circumstances are rather peculiar The well is sunk through a concrete floor, is lined with glazed pipes, and is surrounded by a system of brickwork vaults Under these circumstances it seems quite likely that cavities may occur containing imprisoned air in sufficient quantity to produce the observed results. The effects observed by Latham may also have been produced in some such way

A third possibility is that the phenomenon may be the result of the distortion of the earth's crust and the water-plane under variations of pressure.

Sir G. H. Darwin* in 1882 showed that between two places on an otherwise flat earth of the rigidity of glass a difference of level of about 9 cm. would exist as the result of a difference of pressure equivalent to 5 cm. of mercury. Over the free ocean the water would be depressed 13·6 cm. for each centimetre of mercury increase of pressure. If the subterranean water plane were distorted to the same extent the depression per centimetre of mercury measured by an instrument on the surface would be 13·6 − 1·8 cm. or 11·8 cm., which corresponds to a value of $\alpha = -15{\cdot}7$ mm./mb. The actual changes of level measured at Kew were considerably smaller than this, which implies either that the rigidity of the earth is much less than that assumed by Darwin or that the distortion of the subsoil water surface is a good deal less than over the free ocean. Darwin concluded that the elastic compression of the ground must take place without sensible delay, but it is at least doubtful whether the same statement is applicable to the subsoil water. Again, it is not easy to see why the observed effect should depend to so marked an extent upon the condition of the soil.

The above calculation may perhaps serve to give some rough idea of the order of magnitude of the effects which might be observed under certain circumstances, but the actual conditions are radically different from those postulated in Darwin's problem. The effects of various distributions of load upon the earth's surface have been worked out by Dr. C. Chree† and others in recent years, but so far as I know the solution for the case in which we have porous matter permeated with liquid up to a certain level is not yet available. It seems quite likely that the effects of elastic compression may be at least partially concerned in producing the results which I have detailed, and a mathematical treatment of the subject would be of great interest.

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* G. H. Darwin, "On the Mechanical Effects of Pressure on the Earth's Surface,' 'B.A. Report,' 1882, p. 106.

† C. Chree, "Applications of Physics and Mathematics to Seismology," 'Phil. Mag.,' 1897, p. 173, *et seq.*

Address of the President, Sir J J Thomson, O.M, at the Anniversary Meeting, November 30, 1917.

Fifteen Fellows and three Honorary Members have died since the last Anniversary Meeting

The losses on the Home List are —

Prof Oliver	Prof McKenny Hughes
Mr Clement Reid	Mr R. Bell
Mr W Ellis	Sir W D Niven, K C B
Prof A M. Worthington	Prof Hull
Prof J Purdie	Lord Cromer
Sir E B Tylor	Lord Allerton
Mr Pickard-Cambridge	Mr W Duddell
Dr W H Besant	

From the list of Foreign Members we have lost —

Prof Gaston Darboux
Prof Chauveau
Prof von Baeyer.

By the death of Prof GASTON DARBOUX, the Secretary of the Academy of Sciences of the Institute of France, we have lost a great Mathematician and a great Ambassador of Science, who by his own researches, his official position, and his striking personality, had done much to advance the progress of science We offer our sympathies to our Sister Academy on the loss of one who for 17 years had guided its policy with conspicuous ability and distinction, and we rejoice that our Society was able to show its appreciation of Prof DARBOUX's work by the award, at our last Anniversary Meeting, of the Sylvester Medal

Prof J B CHAUVEAU, who died at the age of 90 in January of this year, was a famous and many-sided biologist and one of the makers of Physiology in the latter half of the last century

Prof VON BAEYER, the successor of Liebig in the Professorship of Chemistry at Munich, was one of the founders of modern Organic Chemistry His activities covered a very wide range and included the introduction of new ideas of great theoretical importance as well as some of the most striking and important applications of Organic Chemistry to industrial purposes.

From the Home list we have lost Prof. **Oliver**, who had been a Fellow of the Society for 53 years and Keeper of the Herbarium at Kew for more than a quarter of a century, he was the predecessor of his son, our Fellow, in the Professorship of Botany at University College He did much to advance Botanical Science, both by his own researches and the unstinted assistance he gave to those of others

Mr **Clement Reid**, a distinguished member of the Geological Survey, had made many notable and widely discussed contributions to our knowledge of fossil plants.

By the death of Mr **William Ellis** we have lost a veteran in the service of Meteorology and Astronomy, who had made important contributions to our knowledge of the connection between sunspot frequency and terrestrial magnetic disturbances The length of his connection with official Astronomy may be gathered from the fact that he attended for 75 years consecutively the Annual Visitation of Greenwich Observatory

Prof A. M **Worthington**, when a Master at Clifton College, was a pioneer in the introduction into the teaching of science in schools of a carefully planned scheme of practical work His work on the tensile strength of liquids and on splashes has enriched science with results of great beauty and high scientific importance

Prof **Thomas Purdie**, for 25 years Professor of Chemistry at the University of St. Andrews, endowed that University with a fully equipped Research Chemical Laboratory He possessed, to an exceptional extent, the power of winning the confidence and affection of his pupils.

Sir E B **Tylor** had been a Fellow of the Society for 46 years As Keeper of the University Museum, as Reader in and subsequently first Professor of Anthropology at Oxford he exerted a wide influence over the progress of that science. His great erudition and charming literary style made him "the first and foremost exponent of the comparative method in this country."

By the death of Prof. **McKenny Hughes**, the successor of Adam Sedgwick in the Woodwardian Professorship of Geology at Cambridge, we have lost a man of wide interests and activities. He was the head of a most successful School of Geology, a collector, an antiquary, and it is to his exertions that the New Geological Museum at Cambridge owes much of its amplitude and completeness

The toll of geologists during the past year has been a heavy one, we have to mourn the loss, at the age of 69, of the well-known geologist, **Edward Hull**, the Director of the Geological Survey of Ireland, and of Dr. **Robert Bell**, the Director of the Canadian Survey.

Sir W. D. **Niven**, for many years Director of Naval Education and the editor of Maxwell's 'Collected Papers' and of the second edition of Maxwell's 'Electricity and Magnetism,' did much when a lecturer at Trinity College to promote the study of Mathematical Physics at Cambridge Though not a fluent lecturer, he possessed the power of exciting the keen interest of his pupils, among whom I am proud to count myself He had a genius for friendship, and no one looked to him in vain for assistance and encouragement.

By the death of Dr BESANT, at the age of 89, we have lost one who in the days when there were Senior Wranglers used to take an active part in their training

Mr PICKARD-CAMBRIDGE belonged to a class which is characteristic of this country, the man of science who does not hold any professorship or official post in connection with science. A country clergyman, he was the leading authority on one branch of Entomology.

The tragic death, at the age of 45, of WILLIAM DUDDELL has deprived English Applied Science of one who was great both by performance and promise His invention of the oscillograph and the singing arc gave us instruments of singular scientific interest and great practical importance When war broke out he threw all his energies into developing applications of science to military needs He was indefatigable in considering and criticising the numerous proposals made for this purpose, and his wide knowledge, insight, and sound judgment were of the utmost value His health broke down under the strain of this work, and by his death we lose one who rendered good service to the country in its time of trial

In less momentous times it has been the privilege of the President on these occasions to bring before the notice of the Society the results of the work of the Fellows in their efforts to unravel the secrets of Nature. In these days duties higher even than that of scientific investigation have occupied our attention The beneficent work of extending the empire of Man over Nature has been replaced by a struggle to prevent civilisation coming under an empire which, we believe, would deprive it of freedom and justice—of what is the soul as it should be the goal of humanity

The extent to which men of science in this country are engaged on investigations connected with the war is hardly realised, except by those who have to try to find the men for any new piece of work of this kind which may have to be put in hand It is a matter of the greatest difficulty to find any competent person who is not already engaged on such work. Professors from our Dominions have come back to help at home, and in some cases have brought their demonstrators and senior students with them The importance of having an ample supply of trained scientific workers, and the necessity for this country to increase its supply in the future, could hardly be proved more incisively than by our experience in this connection.

A large amount of work has been done by Committees of the Royal Society during the past year in connection with the urgent question of food production and distribution, and the prevention of the destruction of grain by insect and other pests, you will have learned from the Report of the Council that these have been in close touch with the Food Controller The functions of the Engineering and Physics Committees which were formed at the beginning of the war to develop the application of science to military purposes have to a large extent been undertaken by departments created by the State in connection with the Army and Navy, furnished with large resources and in close touch with the Services, most of the members of the Physics and Engineering Committees are actively engaged in connection with these departments

The work of the National Physical Laboratory, which has greatly extended during the past year, has been almost entirely in connection with the war, and we have the satisfaction of receiving testimony from all sides that the services the Laboratory has rendered to the country have been of vital importance

The application of science to the purposes of the War, on which so many of our Fellows have been engaged, has involved more than the adaptation of previously known scientific results to this purpose The channels along which the main currents of investigation flowed before the war were those which seemed of most importance in connection with the development of the theory of the subject or with its industrial applications, other parts of the subject formed backwaters, which were often comparatively neglected The needs of the Army, the Navy, and the Air Service have created new types of problems, phenomena which had not hitherto attracted much attention have become of vital importance, and have required further investigation before they could be used to the best advantage

The question of International Science, bound up as it is with International Scientific Academies and Societies, is profoundly affected by the war So much depends upon what happens before Peace is declared that any definite decision as to future action would be premature But to ensure unanimity it is important that the men of science in each of the allied nations should know the views held by their colleagues. Proposals which it is hoped will effect this object have occupied the attention of the Council.

Immediately after our last Anniversary Meeting, Lord Crewe, who was then President of the Council, announced to a deputation of the Conjoint Board of Scientific Societies that a large sum of money had been allotted by the Government for the purpose of promoting the application of science to our industries It was pointed out by some members of the deputation that for the creation of new industries pure science—science undertaken without any definite idea of practical application—was in the long run even more important than the application of science to solve definite technical problems. Lord Crewe, in his speech to the deputation, spoke sympathetically of the claims of pure science, may we hope that this feeling will influence the action of those who have the administration of the fund?

I wish to call attention to a point in the discussion which has been continued through the past year on the vital question of the position of Science in Education The need for a greater appreciation of the value of science has been brought into such prominence by the war that most of those who have advocated the claims of science in education have not unnaturally laid the greatest stress on the importance of science to the welfare, the power, and even the safety, of the nation The supporters of literary studies have, on the other hand, dwelt mainly on the fact that literature broadens a man's horizon, and gives him new interests and pleasures, that it teaches him how to live, if not how to make a living The result of this divergence of appeal has made the discussion appear, to those who watch it from outside, almost like a discussion between Spirituality and Materialism, or between a saint and a man of business Echoes of this sentiment are to be found in the opinions expressed by some members of the Labour Party, there is a tendency to regard science teaching with suspicion, as being intended to make the working man more valuable to his employer rather than to increase the brightness and interest of his own life.

I recognise—and I know no man of science who does not—the necessity of literary studies as a part of the education of every boy and girl, but I must protest against the idea that literature has a monopoly in the mental development of the individual. The study of science widens the horizon of his intellectual activities, and helps him to appreciate the beauty and mystery which surround him. It opens up avenues of constant appeal to his intellect, to his imagination, to his spirit of inquiry, to his love for truth So far from being entirely utilitarian, it often lends romance and interest to things which to those ignorant of science make no appeal to the intellect or imagination, but are regarded by them from an exclusively utilitarian point of view A knowledge of science brightens and widens the intellectual life, and is a constant stimulus to the intellect and imagination.

The question of the position of science in schools is of vital importance, I think that we ought also to pay attention to the need for sustaining and stimulating in after-life the interest in science which we hope will have been aroused at school. We should encourage and develop efforts to bring to the notice of the public those results of science which are of general interest. I am not sure that we do all that is possible in this direction, and yet it seems our duty to the community to give it everything which can add interest to life and stimulate the intelligence, to do everything in our power to increase appreciation and interest in science among our citizens, without such appreciation a full utilisation of the resources of science and adequate encouragement for its development is impossible in a democratic country.

There are many results of general interest embodied in papers which could not be read by anyone who was not a specialist in the subject. I will give one instance, taken from what might seem a somewhat unpromising branch of science, Arithmetic. If we take the numbers in order 1, 2, 3, we see that there are some, such as 3, 5, 7, 11, which cannot be divided by any number smaller than themselves, these are called prime numbers, the number of such primes which are less than a given number is a matter of very considerable importance, and Gauss, many years ago, gave, without any rigorous proof, a rule about it. The rule was tested by actual trial for numbers up to a thousand millions, and, as it was found to be true over that immense range, it was accepted as universally correct in spite of the absence of a satisfactory proof. Quite recently, however, Mr Littlewood, one of our Fellows, has shown that, in spite of this apparently overwhelming evidence in its favour, the result is not general, but the numbers for which it breaks down are so enormous that it would be quite beyond the powers of human endurance to detect its failure by actual trial. I may say, in passing, that, enormous as these numbers are, they are mere nothings compared with what we have to deal with in many branches of Physics. Here, then, we have a result which has satisfied, and apparently always will satisfy, any direct test that can be applied to it, and yet is not generally true, there seems to me to be something of a tragedy, perhaps the suspicion of a sermon, in this investigation, which is in a paper of a highly technical character, quite unintelligible to anyone who was not an expert mathematician.

There are many results of this kind, known only to specialists, but which would interest a very much wider circle of readers if they could be brought to their notice. Unfortunately, there does not seem to be at present any recognised method of doing this. There are excellent periodicals with special circles of readers which might find a place for some of them, but these only reach a minute fraction of the educated public. There is room, I think, for a periodical which would appeal to a much wider circle, which should contain interesting and trustworthy accounts of results of interest, not only in science, but in the other subjects included in a general education.

The desirability of a journal of this kind was recently brought before the notice of the Executive Committee of the Conjoint Board of Scientific Studies If it could be established, it would, I believe, do good work by stimulating the intellectual life of the nation and increasing the appreciation of science throughout the country

The Copley Medal is awarded to M Émile Roux.

M Émile Roux, Pasteur's chief collaborator, succeeded him as the Director of the Institut Pasteur, which he has successfully developed and maintained as the foremost school of bacteriology, both for teaching and for research. From the early eighties, when he was associated with Pasteur and Chamberland in the study of anthrax and production of vaccines against this disease, he has played a leading part in the development of our knowledge of the processes of immunity

His work with the distinguished veterinarian Nocard upon the contagious pleuro-pneumonia of cattle was the first demonstration of the existence of "ultra-microscopic" or, as they are now termed, filterable viruses as disease-producing agencies, his work with Yersin, the first full study of the bacillus of diphtheria and of its toxins He shares with the late Prof. Behring, of Marburg, in the introduction of diphtheria antitoxin as a practical means of prophylaxis and cure, and with him as co-founder of serum therapeutics was awarded the Nobel prize. All the leading French bacteriologists of our generation have been his pupils

A Royal Medal is awarded to Dr. John Aitken.

Dr Aitken is distinguished for his life-long researches on the nuclei of cloudy condensation, embodied in a series of memoirs communicated to the Royal Society of Edinburgh. The latest of these appeared in the present year

Dr Aitken's discoveries opened up a new field of investigation in physics, and constitute a chapter of knowledge of great importance intrinsically and in their relation to the physics of meteorology.

Dr Aitken, who has pursued his work as an amateur, has displayed great experimental ingenuity, and his remarkable construction of the "dust-counter" has provided a permanent scientific appurtenance of precision to the physicist and climatologist.

Among other contributions to science, Dr. Aitken has made important advances in our knowledge of the formation of dew.

A Royal Medal is awarded to Dr Arthur Smith Woodward.

Dr. Smith Woodward has been for many years Keeper of the Department of Geology in the British Museum, and has published a very large number of valuable memoirs on Fossil Vertebrates, especially Fishes He has also published an important 'Catalogue of Fossil Fishes in the British Museum,' and his 'Outlines of Vertebrate Palæontology,' published in 1898, is a standard text-book on the subject

Dr. Smith Woodward's original memoirs are too numerous to mention, but they have secured for him a world-wide reputation, and he is universally regarded as one of the highest authorities on vertebrate palæontology.

Dr Smith Woodward was elected a Fellow of the Royal Society in 1901, and served on the Council from 1913 to 1915

The Davy Medal is awarded to M Albin Haller

Professor of Organic Chemistry at the Sorbonne, Paris, Founder and First President of the International Association of Chemical Societies, and at the present time the most representative chemist of France, he is distinguished for his many and important contributions to chemical science during the past forty years His investigations have covered a very wide field in the domain of organic chemistry, the most important being those dealing with compounds belonging to the camphor group

He has maintained over a long period of years the reputation of the Sorbonne School of Chemical Research, created by Dumas and Wurtz, his predecessors in the Chair

The Buchanan Medal is awarded to Sir Almroth Edward Wright

Almroth Wright was the first (1896) to apply laboratory knowledge on typhoid immunity to the protection of human beings against enteric fever Against formidable opposition he carried out a long series of observations with the highest scientific acumen and unsurpassed technique, and laid the foundations for the effective elimination of enteric fever from the armies of the world. Nothing of importance has been added to his work down to the present time.

The Hughes Medal is awarded to Prof. Charles Glover Barkla.

Prof. Barkla's investigations have mainly dealt with X-rays, and their absorption and secondary emission by solid substances. He showed that secondary emission of X-rays was of two varieties In one of these the X-rays are scattered, without change of quality. The scattered rays were shown by examining tertiary emission to be polarised, and this was a fundamental result for the classification of X-rays with ordinary radiation, at that time doubtful.

Barkla's other kind of secondary emission is characteristic of the secondary radiator, and is accompanied by selective absorption of the primary rays. He showed that each chemical element emitted more than one definite kind of secondary fluorescent radiation. Concentrating attention on, say, the less penetrating kind, it was found to vary in quality by definite steps with the atomic weight of the secondary radiator.

These results have been confirmed and extended by others since the discovery of the diffraction of X-rays by crystals. This, of course, in no way detracts from the merit of the original discoveries made without this powerful assistance.

The Zeros of Bessel Functions.

By G. N. Watson, M.A., D.Sc., Assistant Professor of Pure Mathematics at University College, London.

(Communicated by Prof. J. W. Nicholson, F.R.S. Received August 17, 1917.)

1. Although many results are known concerning the zeros of Bessel functions,[*] the greater number of these results are of practical importance only in the case of functions of comparatively low order.

For example, McMahon has given a formula[†] for calculating the zeros of the Bessel function $J_n(x)$, namely that, if $k_1, k_2, k_3, \dots$, are the positive zeros arranged in ascending order of magnitude, then

$$k_s = \beta - \frac{4n^2-1}{8\beta} - \frac{4(4n^2-1)(28n^2-31)}{3\,(8\beta)^3} - \dots,$$

where $$\beta = \tfrac{1}{4}\pi(2n+4s-1).$$

Now, this formula is derived from Hankel's well-known asymptotic expansion of $J_n(x)$, and the expansion is available for numerical calculations only when $8x$ is large compared with $4n^2$, and so McMahon's formula fails to fulfil its purpose in the case of the smaller zeros of functions of high order, which are frequently of more importance from the physical point of view than the larger zeros of these functions.

It is, however, possible to obtain a number of quite general existence theorems and descriptive properties concerning Bessel functions of any order; and these results, unlike results obtained by the more classical methods, are of peculiar importance in connection with functions of high order. The theorems in question are derived by a consideration of certain contour integrals which arose* in Debye's investigations of asymptotic expansions of Bessel functions by the method of steepest descents.

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[*] See Gray and Mathews, 'Bessel Functions,' pp. 34-50, 241-244; Whittaker and Watson, 'Modern Analysis,' Ch. XVII.

[†] 'Annals of Math.,' vol. 9, pp. 23-30 (1894).

2. As it will be necessary to introduce Bessel functions of the second kind, for which various notations are employed, it is convenient to state here the relations connecting the several functions of the second kind with the fundamental functions $J_{\pm n}(x)$ When n is an integer, the limiting forms of the relations are to be taken

Hankel's function $Y_n(x)$ is defined† by the equation

$$Y_n(x) = 2\pi e^{n\pi i}\{\cos n\pi \; J_n(x) - J_{-n}(x)\} \operatorname{cosec} 2n\pi,$$

and the functions $\Upsilon_n(x)$, $G_n(x)$ are connected with it by the relations‡

$$\pi\Upsilon_n(x) = -2G_n(x) = e^{-n\pi i} \cos n\pi \; Y_n(x)$$

The two Hankel-Nielsen functions $H_n^{(1)}(x)$, $H_n^{(2)}(x)$, which form a fundamental pair of solutions of Bessel's equation, are defined§ by the equations

$$H_n^{(1)}(x) = J_n(x) + i\Upsilon_n(x), \qquad H_n^{(2)}(x) = J_n(x) - i\Upsilon_n(x),$$

so that

$$J_n(x) = \tfrac{1}{2}\{H_n^{(1)}(x) + H_n^{(2)}(x)\},$$

$$J_{-n}(x) = \tfrac{1}{2}\{e^{n\pi i} H_n^{(1)}(x) + e^{-n\pi i} H_n^{(2)}(x)\}$$

We shall only consider positive values of n and x, and as it is known|| that $J_n(x)$ has no zeros when $0 < x \leq n$, we shall, in general, take $x \geq n$, when this is the case, we shall define β to be $\cos^{-1}(n/x)$, where the principal value (*i.e* the value between 0 and $\frac{1}{2}\pi$) of the inverse cosine is to be taken, and we shall write x and $n \sec \beta$ indifferently

It is to be noted that $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ are conjugate complex numbers when n and x are real.

3 We now introduce Debye's integral in a form suitable for our analysis, we take the integral of Sommerfield's type, namely,

$$H_n^{(1)}(x) = \frac{1}{\pi i} \int_{-\infty}^{\infty + \pi i} e^{x \sinh w - nw}\, dw,$$

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* 'Math Ann.,' vol. 67, pp 535-558 (1910).

† Hankel, 'Math. Ann,' vol 1, p 472 This must, of course, be distinguished from Neumann's function $Y^{(n)}(x)$, which is $(\log 2 - \gamma)\, J_n(x) + \frac{1}{2} Y_n(x)$.

‡ The function $\Upsilon_n(x)$, which is equal to $Y_n(x)/\pi$ when n is an integer, is employed by Gray and Mathews; extensive tables of $G_n(x)$ are published in 'Brit. Ass. Reports,' 1913-1916.

§ Hankel, 'Math. Ann.,' vol. 1, p. 472, Nielsen, 'Cylinderfunktionen,' p 17

|| 'Lond Math Soc. Proc.,' (2), vol. 16, pp. 153, 165.

and, observing that $n\sec\beta\sinh w - nw$, *quâ* function of w, has stationary points at $w = \pm i\beta$, we take as the contour of integration a portion of the curve on which

$$\mathrm{I}(\sec\beta\sinh w - w) = \mathrm{I}(\sec\beta\sinh i\beta - i\beta),$$

in accordance with the principles of the method of steepest descents

Next define a new complex variable t by the equation $w = t + i\beta$, and we get

$$\mathrm{H}_n^{(1)}(n\sec\beta) = \frac{1}{\pi i}e^{ni(\tan\beta-\beta)}\int_{-\infty-i\beta}^{\infty+\pi i-i\beta} e^{-n\tau}\,dt,$$

where
$$-\tau = \sinh t - t + i\tan\beta(\cosh t - 1)$$

To investigate the form of the contour we write $u+iv$ for t, where u, v, are real regarding u, v, as Cartesian co-ordinates, the equation of the contour is

$$\sin(\beta+v)\cosh u = \sin\beta + v\cos\beta$$

This curve has a double point at the origin,[*] and the lines $v = -\beta$, $v = \pi-\beta$, are asymptotes, the branch of the curve which passes from $(-\infty, -\beta)$ to the origin and thence to $(\infty, \pi-\beta)$ forms a contour of the requisite type, at any point of it u and v are both positive or both negative, while dv/du is always positive. As t describes the contour, τ is real and decreases from $+\infty$ to 0 (at the origin), and then increases to $+\infty$; the value of τ on the contour is

$$u - \sec\beta\sinh u\cos(v+\beta)$$

4. We can now embark on an investigation of the existence of zeros of $\mathrm{J}_n(x)$, and obtain a rough approximation to their positions

Since each of the integrals

$$\int_{-\infty}^{\infty} e^{-n\tau}\,du, \qquad \int_{-\beta}^{\pi-\beta} e^{-n\tau}dv,$$

is positive, it is clear that, if we regard β as a variable, and if we *define*

$$\arg\int_{-\infty-i\beta}^{\infty+\pi i-i\beta} e^{-n\tau}\,dt$$

to be a positive acute angle when $\beta = 0$, and to vary continuously with β, it will *always* be a positive acute angle, we shall call this angle[†] χ, and then define the angle ψ by the equation

$$\psi = n(\tan\beta - \beta) + \chi.$$

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[*] This statement, together with those immediately following, is contained in Debye's paper I have proved further, 'Camb. Phil. Soc. Proc,' vol. 19, p. 105 (1917), that dv/du cannot exceed $\sqrt{3}$

[†] From the footnote to §3, it follows that $0 < \chi < \frac{1}{2}\pi$.

It is clear that

$$i\,\mathrm{H}_n^{(1)}(n\sec\beta) = \rho e^{\psi i}$$

where ρ is positive; the precise value of ρ is

$$\left|\frac{1}{\pi}\int_{-\infty-i\beta}^{\infty+\pi i-i\beta} e^{-n\tau}\,dt\right|.$$

It is evident that

$$\mathrm{J}_n(x) = \rho\sin\psi, \qquad \mathrm{Y}_n(x) = -\rho\cos\psi,$$

and so the *only* zeros of $\mathrm{J}_n(x)$ are derived[*] from the values of π, 2π, 3π, ..., of ψ.

It will now be shown that, as x increases, ψ also increases, and so to each of the specified values of ψ corresponds *one and only one positive zero of* $\mathrm{J}_n(x)$.

Since $\qquad \tan\psi = -\mathrm{J}_n(x)/\mathrm{Y}_n(x),$

it is clear that

$$\frac{d\psi}{dx} = \frac{\mathrm{J}_n(x)\,\mathrm{Y}_n'(x) - \mathrm{J}_n'(x)\,\mathrm{Y}_n(x)}{\mathrm{J}_n^2(x) + \mathrm{Y}_n^2(x)} = \frac{2/(\pi x)}{\mathrm{J}_n^2(x) + \mathrm{Y}_n^2(x)},$$

by a well known formula. The last expression is positive, and so ψ is an increasing function of x.

Hence, as x increases while n remains fixed, ψ increases steadily, and so it passes through the value $m\pi$ once only (m being any one of the integers 1, 2, 3, ..), and, when $\psi = m\pi$, β has a value such that $n(\tan\beta - \beta)$ lies between $(m-\frac{1}{2})\pi$ and $m\pi$.

For example, taking $n = 1000$, $\beta = \frac{1}{3}\pi$, we get $n\pi^{-1}(\tan\beta - \beta)$ equal to 218.0, and the value of $\psi\pi^{-1}$ with these values of n and β exceeds this by a positive number less than $\frac{1}{8}$. Hence $\mathrm{J}_{1000}(x)$ has 218 zeros between 1000 and 2000.

5. The results just obtained concerning the zeros of $\mathrm{J}_n(x)$ are derived from the theorem that $d\psi/dx$ is not negative. More precise results can be obtained from the deeper theorem that $d\chi/dx$ is not negative; this theorem we now proceed to prove.

It is easy to show that

$$\begin{aligned} d\chi/dx &= d\psi/dx - n\,d(\tan\beta - \beta)/dx \\ &= \frac{2}{\pi x\{\mathrm{J}_n^2(x) + \mathrm{Y}_n^2(x)\}} - \frac{\sqrt{(x^2 - n^2)}}{x}, \end{aligned}$$

and so the required theorem is equivalent to the theorem that

$$(x^2 - n^2)^{\frac{1}{2}}\{\mathrm{J}_n^2(x) + \mathrm{Y}_n^2(x)\} \leq 2/\pi.$$

---

[*] Since $\chi > 0$ and $\tan\beta - \beta \geq 0$, it is evident that ψ is never zero, and since $\mathrm{J}_n(-x) = e^{\pm n\pi i}\,\mathrm{J}_n(x)$ it follows that there is a one-one correspondence between the positive and the negative zeros of $\mathrm{J}_n(x)$.

Now, if $x\to\infty$ while n is fixed, the limit of the function on the left is $2/\pi$, and, consequently, if we can prove that $(x^2-n^2)^{\frac{1}{2}}\{J_n^2(x)+Y_n^2(x)\}$ is an increasing function of x (i.e., that its differential coefficient is not negative), we shall obtain the desired result.

The theorem that $(x^2-n^2)^{\frac{1}{2}}\{J_n^2(x)+Y_n^2(x)\}$ is an increasing function of x seems to be of a deeper nature than might have been anticipated; for example, the proof of it is much more elaborate than the proof of the theorem* that

$$J_n^2(x)+\{xJ_n'(x)\}^2/(x^2-n^2)$$

is a decreasing function of x.

Take the formula given by Nicholson†

$$J_n^2(x)+Y_n^2(x) = 8\pi^{-2}\int_0^\infty K_0(2x\sinh\lambda)\cosh 2n\lambda\,d\lambda,$$

whence the derivate of $(x^2-n^2)^{\frac{1}{2}}\{J_n^2(x)+Y_n^2(x)\}$ is a positive multiple of

$$\int_0^\infty \{xK_0(2x\sinh\lambda)+2(x^2-n^2)\sinh\lambda\, K_0'(2x\sinh\lambda)\}\cosh 2n\lambda\,d\lambda.$$

To prove that this is positive when $x>n$, we transform it by integrating a portion twice by parts, thus

$$\begin{aligned}
&2n^2\int_0^\infty \sinh\lambda\, K_0'(2x\sinh\lambda)\cosh 2n\lambda\,d\lambda \\
&\quad = \Big[n\sinh\lambda\, K_0'(2x\sinh\lambda)\sinh 2n\lambda\Big]_0^\infty - n\int_0^\infty\{\cosh\lambda\, K_0'(2x\sinh\lambda) \\
&\qquad\qquad + 2x\sinh\lambda\cosh\lambda\, K_0''(2x\sinh\lambda)\}\sinh 2n\lambda\,d\lambda \\
&\quad = -n\int_0^\infty 2x\sinh\lambda\cosh\lambda\, K_0(2x\sinh\lambda)\sinh 2n\lambda\,d\lambda \\
&\quad = \Big[-x\sinh\lambda\cosh\lambda\, K_0(2x\sinh\lambda)\cosh 2n\lambda\Big]_0^\infty \\
&\qquad\qquad + \int_0^\infty\{x\cosh 2\lambda\, K_0(2x\sinh\lambda) \\
&\qquad\qquad + 2x^2\sinh\lambda\cosh^2\lambda\, K_0'(2x\sinh\lambda)\}\cosh 2n\lambda\,d\lambda,
\end{aligned}$$

the result has been simplified after the first step by using the differential equation satisfied by K_0.

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* 'Lond. Math. Soc. Proc.,' (2), vol. 16, p. 169.

† 'Quart. Journ. Math.,' vol. 42, p. 221 (1911); a complete proof of this formula requires an investigation similar to that given below in §8. As usual, $K_0(\xi)$ is the Bessel function of Basset's type; it is equal to $\int_0^\infty e^{-\xi\cosh\phi}\,d\phi$, and it satisfies the differential equation

$$\xi K_0''(\xi)+K_0'(\xi)-\xi K_0(\xi)=0.$$

Consequently we have

$$\tfrac{1}{8}\pi^2(x^2-n^2)^{\frac{1}{2}}\frac{d}{dx}\left[(x^2-n^2)^{\frac{1}{2}}\{J_n^2(x)+\mathbf{Y}_n^2(x)\}\right]$$

$$=\int_0^\infty\{-2x\sinh^2\lambda\,K_0(2x\sinh\lambda)-2x^2\sinh^3\lambda\,K_0'(2x\sinh\lambda)\}\cosh 2n\lambda\,d\lambda$$

$$=\int_0^\infty\Big\{-2x\sinh^2\lambda\,K_0(2x\sinh\lambda)$$

$$-x\sinh^3\lambda\operatorname{sech}\lambda\frac{d}{d\lambda}K_0(2x\sinh\lambda)\Big\}\cosh 2n\lambda\,d\lambda$$

$$=\int_0^\infty x\,K_0(2x\sinh\lambda)\Big\{-2\sinh^2\lambda\cosh 2n\lambda$$

$$+\frac{d}{d\lambda}(\sinh^3\lambda\operatorname{sech}\lambda\cosh 2n\lambda)\Big\}d\lambda,$$

on integrating the second portion of the integral by parts

Now $K_0(2x\sinh\lambda)$ is positive, and

$$-2\sinh^2\lambda\cosh 2n\lambda+\frac{d}{d\lambda}(\sinh^3\lambda\operatorname{sech}\lambda\cosh 2n\lambda)$$

$$=\tanh^2\lambda\cosh 2n\lambda+2n\sinh^3\lambda\operatorname{sech}\lambda\sinh 2n\lambda\geqq 0,$$

so that the last integral has an integrand which is not negative, *i.e*

$$\frac{d}{dx}[(x^2-n^2)^{\frac{1}{2}}\{J_n^2(x)+\mathbf{Y}_n^2(x)\}]\geqq 0$$

It follows at once that

$$(x^2-n^2)^{\frac{1}{2}}\{J_n^2(x)+\mathbf{Y}_n^2(x)\}\leqq\lim_{x\to\infty}[(x^2-n^2)^{\frac{1}{2}}\{J_n^2(x)+\mathbf{Y}_n^2(x)\}]=2/\pi,$$

and so, from the formula given at the beginning of the section,

$$d\chi/dx\geqq 0$$

6 It may be observed, in passing, that $K_0(2x\sinh\lambda)$ is a decreasing function* of x, when λ is assigned, if $x\geqq 0$ (whether x be greater or less than n), it follows from Nicholson's integral that

$$J_n^2(x)+\mathbf{Y}_n^2(x)$$

is a decreasing function of x

Since $J_n(x)$ is an increasing function of x when $0<x\leqq n$,† it follows *a fortiori* that $\mathbf{Y}_n^2(x)$ is a decreasing function of x in this range, and since

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* This is evident from the formula

$$K_0(\xi)=\int_0^\infty e^{-\xi\cosh\phi}\,d\phi$$

† 'London Math. Soc. Proc,' (2), vol 16, p 165.

$\mathrm{Y}_n(+0) = -\infty$, it is now evident that, when $0 \le x \le n$, $-\mathrm{Y}_n(x)$ is a positive decreasing function of x.

7. From the result of § 5 we can estimate bounds for χ when n is given. We have

$$\mathrm{J}_n(x) + i\mathrm{Y}_n(x) = i\rho e^{i(n\tan\beta - n\beta + \chi)}$$

and, from Hankel's asymptotic expansion, when x is large,

$$\mathrm{J}_n(x) + i\mathrm{Y}_n(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} e^{ix - \frac{1}{2}(n+\frac{1}{2})\pi i}\{1 + O(1/x)\}$$

On comparing the arguments of these two expressions, we see that, when x is large and n remains fixed,

$$\chi = x - (\tfrac{1}{2}n + \tfrac{1}{4})\pi - \sqrt{(x^2 - n^2)} + n\cos^{-1}(n/x) + \tfrac{1}{2}\pi + O(1/x)$$

Hence $$\lim_{x\to\infty} \chi = \tfrac{1}{4}\pi$$

Also, when $x = n$, we have

$$-i\rho e^{i\chi} = \mathrm{J}_n(n) + i\mathrm{Y}_n(n),$$

and so, when $x = n$, $\quad \chi = \tan^{-1}\{-\mathrm{J}_n(n)/\mathrm{Y}_n(n)\}$,

the expression on the right being a positive acute angle.

Since $d\chi/dx$ is not negative, it follows that, when $x \ge n$, we have

$$\tan^{-1}\{-\mathrm{J}_n(n)/\mathrm{Y}_n(n)\} \le \chi < \tfrac{1}{4}\pi$$

To form an idea of the magnitude of the lower bound of χ we write down Debye's formulæ, valid for large values of n, namely,

$$\mathrm{J}_n(n) = \frac{1}{2\pi\sqrt{3}}\left\{\frac{\Gamma(\frac{1}{3})}{(\frac{1}{6}n)^{1/3}} - \frac{\Gamma(\frac{2}{3})}{420(\frac{1}{6}n)^{5/3}} - \quad\right\}$$

$$-\mathrm{Y}_n(n) = \frac{1}{2\pi}\left\{\frac{\Gamma(\frac{1}{3})}{(\frac{1}{6}n)^{1/3}} + \frac{\Gamma(\frac{2}{3})}{420(\frac{1}{6}n)^{5/3}} - \quad\right\},$$

whence it follows that

$$\lim_{n\to\infty}\{-\mathrm{J}_n(n)/\mathrm{Y}_n(n)\} = 1/\sqrt{3} = \tan\tfrac{1}{6}\pi$$

We write $$\tan^{-1}\{-\mathrm{J}_n(n)/\mathrm{Y}_n(n)\} = \gamma_n,$$

so that $$\lim_{n\to\infty}\gamma_n = \tfrac{1}{6}\pi,$$

and it appears from Debye's formulæ that, when n is sufficiently large, γ_n is an increasing function of n.

The following Table gives the sexagesimal measure (to the nearest half-minute) of the angle whose circular measure is γ_n; it exhibits the closeness of γ_n to its limit, even when n is quite small:—

n	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	3
γ_n	0	26° 34′	29° 23½′	29° 38′	29° 45′	29° 51′

n	4	6	8	10	12	∞
γ_n	29° 54′	29° 56½′	29° 57½′	29° 58′	29° 58½′	30°

A more precise estimate than the estimate of § 4 can now be made for the positions of the zeros of $J_n(n\sec\beta)$, where β is regarded as the variable; for $\psi = m\pi$ when β has some value for which $n(\tan\beta-\beta)$ lies between $(m-\frac{1}{4})\pi$ and $m\pi-\gamma_n$ and the difference between those two expressions (when n is not small) slightly exceeds $\frac{1}{12}\pi$.

8. The Table just given indicates the possibility of the truth of the proposition that $-J_n(n)/Y_n(n)$ is an increasing function of n for all positive values of n. Of this proposition we shall now give a formal proof.

It may first be pointed out that (with the notation of § 3)

$$J_n(n)+iY_n(n) = \frac{1}{\pi i}\int_{-\infty}^{\infty+\pi i} e^{-n\tau}\,dt,$$

where the contour consists of the negative part of the real axis and a curve on the right of the imaginary axis, it follows that

$$J_n(n) = \frac{1}{\pi}\int_0^{\pi} e^{-n\tau}\,dt$$

$$-Y_n(n) = \frac{1}{\pi}\int_{\infty}^{0} e^{-n\tau}\,du+\frac{1}{\pi}\int_0^{\infty} e^{-n\tau}\,du,$$

and the presence of the first integral in the expression for $-Y_n(n)$ seems to render it impossible to apply successfully a method which has been employed* in proving theorems concerning pairs of Bessel functions of the first kind.

It is, however, possible to construct a proof with the aid of Hardy's theory† of divergent integrals. The theorem to be proved is, effectively, that

$$J_n(n)\frac{d}{dn}Y_n(n)-Y_n(n)\frac{d}{dn}J_n(n)\geq 0.$$

Now the expression on the left is equal to

$$\left\{J_n(x)\frac{\partial}{\partial n}Y_n(x)-Y_n(x)\frac{\partial}{\partial n}J_n(x)\right\}+\left\{J_n(x)\frac{d}{dx}Y_n(x)-Y_n(x)\frac{d}{dx}J_n(x)\right\},$$

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* 'Lond. Math. Soc. Proc.,' (2), vol. 16, pp. 161–165.

† 'Quart. Journ. Math.,' vol. 35, pp. 22–66 (1904); 'Camb. Phil. Soc. Trans.,' vol. 21, pp. 1–48 (1908).

where x has to be put equal to n after the differentiations have been performed

If we replace the Bessel functions in the first part of this expression by Hankel-Nielsen functions and apply the theorem quoted in § 4 to the second part of the expression, it is evident that we need to prove that

$$\tfrac{1}{2}n\pi i\left[H_n^{(1)}(x)\frac{\partial}{\partial n}H_n^{(2)}(x)-H_n^{(2)}(x)\frac{\partial}{\partial n}H_n^{(1)}(x)\right]_{x=n}+1=0$$

Take the formulae

$$H_n^{(1)}(x)=\frac{1}{\pi i}\int_{-\infty}^{\infty+\pi i}e^{x\sinh w-nw}\,dw,$$

$$H_n^{(2)}(x)=-\frac{1}{\pi i}\int_{-\infty}^{\infty-\pi i}e^{x\sinh W-nW}\,dW$$

in which it is permissible to differentiate with regard to n under the sign of integration

From the manner in which the integrand tends to zero as $|w|\to\infty$ on the contour specified, it is easy to see that we may introduce an exponential factor of Hardy's type, so that we get*

$$H_n^{(1)}(x)=\operatorname*{Lim}_{\lambda\to+0}\frac{1}{\pi i}\int_{-\infty}^{\infty+\pi i}\exp(-\lambda w^2)\,e^{x\sinh w-nw}\,dw$$

By Cauchy's theorem the contour may be deformed into the line $I(w)=\frac{1}{2}\pi$, so long as $\lambda\neq0$ writing $u+\frac{1}{2}\pi i$ for w, where u is real, we see that

$$H_n^{(1)}(x)=\operatorname*{Lim}_{\lambda\to+0}\frac{1}{\pi i}e^{-\frac{1}{2}n\pi i}\int_{-\infty}^{\infty}\exp\{-\lambda(u+\tfrac{1}{2}\pi i)^2\}\,e^{ix\cosh u-nu}\,du$$

As $|\exp(ix\cosh u-nu)|$ does not tend to zero as $u\to-\infty$, it is no longer necessarily permissible† to interchange the limiting process with the process of integration

In Hardy's notation the last result is written

$$H_n^{(1)}(x)=\frac{1}{\pi i}e^{-\frac{1}{2}n\pi i}\quad G\int_{-\infty}^{\infty}e^{ix\cosh u-nu}\,du,$$

the expression on the right being described as a "generalised integral"

In like manner, we find that

$$H_n^{(2)}(x)=-\frac{1}{\pi i}e^{\frac{1}{2}n\pi i}\quad G\int_{-\infty}^{\infty}e^{-ix\cosh U-nU}\,dU$$

with an implied exponential factor, $\exp\{-\lambda(U-\frac{1}{2}\pi i)^2\}$

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* That is to say, the sign Lim is commutative with the integral sign this will not be the case with subsequent integrals.

† *Cf* Bromwich 'Theory of Infinite Series, p 422

By considering the contour integrals quoted earlier in the section, it is easy to see that the differentiation with regard to n is commutative with the G-integral signs.

We now substitute the G-integrals in the expression

$$H_n^{(1)}(x)\frac{\partial}{\partial n}H_n^{(2)}(x)-H_n^{(2)}(x)\frac{\partial}{\partial n}H_n^{(1)}(x)$$

and since the necessary conditions concerning convergence are satisfied for any assigned positive value of λ we may regard the product of two integrals such as

$$\int_{\infty}^{\infty} e_1(u)f(u)\,du \times \int_{\infty}^{\infty} e_2(U)F(U)\,dU,$$

(where e_1, e_2, denote the exponential factors) as a *double integral*

$$\int_{\infty}^{\infty}\int_{\infty}^{\infty} e_1(u)e_2(U)f(u)F(U)\quad(dU\,du)$$

We thus find that

$$\pi^2\left|H_n^{(1)}(x)\frac{\partial}{\partial n}H_n^{(2)}(x)-H_n^{(2)}(x)\frac{\partial}{\partial n}H_n^{(1)}(x)\right|_{x=a}$$

$$= G\int_{-\infty}^{\infty}\int_{\infty}^{\infty}(\pi i-U+u)\,e^{ia(\cosh u-\cosh U)-n(u+U)}\,(dU\,du),$$

where the factor $\exp\{-\lambda(u+\frac{1}{2}\pi i)^2-\lambda(U-\frac{1}{2}\pi i)^2\}$ is implied by the symbol G.

We now change the variables by writing

$$U=\xi-\eta\qquad u=\xi+\eta,$$

the usual transformation formula holds, in view of the presence of the exponential factor, and the integral becomes

$$2G\int_{\infty}^{\infty}\int_{\infty}^{\infty}(\pi i+2\eta)\,e^{2ia\sinh\xi\sinh\eta-2n\xi}\,(d\xi\,d\eta),$$

with an implied factor

$$\exp\{-2\lambda\xi^2-2\lambda(\eta+\tfrac{1}{2}\pi i)^2\}$$

In view of the absolute convergence of the integral, we may replace it by the repeated integral in which the integration with regard to η is performed first.

We therefore consider

$$\int_{-\infty}^{\infty}(\pi i+2\eta)\exp\{-2\lambda(\eta+\tfrac{1}{2}\pi i)^2+2ia\sinh\xi\sinh\eta\}\,d\eta$$

The modulus of this integral cannot exceed

$$\int_{-\infty}^{\infty}(\pi+2|\eta|)\exp(-2\lambda\eta^2+\tfrac{1}{2}\lambda\pi^2)\,d\eta=\{(2\lambda)^{-1/2}\pi^{3/2}+\lambda^{-1}\}\exp(\tfrac{1}{2}\lambda\pi^2)$$

Further, when ξ is positive, we may deform the η-path of integration into the contour $I(\eta) = \frac{1}{2}\pi i$, on writing $\eta = v + \frac{1}{2}\pi i$, where v is real, the integral becomes

$$2\int_{-\infty}^{\infty}(v+\pi i)\exp\{-2\lambda(v+\pi i)^2 - 2n\sinh\xi\cosh v\}\,dv$$

$$= 2\exp(2\lambda\pi^2)\int_{-\infty}^{\infty}(v+\pi i)e^{-2n\sinh\xi\cosh v}\,dv$$

$$+2\exp(2\lambda\pi^2)\,.\int_{-\infty}^{\infty}(v+\pi i)\{1-\exp(-2\lambda v^2-4\lambda v\pi i)\}e^{-2n\sinh\xi\cosh v}\,dv$$

$$= 4\pi i\exp(2\lambda\pi^2)\int_0^{\infty}e^{-2n\sinh\xi\cosh v}\,dv$$

$$+4\exp(2\lambda\pi^2)\int_0^{\infty}[iv\sin(4\lambda\pi v)\exp(-2\lambda v^2)$$

$$+\pi i\{1-\cos(4\lambda\pi v)\exp(-2\lambda v^2)\}]e^{-2n\sinh\xi\cosh v}\,dv,$$

on bisecting the ranges of integration and combining the integrals so obtained

We now approximate to

$$v\sin(4\lambda\pi v)\exp(-2\lambda v^2)+\pi\{1-\cos(4\lambda\pi v)\exp(-2\lambda v^2)\}$$

The modulus of this expression does not exceed

$$|v\sin(4\lambda\pi v)\exp(-2\lambda v^2)+\pi\{1-\cos(4\lambda\pi v)\}$$

$$+\pi|\cos(4\lambda\pi v)|\{1-\exp(-2\lambda v^2)\}$$

$$\le 4\lambda\pi v^2+2\pi\sin^2(2\lambda\pi v)+\pi\quad 2\lambda v^2$$

$$\le v^2(6\lambda\pi+8\lambda^2\pi^3)$$

Hence, when $\xi > 0$, we have (for some value of $|\theta|$ not exceeding 1)

$$\int_{-\infty}^{\infty}(\pi i+2\eta)\exp\{-2\lambda(\eta+\tfrac{1}{2}\pi i)^2+2ni\sinh\xi\sinh\eta\}\,d\eta$$

$$= 4\pi i\exp(2\lambda\pi^2)\int_0^{\infty}e^{-2n\sinh\xi\cosh v}\,dv$$

$$+2i\theta(6\lambda\pi+8\lambda^2\pi^3)\exp(2\lambda\pi^2)\int_{-\infty}^{\infty}v^2e^{-2n\sinh\xi\cosh v}\,dv$$

$$= 4\pi i\exp(2\lambda\pi^2)K_0(2n\sinh\xi)$$

$$+2i\theta\quad(6\lambda\pi+8\lambda^2\pi^3)\exp(2\lambda\pi^2)\left[\frac{\partial^2}{\partial m^2}\int_{-\infty}^{\infty}e^{-mv-2n\sinh\xi\cosh v}\,dv\right]_{m=0}$$

The last integral is equal to $2\sec(m\pi)K_m(2n\sinh\xi)$, where K_m denotes the Bessel function of Basset's type expressible in terms of "the Bessel functions with imaginary argument" by the formula

$$K_m = \tfrac{1}{2}\pi\cot(m\pi)\quad(I_{-m}-I_m)$$

We shall write

$$\left[\frac{\partial^2}{\partial m^2}\{\sec(m\pi)\;\mathrm{K}_m(2n\sinh\xi)\}\right]_{m=0}=\mathrm{F}_0(2n\sinh\xi)$$

By using contour integrals of Barnes' type* involving Gamma-functions, it is easy to see that, when ζ is small,

$$\mathrm{F}_0(\zeta)=O\{(\log\zeta)^3\},$$

while, when ζ is large

$$\mathrm{F}_0(\zeta)=O(e^{-\zeta}\zeta^{-3/2})$$

Thus, when $\xi>0$ we have

$$\int_{-\infty}^{\infty}(\pi i+2\eta)\exp\{-2\lambda(\eta+\tfrac{1}{2}\pi i)^2+2ni\sinh\xi\sinh\eta\}\,d\eta$$

$$=4\pi i\exp(2\lambda\pi^2)\;\mathrm{K}_0(2n\sinh\xi)$$

$$+4i\theta\;(6\lambda\pi+8\lambda^2\pi^3)\exp(2\lambda\pi^2)\;\mathrm{F}_0(2n\sinh\xi),$$

where $|\theta|\leq 1$

Next consider

$$\int_{-\infty}^{\infty}(\pi i+2\eta)\exp\{-2\lambda(\eta+\tfrac{1}{2}\pi i)^2+2ni\sinh\xi\sinh\eta\}\,d\eta,$$

when $\xi<0$, we may deform the path of integration into the contour $\mathrm{I}(\eta)=-\frac{1}{2}\pi i$ on writing $v-\frac{1}{2}\pi i$ for η, where v is real the integral becomes

$$\int_{-\infty}^{\infty}2v\exp\{-2\lambda v^2+2n\sinh\xi\cosh v\}\,dv=0$$

since the integral is an odd function of v

We now return to the repeated integral

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}(\pi i+2\eta)\exp\{-2\lambda(\eta+\tfrac{1}{2}\pi i)^2+2ni\sinh\xi\sinh\eta\}\,d\eta\right]$$

$$\times\exp(-2\lambda\xi^2-2n\xi)\,d\xi$$

On dividing the ξ-range of integration into three portions, $(-\infty,-\epsilon)$, $(-\epsilon,\epsilon)$, (ϵ,∞), and substituting for the η-integral in each portion we see that the repeated integral is equal to

$$\int_{-\epsilon}^{\epsilon}\theta_1(2\lambda^{-1/2}\pi^{3/2}+\lambda^{-1})\exp(\tfrac{1}{2}\lambda\pi^2)\exp(-2\lambda\xi^2-2n\xi)\,d\xi$$

$$+\int_{\epsilon}^{\infty}\Big[4\pi i\exp(2\lambda\pi^2)\,.\,\mathrm{K}_0(2n\sinh\xi)$$

$$+4i\theta\;(6\lambda\pi+8\lambda^2\pi^3)\exp(2\lambda\pi^2)\,\mathrm{F}_0)(2n\sinh\xi)\Big]\exp(-2\lambda\xi^2-2n\xi)\,d\xi,$$

where $|\theta_1|<1$, and ϵ may be given *any* positive value

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* 'Camb Phil Soc Trans,' vol 20, pp 253–279 (1907)

The first integral is equal to

$$2\epsilon\theta_3(2\lambda^{-1/2}\pi^{3/2}+\lambda^{-1})\exp(\tfrac{1}{2}\lambda\pi^2)\exp 2n\epsilon,$$

where $|\theta_3| < 1$.

Now make $\epsilon \to 0$; we find that the repeated integral is equal to

$$4\pi i\exp(2\lambda\pi^2)\int_0^\infty \mathrm{K}_0(2n\sinh\xi)\exp(-2\lambda\xi^2-2n\xi)\,d\xi$$
$$+4i(6\lambda\pi+8\lambda^2\pi^3)\int_0^\infty \theta\,\mathrm{K}_0(2n\sinh\xi)\exp(-2\lambda\xi^2-2n\xi)\,d\xi.$$

The second integral is convergent and its modulus does not exceed

$$\int_0^\infty \mathrm{K}_0(2n\sinh\xi)\,e^{-2n\xi}\,d\xi;$$

this is finite and independent of λ.

Further, in the first of the two integrals, the operation of proceeding to the limit (when $\lambda \to 0$) is commutative with the integral sign, since K_0 tends exponentially to zero as $n \to \infty$, and therefore we have proved that

$$2\mathrm{G}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\pi i+2\eta)\,e^{2ni\sinh\xi\sinh\eta-2n\xi}\,(d\xi\,d\eta) = 8\pi i\int_0^\infty e^{-2n\xi}\,\mathrm{K}_0(2n\sinh\xi)\,d\xi.$$

Hence we have

$$\tfrac{1}{4}n\pi i\left[\mathrm{H}_n^{(1)}(x)\frac{\partial}{\partial n}\mathrm{H}_n^{(2)}(x)-\mathrm{H}_n^{(2)}(x)\frac{\partial}{\partial n}\mathrm{H}_n^{(1)}(x)\right]_{x=n}+1$$
$$= 1-2n\int_0^\infty e^{-2n\xi}\,\mathrm{K}_0(2n\sinh\xi)\,d\xi$$
$$= 1-\int_0^\infty e^{-\mu}\,\mathrm{K}_0\left(2n\sinh\frac{\mu}{2n}\right)d\mu.$$

But $2n\sinh(\frac{1}{2}\mu/n)$ is a decreasing function of n when μ is assigned, and K_0 is a positive decreasing function of its argument, and so

$$\int_0^\infty e^{-\mu}\,\mathrm{K}_0\left(2n\sinh\frac{\mu}{2n}\right)d\mu$$

is an increasing function of n, and therefore

$$\tfrac{1}{4}n\pi i\left[\mathrm{H}_n^{(1)}(x)\frac{\partial}{\partial n}\mathrm{H}_n^{(2)}(x)-\mathrm{H}_n^{(2)}(x)\frac{\partial}{\partial n}\mathrm{H}_n^{(1)}(x)\right]_{x=n}+1$$

is a decreasing function of n.

Now the last expression is equal to

$$\tfrac{1}{4}n\pi i\left[\mathrm{H}_n^{(1)}(n)\frac{d}{dn}\mathrm{H}_n^{(2)}(n)-\mathrm{H}_n^{(2)}(n)\frac{d}{dn}\mathrm{H}_n^{(1)}(n)\right],$$

and so this expression is greater than its limit when $n \to \infty$.

But, by Debye's formulæ, when n is large,

$$H_n^{(1)}(n) \sim -\frac{2}{3\pi}\Big[6^{1/3} e^{2\pi i/3} \sin\left(\frac{1}{3}\pi\right)\Gamma\left(\frac{1}{3}\right)n^{-1/3}$$
$$+\frac{1}{280}\ 6^{5/3} e^{10\pi i/3} \sin\left(\frac{5}{3}\pi\right)\Gamma\left(\frac{5}{3}\right)n^{-5/3}+ \quad \Big],$$

while $H_n^{(2)}(n)$ is the conjugate of this complex number, and so, since term-by-term differentiations are permissible, we have

$$\tfrac{1}{4}n\pi i\left[H_n^{(1)}(n)\frac{d}{dn}H_n^{(2)}(n)-H_n^{(2)}(n)\frac{d}{dn}H_n^{(1)}(n)\right] \sim 1/(105n^2) \to 0$$

as $n \to \infty$

Hence, for finite values of n

$$\tfrac{1}{4}n\pi i\left[H_n^{(1)}(x)\frac{\partial}{\partial n}H_n^{(2)}(x)-H_n^{(2)}(x)\frac{\partial}{\partial n}H_n^{(1)}(x)\right]_{x=n}+1>0,$$

and we have already seen that this is the condition that $-J_n(n)/Y_n(n)$ should be an increasing function of n

We have therefore proved the property of the Bessel function which was suggested by the Table given in §7

9 So far we have been concerned with what might be regarded as theoretical properties of the zeros of $J_n(x)$, as regards their actual calculation when a rough estimate has been obtained[*] for the value of the particular zero required, if β is not small, Debye's asymptotic expansion is available, the first two terms of this expansion are given by the formula

$$J_n(n\sec\beta) \sim \left(\frac{2}{n\pi\tan\beta}\right)^{\frac{1}{2}}\Big[\cos\left\{n\tan\beta-\beta)-\frac{1}{4}\pi\right\}$$
$$+n^{-1}\left(\frac{1}{8}+\frac{5}{24}\cot^2\beta\right)\cot\beta\cos\left\{n(\tan\beta-\beta)-\frac{3}{4}\pi\right\}+ \quad \Big],$$

When β is small, an approximation which I have recently communicated to the Cambridge Philosophical Society may be used, namely that, if $\beta \leq \frac{1}{4}\pi$,

$$J_n(n\sec\beta) \sim \tfrac{1}{3}\tan\beta\cos\{n(\tan\beta-\tfrac{1}{3}\tan^3\beta-\beta)-\tfrac{1}{6}\pi\}\quad J_{-1/3}(\tfrac{1}{3}n\tan^3\beta)$$
$$+\tfrac{1}{3}\tan\beta\cos\{n(\tan\beta-\tfrac{1}{3}\tan^3\beta-\beta)+\tfrac{1}{6}\pi\}\quad J_{1/3}(\tfrac{1}{3}n\tan^3\beta)$$

with an error less than $24/n$, which is of lower order of magnitude than the approximation (when n is large), except near the zeros of the function

To apply this formula it is necessary to have tables of the functions $J_{\pm\frac{1}{3}}$, some Tables have been given by Dinnik[†] of the functions obtained by omitting the factors $\Gamma(1 \pm \frac{1}{3})$ These are four-figure Tables with intervals 0 2, on account of the oscillation of the functions (with period roughly equal to 2π) interpolation is impracticable I have therefore constructed the following two Tables, involving not only the Bessel functions but also the associated functions $U_m(x)$, $V_m(x)$, defined by the equations

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[*] By the result obtained in §7 or otherwise

[†] 'Archiv der Math und Phys,' (3), vol 18, pp 337–338 (1911) Tables of these functions are of importance, as the functions occur in various approximate formulæ and also in the theory of elasticity

$$J_m(x) = U_m(x)\cos(x-\tfrac{1}{2}m\pi-\tfrac{1}{4}\pi)-V_m(x)\sin(x-\tfrac{1}{2}m\pi-\tfrac{1}{4}\pi)$$

$$J_{-m}(x) = U_m(x)\cos(x+\tfrac{1}{2}m\pi-\tfrac{1}{4}\pi)-V_m(x)\sin(x+\tfrac{1}{2}m\pi-\tfrac{1}{4}\pi)$$

When x is large $U_m(x)$ and $V_m(x)$ have the well-known asymptotic expansions

$$U_m(x)\sim\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left\{1-\frac{(4m^2-1^2)(4m^2-3^2)}{2!\,(8x)^2}+\ldots\right\},$$

$$V_m(x)\sim\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left\{\frac{4m^2-1^2}{1!\,(8x)}-\ldots\right\},$$

x	$J_{\frac{1}{3}}(x)$	$J_{-\frac{1}{3}}(x)$	$U_{\frac{1}{3}}(x)$	$-V_{\frac{1}{3}}(x)$	$-V_{\frac{1}{3}}(x)/U_{\frac{1}{3}}(x)$
0·00	0·0000	∞	∞	∞	0·2679
0·05	0·3270	2·5232	2·6937	0·5246	0·1948
0·10	0·4118	1·9971	2·0801	0·3469	0·1668
0·15	0·4703	1·7364	1·7768	0·2629	0·1480
0·20	0·5159	1·5672	1·5833	0·2117	0·1337
0·25	0·5534	1·4425	1·4447	0·1768	0·1224
0·30	0·5850	1·3440	1·3392	0·1517	0·1133
0·35	0·6120	1·2602	1·2535	0·1319	0·1053
0·40	0·6354	1·1879	1·1832	0·1163	0·0983
0·45	0·6555	1·1224	1·1238	0·1037	0·0923
0·50	0·6728	1·0644	1·0727	0·0934	0·0870
0·55	0·6870	1·0097	1·0280	0·0847	0·0823
0·60	0·7000	0·9582	0·9886	0·0773	0·0782
0·65	0·7103	0·9093	0·9534	0·0709	0·0744
0·70	0·7186	0·8623	0·9217	0·0654	0·0709
0·75	0·7250	0·8170	0·8930	0·0604	0·0677
0·80	0·7295	0·7731	0·8668	0·0562	0·0649
0·85	0·7323	0·7321	0·8427	0·0525	0·0622
0·90	0·7334	0·6883	0·8206	0·0491	0·0598
0·95	0·7329	0·6472	0·8001	0·0460	0·0575
1·00	0·7309	0·6069	0·7811	0·0433	0·0554
1·05	0·7275	0·5672	0·7634	0·0408	0·0535
1·10	0·7225	0·5281	0·7468	0·0386	0·0517
1·15	0·7162	0·4896	0·7312	0·0365	0·0499
1·20	0·7085	0·4516	0·7165	0·0346	0·0484
1·25	0·6995	0·4141	0·7027	0·0329	0·0469
1·30	0·6893	0·3771	0·6896	0·0314	0·0455
1·35	0·6780	0·3408	0·6773	0·0299	0·0441
1·40	0·6654	0·3049	0·6656	0·0285	0·0428
1·45	0·6518	0·2696	0·6545	0·0272	0·0416
1·50	0·6372	0·2349	0·6439	0·0261	0·0405
1·55	0·6215	0·2007	0·6338	0·0250	0·0394
1·60	0·6048	0·1672	0·6242	0·0240	0·0384
1·65	0·5873	0·1343	0·6150	0·0230	0·0374
1·70	0·5688	0·1021	0·6062	0·0221	0·0365
1·75	0·5496	0·0706	0·5977	0·0213	0·0357
1·80	0·5296	0·0397	0·5895	0·0205	0·0348
1·85	0·5088	+0·0090	0·5817	0·0198	0·0340
1·90	0·4874	−0·0196	0·5742	0·0191	0·0332
1·95	0·4654	−0·0482	0·5670	0·0184	0·0326
2·00	0·4429	−0·0758	0·5600	0·0178	0·0318

x	$J_{\frac13}(x)$	$J_{-\frac13}(x)$	$U_{\frac13}(x)$	$-V_{\frac13}(x)$	$-V_{\frac13}(x)/U_{\frac13}(x)$
2·0	0·4429	−0·0758	0·5600	0·0178	0·0318
2·2	0·3483	−0·1774	0·5346	0·0157	0·0293
2·4	0·2488	−0·2636	0·5123	0·0139	0·0272
2·6	0·1480	−0·3330	0·4926	0·0125	0·0253
2·8	+0·0490	−0·3817	0·4749	0·0112	0·0237
3·0	−0·0450	−0·4188	0·4590	0·0102	0·0222
3·2	−0·1311	−0·4336	0·4446	0·0093	0·0210
3·4	−0·2071	−0·4314	0·4314	0·0086	0·0198
3·6	−0·2707	−0·4120	0·4194	0·0079	0·0188
3·8	−0·3205	−0·3795	0·4083	0·0073	0·0179
4·0	−0·3554	−0·3331	0·3981	0·0068	0·0170
4·2	−0·3749	−0·2761	0·3886	0·0063	0·0161
4·4	−0·3780	−0·2111	0·3797	0·0058	0·0153
4·6	−0·3660	−0·1408	0·3714	0·0054	0·0147
4·8	−0·3434	−0·0681	0·3637	0·0051	0·0141
5·0	−0·3064	+0·0044	0·3563	0·0048	0·0136
5·2	−0·2589	0·0788	0·3494	0·0046	0·0131
5·4	−0·2031	0·1378	0·3429	0·0044	0·0127
5·6	−0·1415	0·1939	0·3368	0·0041	0·0123
5·8	−0·0765	0·2406	0·3309	0·0039	0·0119
6·0	−0·0106	0·2763	0·3253	0·0037	0·0114
6·2	+0·0534	0·3001	0·3201	0·0036	0·0111
6·4	0·1133	0·3114	0·3151	0·0034	0·0107
6·6	0·1670	0·3100	0·3103	0·0032	0·0104
6·8	0·2136	0·2967	0·3058	0·0031	0·0100
7·0	0·2484	0·2720	0·3013	0·0029	0·0098
7·2	0·2735	0·2374	0·2972	0·0028	0·0096
7·4	0·2872	0·1944	0·2932	0·0027	0·0093
7·6	0·2893	0·1450	0·2893	0·0026	0·0091
7·8	0·2800	0·0911	0·2856	0·0025	0·0089
8·0	0·2598	0·0350	0·2820	0·0024	0·0087

given by Hankel, these may be used for calculation with great accuracy if $x > 8$ when $m = \frac13$

The integrals of Schafheitlin

$$U_m(x),\ V_m(x) = \frac{2^{m+1}x^m}{\Gamma(m+\frac12)\,\Gamma(\frac12)}\int_0^{\frac12\pi} e^{-2x\tan\theta}\sin^{m+\frac12}\theta\,\sec^{2m+1}\theta\,{\cos\atop\sin}(m-\tfrac12)\,\theta\,d\theta,$$

which are easily derivable from Hankel's contour integrals, show at once that, when $m = \frac13$ and x is positive, $U_{1/3}(x)$ is positive and $V_{1/3}(x)$ is negative

Further, $V_{1/3}(x)/U_{1/3}(x)$ increases algebraically with x, for

$$U_{1/3}(x)\,V_{1/3}'(x) - U_{1/3}'(x)\,V_{1/3}(x)$$

$$= \left\{\frac{2^{4/3}x^{1/3}}{\Gamma(\frac56)\,\Gamma(\frac12)}\right\}^2 \int_0^{\frac12\pi}\int_0^{\frac12\pi} e^{-2x(\tan\theta+\tan\phi)}\,F(\theta,\phi)\,d\theta\,d\phi,$$

where

$$F(\theta,\phi) = 2(\sin\theta\sin\phi)^{1/6}(\cos\theta\cos\phi)^{-5/3}(\tan\phi-\tan\theta)\cos\tfrac16\theta\sin\tfrac16\phi$$

Since $F(\theta,\phi)+F(\phi,\theta)$ is positive, if we interchange the parameters θ, ϕ, in the double integral, and add the results so obtained we find at once that

$$U_{1/3}(x)\,V_{1/3}'(x) - U_{1/3}'(x)\,V_{1/3}(x)$$

is positive

Hence, as x increases from 0 to ∞, it follows that $V_{1/3}(x)/U_{1/3}(x)$ steadily increases from $-\cot\frac{5}{12}\pi$ to 0, the limit of $V_{1/3}(x)/U_{1/3}(x)$ when $x \to 0$ being derived from its expression in terms of $J_{\pm 1/3}(x)$

Now the approximate expression for $J_n\,(n\sec\beta)$ may be written in the form

$$J_n(n\sec\beta) \sim 3^{-\frac{1}{2}}\tan\beta\cos\{n(\tan\beta-\beta)-\tfrac{1}{4}\pi\}\,U_{1/3}(\tfrac{1}{3}n\tan^3\beta)$$
$$-3^{-\frac{1}{2}}\tan\beta\sin\{n(\tan\beta-\beta)-\tfrac{1}{4}\pi\}\,V_{1/3}(\tfrac{1}{3}n\tan^3\beta)$$

Hence, when $\beta \leq \frac{1}{4}\pi$ the zeros of $J_n(n\sec\beta)$ are given approximately by the equation

$$\cot\{n(\tan\beta-\beta)-\tfrac{1}{4}\pi\} = V_{1/3}(\tfrac{1}{3}n\tan^3\beta)/U_{1/3}(\tfrac{1}{3}n\tan^3\beta)$$

If we draw the graphs of the functions

$$V_{1/3}(\tfrac{1}{3}n\tan^3\beta)/U_{1/3}(\tfrac{1}{3}n\tan^3\beta) \qquad \cot\{n(\tan\beta-\beta)-\tfrac{1}{4}\pi\},$$

we see that, near the intersection of the graphs the former has a positive slope and the latter a negative slope, and a good estimate of the value of $\frac{1}{3}n\tan^3\beta$ at the intersection can be obtained by taking values of β on each side of the intersection and then using the principle of proportional parts

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The Electrostatic Problem of a Conducting Sphere in a Spherical Cavity

By ALEXANDER RUSSELL, M A, D Sc, M I E E

(Communicated by Dr C Chree, F R.S Received October 4, 1917)

The mathematical problems which arise in connection with spherical electrodes have often been discussed A *résumé* of them is given in Chap VIII, Vol I, of the second edition of the author's treatise on 'Alternating Current Theory' The problem, however, of a sphere in a spherical cavity has barely been considered. Sir W Thomson* (Lord Kelvin) gives numerical results which he says were calculated for application to the theory of conducting spheres "in the interior of a hollow insulated and electrified conductor" As a matter of fact, however, the numerical results given apply only to two equal conducting spheres external to one another

Solutions for the capacity between two spheres have been given for the case when the centre of the sphere is close to the centre of the cavity These solutions are either incorrect,* or approximately correct† only in special cases. It will be useful, therefore, to show how the complete numerical solution can be readily found in all cases

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* 'Reprint,' p 96.

The solution is required in connection with the following problems, which are of importance in physical and engineering science (1) The calculation of the capacity of a spherical condenser when the centre of the inner sphere does not coincide with the centre of the cavity, (2) the calculation of the electrostatic force acting on the inner sphere, and (3) the calculation of the maximum value of the electric stress on the dielectric The first problem arises when determining the ratio of the electrostatic to the electromagnetic unit of electricity by means of a spherical condenser The solution is also of value when constructing spherical condensers of variable capacity The solution of the second problem is of value, as it enables us to make an absolute electrometer by suspending one sphere inside another The attracting force for a given distance between them is greater in certain cases than when the spheres are external to one another, and errors due to the presence of conducting or insulating materials in the neighbourhood are obviated The final problem—the calculation of the maximum value of the potential gradient between the two spheres—arises when determining the electric strengths of insulating materials by using them as the dielectric of a condenser The electric pressure between the electrodes is increased at a definite rate, and the value of the disruptive voltage is noted The determination of the electric strengths of dielectrics is a problem of great urgency Methods of finding these values are at present being considered and tested experimentally by Panels of the Research Committee of the Institution of Electrical Engineers

1 *The Capacity between the Two Spheres*

Let q_1 and q_2 be the total charges on the inner and outer conductors respectively, and let v_1 and v_2 be their potentials If k_{11} and k_{22} be their self-capacity coefficients, and k_{12} be the mutual capacity coefficient we may write

$$q_1 = k_{11}v_1 + k_{12}v_2, \tag{1}$$

and

$$q_2 = k_{22}v_2 + k_{12}v_1 \tag{2}$$

Since q_1 must vanish when $v_1 = v_2$, we have $k_{11} = -k_{12}$

It is obvious that $-q_1$ is the charge on the inner side of the outer conductor, and therefore q_1+q_2 is the charge on the outer surface. Hence, if K be the capacity of the outside surface, we have

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* J H Jeans, 'Electricity and Magnetism,' 2nd ed ex 47, p 292 L Cohen, 'Alternating Current Problems,' p 90

† E B Rosa and N E Dorsey, "Bulletin of the Bureau of Standards," "The Determination of the Ratio of the Electric Units," vol 3, p 477

$$q_1+q_2 = Kv_2$$

But from (1) and (2)

$$q_1+q_2 = (k_{22}-k_{11})v_2,$$

and hence

$$k_{22} = K+k_{11}, \tag{3}$$

and

$$q_1 = k_{11}(v_1-v_2)$$

In the problem we are considering the value of K is immaterial, and k_{11} is the capacity C of the condenser formed by the inner and outer conductors

Let a and b be the radii of the sphere and of the cavity respectively, and let c be the distance between their centres By Kelvin's method of images we get

$$C = a+\frac{a^2b}{d^2}+\frac{a^3b^2}{d^4-a^2c^2}+\frac{a^4b^3}{d^2(d^4-2a^2c^2)-a^4c^2}$$
$$+\frac{a^5b^4}{d^4(d^4-3a^2c^2)-2a^4c^2d^2+a^4c^2(c^2-a^2)}+R_6, \tag{4}$$

where $d^2 = b^2-c^2$, and R_6 denotes the remainder of the series after the first five terms. If we write $-b$ for b in (4), we get the well known formula for k_{11} for two external spheres Except when both a and c are very small compared with b, (4) is of little use for numerical computation

If we introduce r, the radius of the sphere which cuts the two spheres orthogonally, and two new variables α and β defined by the equations,

$$\sinh\alpha = r/a, \qquad \sinh\beta = r/b, \tag{A}$$

into the formula it simplifies considerably

It is also convenient to denote $\alpha-\beta$ by ω. It is easy to show that

$$c = (b^2+r^2)^{\frac{1}{2}}-(a^2+r^2)^{\frac{1}{2}}, \tag{B}$$

$$\left.\begin{aligned} \cosh\alpha &= (b^2-a^2-c^2)/2ac \\ \cosh\beta &= (b^2+c^2-a^2)/2bc \\ \text{and} \qquad \cosh\omega &= (a^2+b^2-c^2)/2ab \end{aligned}\right\} \tag{C}$$

Writing $2s$ for $a+b+c$ we get

$$ab\sinh\omega = cr = 2\{s(s-a)(b-s)(s-c)\}^{\frac{1}{2}}, \tag{D}$$

$$\left.\begin{aligned} &\sinh(\omega/2) = \{(s-a)(b-s)/ab\}^{\frac{1}{2}}, \qquad \cosh(\omega/2) = \{s(s-c)/ab\}^{\frac{1}{2}}, \\ &\text{and} \\ &\sinh(\alpha/2) = \{s(b-s)/ac\}^{\frac{1}{2}}, \qquad \cosh(\alpha/2) = \{(s-a)(s-c)/ac\}^{\frac{1}{2}} \end{aligned}\right\} \tag{E}$$

We also have

$$\omega = 2\log_e\{\cosh(\omega/2)+\sinh(\omega/2)\}$$

These formulæ, which are adapted for logarithmic computation, enable us to calculate the values of α, β, ω, and r for any given values of a, b, and c easily and rapidly

In terms of r, α and ω, (4) becomes

$$C = c\sum_{n=0}^{n=\infty}\frac{1}{\sinh(\alpha+n\omega)} \tag{5}$$

It will be seen that C, which equals k_{11}, is given by Maxwell's formula, but it must be carefully noticed that $\omega = \alpha-\beta$, and can therefore never be greater than α. In the case of two external spheres, the author (*loc cit ante*) has shown how the series in (5) can be computed in all cases But for external spheres $\omega = \alpha+\beta$, and hence ω cannot be less than α In the problem under consideration ω cannot be greater than α, and so some of the formulæ we have previously given can only be used for this problem in special cases We have to find, therefore, suitable formulæ for the other cases It is easy to prove from (C) that as c increases from zero to $b-a$, α diminishes from infinity to zero, and ω diminishes from $\log(b/a)$ to zero In this case also ω/α increases from zero to c/b

When α is large C can be found from (4) and (5) For instance, if $\alpha+n\omega$ is not less than 5, we can write $\cosh(\alpha+n\omega) = \sinh(\alpha+n\omega)$, the error being less than 1 in 10,000 Hence, to this accuracy at least,

$$R_n = c/\sinh(\alpha+n\omega) \quad e^{\omega}/(e^{\omega}-1), \tag{6}$$

where $$e^{\omega}/(e^{\omega}-1) = \tfrac{1}{2}+\tfrac{1}{2}\{[(b+a)^2-c^2]/[(b-a)^2-c^2]\}^{\frac{1}{2}}$$
$$= \tfrac{1}{2}+\tfrac{1}{2}\{s(s-c)/(s-a)(b-s)\}^{\frac{1}{2}}$$

When α is not less than 5, and b^2 is therefore not less than a^2+c^2+10ac, we have

$$C = \frac{c}{\sinh\alpha} \quad \frac{e^{\omega}}{e^{\omega}-1} = \frac{a}{2}+\frac{a}{2}\left\{\frac{s(s-c)}{(s-a)(b-s)}\right\}^{\frac{1}{2}}$$

Finally, when c is zero,

$$C = ab/(b-a)$$

When α and ω are small, a series very suitable for computing purposes can be found as follows. Expanding the terms in (5) in powers of $e^{-\alpha}$ and $e^{-\omega}$, and summing the series in a different order, we get

$$R_n = a(1-e^{-2\alpha})e^{-n\omega}\{1/(1-e^{-\omega})+e^{-2(\alpha+n\omega)}/(1-e^{-3\omega})$$
$$+e^{-4(\alpha+n\omega)}/(1-e^{-5\omega})+\dots\} \tag{7}$$

Except when α and $n\omega$ are very minute, this series converges with most satisfactory rapidity It can be used in nearly every practical case

Another useful expression which enables us to compute C rapidly can be obtained as follows Writing D for $\partial/\partial\alpha$, we get by Taylor's Theorem

$$
\begin{aligned}
f(\alpha)+f(\alpha+\omega)+f(\alpha+2\omega)+\ \ldots &= \{1+e^{\omega D}+e^{2\omega D}+\ \ldots\}f(\alpha) \\
&= \frac{1}{1-e^{\omega D}}f(\alpha) \\
&= A-\frac{1}{\omega}\int f(\alpha)\,\partial\alpha+\tfrac{1}{2}f(\alpha)-\frac{B_1\omega}{2!}f'(\alpha) \\
&\quad +\frac{B_3\omega^3}{4!}f'''(\alpha)-\ \ldots, \qquad (F)
\end{aligned}
$$

where B_1, B_3, .. are Bernoulli's numbers, and A is a constant which has to be determined in any particular case The limitations of this theorem can be stated by aid of a theorem given in Whittaker and Watson's 'Modern Analysis,' § 7·21 For our purpose it is sufficient to remember that the series on the right-hand side is semi-convergent, but in any case in which it is applicable the value of the last term included in the computation enables us to see the accuracy obtained

Applying theorem (F) to formula (5), we get

$$
\begin{aligned}
C = \frac{t}{\omega}\Big\{&\tfrac{1}{2}\log\frac{\cosh\alpha+1}{\cosh\alpha-1}+\frac{\omega}{2\sinh\alpha}+\frac{\omega^2}{12}\,\frac{\cosh\alpha}{\sinh^2\alpha} \\
&-\frac{\omega^4}{720}\,\frac{\cosh\alpha}{\sinh^2\alpha}\,\frac{\sinh^2\alpha+6}{\sinh^3\alpha} \\
&+\frac{\omega^6}{30240}\,\frac{\cosh\alpha}{\sinh^2\alpha}\,\frac{\sinh^4\alpha+60\sinh^2\alpha+120}{\sinh^4\alpha}-\ \ldots\Big\}, \qquad (8)
\end{aligned}
$$

the constant A being equal to zero That A vanishes can easily be seen by making α very great in (8)

In many cases the simpler formula

$$
\begin{aligned}
C = \frac{t}{\omega}\Big\{&\frac{1}{\cosh\alpha}+\frac{1}{3\cosh^3\alpha}+\frac{1}{5\cosh^5\alpha}+\ \ldots\Big\} \\
&+\frac{\alpha}{2}+\frac{\omega\alpha}{12}\cdot\frac{\cosh\alpha}{\sinh\alpha}-\Big(\frac{\omega\alpha}{12}\,\frac{\cosh\alpha}{\sinh\alpha}\Big)\frac{\omega^2}{60}\,\frac{\sinh^2\alpha+6}{\sinh^3\alpha}+\ \ldots \qquad (9)
\end{aligned}
$$

enables us to find C with high accuracy

If we put $c = 0$, we get from (8)

$$
C = a\left[\frac{1}{\omega}+\frac{1}{2}+\frac{B_1}{2!}\,\omega-\frac{B_3}{4!}\,\omega^3+\ \ldots\right] = a\,e^{\omega}/(e^{\omega}-1) = ab/(b-a)
$$

When α is small the following formula readily deducible from (8) may be used

$$
\begin{aligned}
C = \frac{t}{\omega}\Big\{&-\psi\Big(\frac{\alpha}{\omega}\Big)+\log\frac{2}{\omega}+\frac{\omega^2}{72}+\frac{7\,\omega^4}{43200}+\frac{31\,\omega^6}{3810240}+\ \ldots \\
&+\frac{\alpha\beta}{12}-\frac{7\alpha^2\beta^2}{1440}+\frac{31\alpha^3\beta^3}{90720}-\frac{31\alpha^2\beta^2\omega^2}{181440}+\ \ldots\Big\}, \qquad (10)
\end{aligned}
$$

where $\psi(x)$ is the logarithmic derivate of the gamma function

Since $C = k_{11}$, and for an infinitely thin shell $k_{22} = C+b$, we get by symmetry,

$$C+b = \frac{r}{\omega}\left\{-\psi\left(\frac{\beta}{\omega}\right)+\log\frac{2}{\omega}+\frac{\omega^2}{72}+\frac{7\omega^4}{43200}+\frac{31\omega^6}{3810240}+\right.$$

$$\left.+\frac{\beta(\beta-\omega)}{12}-\frac{7\beta^2(\beta-\omega)^2}{1440}+\frac{31\beta^4(\beta-\omega)^3}{90720}-\frac{31\beta^2\omega^2(\beta-\omega)^2}{181440}-\quad\right\} \qquad (11)$$

If we subtract (10) from (11), and notice that $\alpha = \beta+\omega$, we get, as we ought,

$$b = r\left\{\frac{1}{\beta}-\frac{\beta}{6}+\frac{7\beta^3}{360}-\frac{31\beta^5}{15120}+\quad\right\} = \frac{r}{\sinh\beta}$$

For computing purposes when α and ω are small, either equation (10) or (11) may be used. In the particular case when $\beta = \omega$, and therefore $c = a$, we get

$$C = -b+\frac{r}{\omega}\left\{\gamma+\log\frac{2}{\omega}+\frac{\omega^2}{72}+\frac{7\omega^4}{43200}+\frac{31\omega^6}{3810240}+\quad\right\}, \qquad (12)$$

where $\gamma = 0.5772157$ (Euler's constant).

The series (10) and (11) can also be readily deduced from the general theorem (15) given in a preceding paper.* For the function $F(x)$ previously used we have written $-\psi(1/x)-\log x$. Tables of the values of $\psi(x)$ are published.†

When the distance x between the spheres is small compared with $b-a$, the maximum possible distance between their centres, we get, on substituting $b-a-x$ for c in formulæ (C) and (D), and expanding in powers of $x/(b-a)$, the following approximate equations:—

$$r^2 = \frac{2abx}{b-a}\left\{1+\frac{a^2+ab+b^2}{2ab}\,\frac{x}{b-a}+\frac{(a+b)^2}{2ab}\,\frac{x^2}{(b-a)^2}\right\}, \qquad (13)$$

$$\frac{\alpha}{\omega} = \frac{b}{b-a}+\frac{b+a}{3(b-a)}\,\frac{x}{b-a}+\frac{(b+a)\{25ab-2(a^2+b^2)\}}{90ab(b-a)}\,\frac{x^2}{(b-a)^2} \qquad (14)$$

and writing $k = 1+(b-a)^2/3ab$, we get

$$\frac{r}{\omega} = \frac{ab}{b-a}\left\{1+k\,\frac{x}{b-a}+\frac{3+9k-2k^2}{10}\,\frac{x^2}{(b-a)^2}\right\} \qquad (15)$$

Neglecting the sixth and higher powers of small quantities in (10) we get

$$C = \frac{r}{\omega}\left\{\tfrac{1}{2}\log\frac{2ab}{(b-a)x}-\psi\left(\frac{a}{\omega}\right)+\tfrac{1}{3}(k+\tfrac{1}{4})\frac{x}{b-a}\right.$$

$$\left.+\frac{1}{3600}(331+948k-624k^2)\frac{x^2}{(b-a)^2}\right\} \qquad (16)$$

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* 'Roy. Soc. Proc.,' A, vol. 82, p. 527 (1909).

† *E.g.*, in Russell's 'Alternating Currents,' 2nd ed., vol. 1, p. 241.

The values of r/ω and α/ω can be found from (14) and (15), and there is no difficulty in calculating the value of $\psi(x)$ in every case by means of the formulæ

$$\psi(x) = \log x - 1/2x - B_1/2x^2 + B_3/4x^4 - B_5/6x^6 + \qquad \text{(G)}$$

and

$$\psi(x) = \psi(1+x) - 1/x \qquad \text{(H)}$$

When x is so small that $c/(b-a)$ can be neglected compared with unity, we get

$$C = \frac{ab}{b-a}\left\{\tfrac{1}{2}\log\frac{2ab}{(b-a)c} - \psi\left(\frac{b}{b-a}\right)\right\} \qquad (17)$$

As an illustration of the application of these formulæ, let us first calculate the capacity of the spherical condenser B used by Rosa and Dorsey in their experiments for the determination of v. The dimensions of this condenser are given on p. 459 of their classical paper (*loc. cit. ante*). At a certain temperature they found that $b = 12·67158$ and $a = 8·87391$ cm respectively. When the inner sphere is central we should therefore have

$$C = 29·60933 \quad = C_0 \text{ (say)}$$

They give measurements of the capacity of the condenser when $c/(b-a)$ is small, and they also give the following formula,

$$C = C_0\left\{1 + \tfrac{1}{3}\ \frac{c^2}{(b-a)^2} + \tfrac{1}{5}\ \frac{c^4}{(b-a)^4}\right\}, \qquad (18)$$

for calculating C in this case.

Taking the case when $c = 0·3$ cm., and using (18), we get

$$C = C_0\{1 + 0·002080 + 0·000008\} = 29·67115$$

We shall apply some of the formulæ given above to check this result. Putting $n = 0$ in (7) we get

$$C = 2r\{e^{-\alpha}/(1-e^{-\omega}) + e^{-3\alpha}/(1-e^{-3\omega}) + e^{-5\alpha}/(1-e^{-5\omega}) + \dots\} \qquad (19)$$

We easily find from (D) and (E) that

$$\log r = 2·1333210, \quad \log e^{-\alpha} = \bar{2}·5133018, \quad \text{and } \log e^{-\omega} = 1·8457628$$

Hence we get

$$C = 2r\{0·1090776 + 0·0000529 + 0·0000000 + \dots\} = 29·66861$$

If we put $n = 2$ in (7), we get

$$C = a + a^2b/(b^2-c^2) + 2r\{e^{-(\alpha+2\omega)}/(1-e^{-\omega}) + e^{-3(\alpha+2\omega)}/(1-e^{-3\omega}) + \dots\} \qquad (20)$$

Hence, substituting the numerical values, we find that

$$C = 8·87391 + 6·21788 + 2r\{0·05361190 + 0·0000063 + \dots\} = 29·66862.$$

Similarly, making $n = 3$ and substituting the numerical values in the resulting formula, we get the same answer. To get the same accuracy by Maxwell's series formula (5) we would have to find the value of about 40 terms of the series

If we use formula (8) we get

$$\begin{aligned} C &= 24{\cdot}969052 + 4\ 436955 + 0\ 263186 - 0\ 000567 + 0\ 000002 - \\ &= 29\ 66862 \end{aligned}$$

It will be seen that the value computed by (18) is too great by about 8 5 in 100,000 From fig 11 in the authors' paper the measured value seems to be about $C_0(1\ 0023)$, giving an error, therefore, of about 3 parts in 10,000.

The true formula for C in a series of expanding powers of $c^2/(b-a)^2$ can be deduced without much difficulty from (19) We find that

$$C = C_0\{1 + A_2 c^2/(b-a)^2 + A_2 A_4 c^4/(b-a)^4 + \ \ .\}, \tag{21}$$

where $A_2 = ab/(a^2+ab+b^2)$,

and

$$A_4 = \frac{ab(a^2+ab+b^2)^2 + (a^2+b^2)(b^2-a^2)^2}{(a^2+ab+b^2)\{(a^2+b^2)^2 + ab(a^2-ab+b^2)\}}$$

This is a useful formula when $c^2/(b-a)^2$ is small

For the values of a, b and c given above we get, by (21),

$$C = C_0\{1 + 0\ 0019948 + 0\ 0000077\} = 29\ 668623$$

When b and a are nearly equal to one another formula (21) is practically identical with (18) The formula, however, can only be used when we are neglecting the sixth and higher powers of $c/(b-a)$, and so c must be small compared with $b-a$

As a further numerical example let us suppose that $b = 10$, $a = 7$, and $c = 1$. In this case $c/(b-a)$ is $\frac{1}{3}$ and we should expect that the value of C computed by (21) would be too small. In this case $\cosh\omega = 37/35$, $\sinh\omega = 12/35$, $\cosh\alpha = 25/7$ and $\sinh\alpha = 24/7$. We thus easily find by (9) that

$$C = 20\ 51990 + 3\ 5 + 0{\cdot}20445 - 0\ 00058 + \ \ . = 24{\cdot}22377$$

By (21) we get $C = 24{\cdot}219$; the value is, therefore, too small by about 1 part in 5000.

Let us now suppose that $b = 109$, $a = 98$ and $c = 10{\cdot}8$ In this case $\alpha = 0{\cdot}2025243$, $\beta = 0{\cdot}1823216$, and consequently $\omega = 0{\cdot}0202027$. Maxwell's series would, therefore, converge so slowly that the labour involved in computing C by it would be prohibitive Using (10) we get

$$\begin{aligned} C = r/\omega\ .\ \{-2\ 2543372 + 4{\cdot}5950858 + 0{\cdot}0000057 + 0{\cdot}0000000 + .. \\ + 0{\cdot}0030770 - 0{\cdot}0000066 + ..\ \} = 2318\ 373. \end{aligned}$$

Similarly by (11) we get

$$C+109 = r/\omega \;\{-2{\cdot}1435290+4{\cdot}5950858+0{\cdot}0000057+0{\cdot}0000000+ \\ +0{\cdot}0024631-0{\cdot}00000425+\ldots\},$$

and hence $C = 2318{\cdot}373$

As a further numerical example let us suppose that $b=5$, $a=2$ and $c=2$ In this case, $\cosh n\omega = 2^{n-1}+1/2^{n+1}$, and $\sinh n\omega = 2^{n-1}-1/2^{n+1}$ Hence $e^{\omega}=2$ and $\omega = \log_e 2 = 0{\cdot}6931472$ We also have $r = 3{\cdot}75$ Substituting in (12) we get $C = 3{\cdot}891981$ In this case (10) gives the same answer with very little additional labour and (7) is even more suitable The error introduced by using (21), however, is about 2 per cent, $c/(b-a)$ being $\frac{2}{3}$

As a final example, let us suppose that $b=201$, $a=100$, and $c=100$ We have $x=1$, and hence substituting in (14) and (15) we get $r/\omega = 201{\cdot}33$ and $\alpha/\omega = 2{\cdot}0000$ Hence by (16)

$$C = 201{\cdot}33\times 2{\cdot}5752 = 518{\cdot}465$$

The true value of C found by (12) is $518{\cdot}468$ The agreement is therefore quite satisfactory. When the spheres are very close together, (17) gives accurate results If we write $-b$ for b in (17) we get the value of k_{11} for two external spheres of radii a and b respectively at a microscopic distance x apart.*

2 *The Electrostatic Force on the Inner Sphere*

Let W denote the electrostatic energy stored up between the two spheres, and let v_1 and v_2 be their potentials. If C be the capacity between them, we have

$$W = \tfrac{1}{2}C(v_1-v_2)^2,$$

and thus, if F be the attractive force when the potential difference is maintained constant, we have

$$F = \tfrac{1}{2}(v_1-v_2)^2\,\partial C/\partial c$$

Let us first suppose that $c/(b-a)$ is small compared with unity In this case we get, by (21),

$$F = \frac{a^2b^2c\,(v_1-v_2)^2}{(b^3-a^3)(b-a)^2}\left\{1+2A_4\frac{c^2}{(b-a)^2}\right\}\ldots \tag{22}$$

approximately Hence, for small values of $c^2/(b-a)^2$, the attractive force varies as the distance between the centre of the sphere and the centre of the cavity

It is easy to see, from the definitions of α, β, ω, and r, that

$$\partial\alpha/\partial c = -\sinh\alpha\cosh\beta/r\sinh\omega, \qquad \partial\beta/\partial c = -\sinh\beta\cosh\alpha/r\sinh\omega,$$
$$\partial\omega/\partial c = -1/r; \qquad \text{and} \qquad \partial r/\partial c = -\cosh\alpha\cosh\beta/\sinh\omega.$$

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* Russell, 'Alternating Currents,' 2nd. ed., vol. 1, p. 241

We thus get by (5)

$$\frac{2\mathrm{F}}{(v_1-v_2)^2}=\sum_1^\infty n\frac{\cosh(\alpha+n\omega)}{\sinh^2(\alpha+n\omega)}-\frac{\cosh\alpha\cosh\beta}{\sinh\omega}\sum_0^\infty\frac{1}{\sinh(\alpha+n\omega)}$$
$$+\frac{\sinh\alpha\cosh\beta}{\sinh\omega}\sum_0^\infty\frac{\cosh(\alpha+n\omega)}{\sinh^2(\alpha+n\omega)} \qquad (23)$$

When ω is small, the summation of the three series given in this formula is laborious It is therefore better to convert them into semi-converging series by (F) We thus readily find that

$$\frac{2\mathrm{F}}{(v_1-v_2)}=\frac{1}{2\omega^3}\log\frac{\cosh\alpha+1}{\cosh\alpha-1}+\frac{\cosh\alpha}{12\sinh^2\alpha}-\frac{\omega^2}{240}\quad\frac{\cosh\alpha(\cosh^2\alpha+5)}{\sinh^4\alpha}+$$
$$-\frac{\cosh\beta}{\sinh\omega}\left\{\frac{\cosh\alpha}{2\omega}\log\frac{\cosh\alpha+1}{\cosh\alpha-1}-\frac{1}{\omega}\right.$$
$$\left.-\frac{\omega}{12\sinh^2\alpha}+\frac{\omega^3}{720}\quad\frac{13\cosh^2\alpha+5}{\sinh^4\alpha}-\quad\right\} \qquad (24)$$

This formula will be found suitable for computing, except when $c/(b-a)$ is nearly equal to unity In this case we get, by (14), (15), and (16),

$$\frac{2\mathrm{F}}{(v_1-v_2)^2}=\frac{r}{\omega}\left[\frac{1}{2x}+\psi'\left(\frac{a}{\omega}\right)\cdot\frac{b+a}{3(b-a)^2}\quad\left\{1+\frac{25ab-2(a^2+b^2)}{30ab}\,\frac{x}{b-a}\right\}\right.$$
$$\left.-\frac{1}{3}(k+\tfrac{1}{4})\frac{1}{b-a}-\frac{1}{1800}(331+948k-624k^2)\frac{x}{(b-a)^2}\right]$$
$$-\frac{\mathrm{C}\omega}{r}\quad\frac{ab}{(b-a)^2}\cdot\left\{h+\frac{3+9k-2k^2}{10}\cdot\frac{2x}{b-a}\right\} \qquad (25)$$

C can be computed by the formulæ given earlier in the paper and $\psi'(x)$* is found from

$$\psi'(x)=\frac{1}{x}+\frac{1}{2x^2}+\frac{\mathrm{B}_1}{x^3}-\frac{\mathrm{B}_2}{x^5}+\frac{\mathrm{B}_3}{x^7}-. \qquad (\mathrm{I})$$

If this converge too slowly we can write

$$\psi'(x)=\psi'(x+n)+\sum_0^{n-1}\frac{1}{(x+n)^2}. \qquad (\mathrm{J})$$

and use (I) to find $\psi'(x+n)$

Finally, when x is very small, we have

$$\mathrm{F}=ab\,.\,(v_1-v_2)^2/4(b-a)x. \qquad (26)$$

In the case when the spheres are external to one another and the distance between them is microscopic, we previously found (*loc. cit. ante*) that

$$\mathrm{F}=ab\quad(v_1-v_2)^2/4(b+a)x$$

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* A short table of the values of $\psi'(x)$ is given in 'Funktionentafeln,' Jahnke and Emde, p. 31.

When arranged as a spherical condenser, therefore, the attractive force is greater in the ratio of $b+a$ to $b-a$ when the distance apart is very minute.

When the electrostatic charge q on the inner condenser remains constant, the attractive force F′ is given by

$$F' = q^2/C^2 \quad \partial C/\partial c, \tag{27}$$

and formulæ for computation are obtained in exactly the same way as for the preceding problem

In formula (26) if v_1-v_2 is given in electrostatic units, and a and b are in centimetres, then F is given in dynes. If v_1-v_2 is given in volts, we divide it by 300 to reduce it to electrostatic units Similarly, if q in (27) be given in coulombs, we must multiply its value by 3×10^9

3 *The Maximum Value of the Electric Stress*

In determining the voltage at which the disruptive discharge inside the cavity takes place, a knowledge of the maximum value R_{max} of the electric stress on the dielectric is essential In the case of spherical electrodes external to one another, the author* has shown how the value of this stress can be computed In a similar way, it can readily be shown by the method of images that

$$R_{max} = \frac{v_1-v_2}{a} \quad \frac{\cosh^2\frac{1}{2}\alpha}{\sinh\frac{1}{2}\alpha}\left\{\frac{\sinh\frac{1}{2}\alpha}{\cosh^2\frac{1}{2}\alpha}+\frac{\sinh(\frac{1}{2}\alpha+\omega)}{\cosh^2(\frac{1}{2}\alpha+\omega)} + \frac{\sinh(\frac{1}{2}\alpha+2\omega)}{\cosh^2(\frac{1}{2}\alpha+2\omega)}+ \quad \right\} \tag{28}$$

When $c = 0$, α is infinite and ω is $\log(b/a)$ In this case, therefore

$$R_{max} = \frac{v_1-v_2}{a}\left\{1+\frac{1}{e^{\omega}}+\frac{1}{e^{2\omega}}+ \ . \right\} = \frac{b}{a}\cdot\frac{v_1-v_2}{b-a}.$$

Expanding the third and subsequent terms of (28) in series, ascending by powers of $e^{-\alpha}$ and $e^{-\omega}$, and then summing the terms of the resulting series in a different order, we can show that

$$R_{max} = \frac{v_1-v_2}{a}\left[1+\frac{ab\{b^2-c(c-a)\}}{\{b^2-c(c+a)\}^2} + e^{-\omega}\frac{(1+e^{-\alpha})^2}{1-e^{-\alpha}} \times\left\{\frac{e^{-\omega}}{1-e^{-\alpha}} - 3\frac{e^{-3\omega}}{1-e^{-3\omega}}e^{-(\alpha+2\omega)} + 5\frac{e^{-5\omega}}{1-e^{-5\omega}}e^{-2(\alpha+2\omega)} - \ . \ . \right\}\right] \tag{29}$$

Except when $e^{-\omega}$ is nearly unity, this series is most suitable for computation. As an example, let us suppose that $b = 5$, $a = 2$, and $c = 2$. We find that $e^{-\omega} = 1/2$ and $e^{-\alpha} = 1/4$, and so, substituting in (29), we get

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* 'Proc. Phys. Soc Lond.,' vol 24, p. 22.

$$\begin{aligned}R_{\text{max.}} &= \tfrac{1}{2}(v_1 - v_2)\{1 + 0\cdot 8650520 + \tfrac{25}{24}(1 - 0\cdot 0267857 + 0\cdot 0006301 \\ &\qquad - 0\cdot 0000135 + 0\cdot 0000003 - \quad)\} \\ &= 1\cdot 4397297\,(v_1 - v_2)/x,\end{aligned}$$

where $x =$ the distance between the spheres $= 1$.

When ω is small we can use the following formula deduced from (F) and (28)

$$\begin{aligned}R_{\max} = \frac{v_1 - v_2}{a}\Bigg[&\frac{1}{\omega}\;\frac{1 + \cosh\alpha}{\sinh\alpha} + \frac{1}{2} - \frac{\omega}{6}\;\frac{1 - \sinh^2\frac{1}{2}\alpha}{\sinh\alpha} \\ &- \frac{\omega^3}{720}\cdot\frac{1}{\sinh\frac{1}{2}\alpha\cosh^3\frac{1}{2}\alpha}\{\sinh^4\tfrac{1}{2}\alpha - 18\sinh^2\tfrac{1}{2}\alpha + 5\} \\ &- \frac{\omega^5}{30240}\cdot\frac{1}{\sinh\frac{1}{2}\alpha\cosh^5\frac{1}{2}\alpha} \\ &\qquad \times\{179\sinh^4\tfrac{1}{2}\alpha + 61 - \sinh^6\tfrac{1}{2}\alpha - 479\sinh^2\tfrac{1}{2}\alpha\}\Bigg] \qquad (30)\end{aligned}$$

For example, when $b = 5$, $a = 2$, and $c = 2$, so that x is 1, we have

$$\begin{aligned}R_{\max} &= \tfrac{1}{2}(v_1 - v_2)\{1\cdot 202246 + 0\cdot 25 - 0\cdot 013478 + 0\cdot 000759 + 0\cdot 000176\} \\ &= 1\cdot 43970\,(v_1 - v_2)/x\end{aligned}$$

In this case, therefore, where ω is nearly equal to 0·7, the error is about 1 in 48,000.

If in (30) we write $b - a - x$ for c, and neglect cubes and higher powers of $x/(b-a)$, we get, after lengthy algebraical work, that

$$R_{\max} = \{1 + (2b + a)\,x/3ab + [4\,(a + b)^2 - ab]\,x^2/45a^2b^2 + \,.\,.\}(v_1 - v_2)/x. \qquad (31)$$

As an example let $b = 5$, $a = 2$, and $x = 1$ Substituting in (31) we get

$$R_{\max} = 1\cdot 4413\,(v_1 - v_2)/x.$$

Hence, although $x/(b-a)$ is as great as $1/3$, the error is only about 1 in 700 If we write $-b$ for b in (31), we get the formula applicable to the case of external spheres. For external spheres, therefore,

$$R_{\max} = \{1 + (2b - a)\,x/3ab + [4\,(a - b)^2 - ab]\,x^2/45a^2b^2\}\,(v_1 - v_2)/x \qquad (32)$$

Comparing (31) and (32), we see that, for a given value of a and of $v_1 - v_2$, the maximum electric stress is greater for the spherical condenser

If we put $b = a$ in (34) we get

$$R_{\max} = \{1 + x/3a + x^2/45a^2 + \;.\;\}\,(v_1 - v_2)/x,$$

a useful formula, which is applicable to the case of equal spherical electrodes when they are close together This formula was first given by Schuster *

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* 'Phil Mag,' vol. 29, p. 192 (1890)

On a New Gyroscopic Phenomenon

By E. E. Tournay Hinde.*

(Communicated by Arthur Schuster, Sec. R.S. Received November 9, 1917.)

I should like to submit to the Royal Society an experimental phenomenon connected with gyration, which I believe to be novel.

The accompanying diagram and description will explain the circumstances under which the phenomenon takes place.

ABCD represents a wooden frame, which must be firmly secured in a vertical position; fixed in this frame is a 0.15-inch diameter smooth vertical rod E. G is an eye or ring of round metal, having an internal diameter of about 0.165 inch; to this ring is securely attached a spindle carrying the fly-wheel of a gyroscope F (a well known type of small toy gyroscope, weighing about 3¾ oz., and altered as required, was used), the gyroscope and its spindle being thus so arranged that they together can slide freely up and down the rod E, and also freely revolve, horizontally, round the same. The axis of the ring G is slightly inclined to the axis of the spindle, so that the latter becomes more approximately horizontal when at rest (see enlarged section). H^1 and H^2 are fixed sleeves on the rod E, and serve merely as stops, to prevent the gyroscope striking the wooden frame when experimenting, and I is a small loose washer to reduce friction.

In order to observe the phenomenon to which I wish to draw attention, the gyroscope is (by means of a cord on its hub) first spun as rapidly as possible, say in the direction marked J (see elevation), and then a slight horizontal motion is imparted to the free end of the spindle at K, in the direction of L (see plan), *i.e.*, in the reverse direction to the corresponding gravitational precession. The gyroscope will at first slowly, and then with an accelerated action, continue to revolve round the vertical rod, and at the same time rise bodily up the rod, with a steady spiral and vibratory motion, until it arrives at the upper stop H^2, as shown by dotted lines, where it will continue to revolve until, the momentum of the fly-wheel becoming exhausted, it falls back to its starting point.

The experiment described was the first among a series of many to give such a definite rising action, and subsequently I have found, by making further investigation, that—

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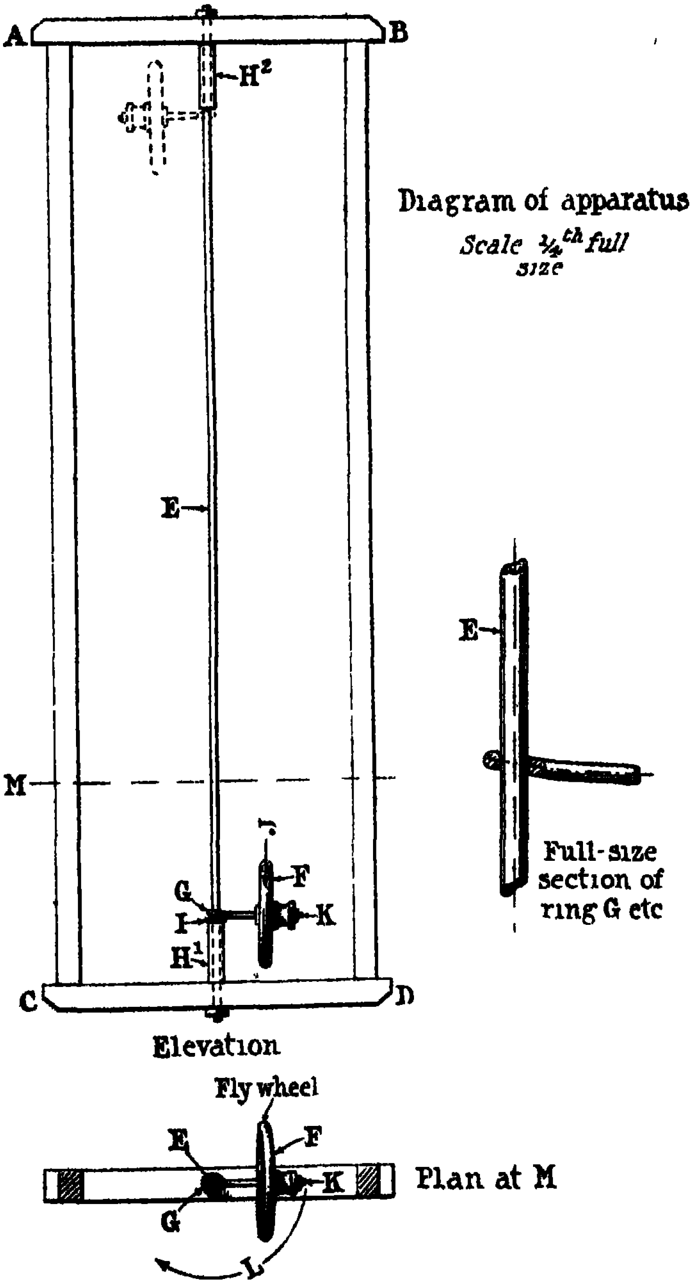
* A communication in the form of a letter, dated March 8, from Sydney, N.S.W., reached the Royal Society in the course of the spring. The present paper contains the essential parts of the letter.—A. S.



(*a*) The horizontal rotation of the gyroscope round the vertical rod is not essential to the production of the peculiar rising action described, for, if in place of the vertical round rod E and the ring G (see diagram) a fixed square vertical rod and a square eye be substituted, so as to prevent any horizontal rotation, the rising action can still be obtained

(*b*) By varying the intensity of the gyroscope's vibrations, the rising action may be accelerated or retarded, or, if the vibrations are eliminated as far as practicable, the rising action also ceases

Note on Mr Hinde's Paper

By ANDREW GRAY, F R S

The arm shown in Mr Hinde's figure revolves round the vertical rod with moderate angular speed μ in the direction opposed to the slower of the two precessions which the gravity couple could produce in an ordinary top The precession therefore tends to increase the inclination of the arm to the upward vertical, that is, to tilt the outer end of the arm down The ring is therefore tilted so as to bear on the vertical rod with its upper edge on the side remote from the gyroscope, and with its lower edge on the same side as the gyroscope The arm is thus locked on the stem as regards tilting Apart from the variations to be dealt with presently, it is prevented from further tilting by a couple, and its sliding down is impeded by an upward force due to friction at the contacts with the rod

We have now to show how the upward motion is produced. With the model constructed in my workshop I have been able to confirm Mr. Hinde's statements on the whole I have also been able to try different small gyroscopes on the arm, and have found that if the wheel of the gyroscope is perfectly balanced about the axis of rotation, no climbing takes place There seems therefore no doubt that the effect is due to the centrifugal couples which arise from want of exact balance The action of these may be assisted also by lateral vibration of the vertical stem and the corresponding vibration of the gyroscope

As the wheel revolves the plane of the centrifugal couple revolves with it about the arm, while the arm turns, of course comparatively slowly, about the vertical stem If the turnings be as in Mr Hinde's figure, the side of excess mass of the wheel will be successively on the right, at the top, on the left, and at the bottom, to an observer looking at the arm from the outside beyond the wheel. The axis of the centrifugal couple is then to be drawn upwards, to the left, downwards, and to the right respectively, for the four positions specified We shall call these positions 1, 2, 3, 4

Take position 1 The couple produces angular momentum (A M) about the axis of the couple, and hence the arm begins to tilt upwards, so that, relative to the former condition of things, an upward component of the A.M. of the fly-wheel may be brought into existence.

But as the fly-wheel passes on to position 2, in which the couple axis is towards the left, the arm is accelerated in the precessional motion so as to cause a component of A M. towards the left to grow up to correspond to the growing couple But the couple is to be regarded as the accompaniment of an upward force applied at the centroid of the revolving arm and attached wheel, and the system slides upward a certain distance, while retaining its upward tilted position, in which the locking action of the ring has been relieved

As the wheel passes on through the next two positions the same actions as before take place, but in the contrary directions The downward tilting in position 3 locks the ring, and so there is no sliding down in position 4

The oscillatory variations of speed of the azimuthal motion are essential to the action

On the whole the centre of the wheel is raised in each revolution about the axis of rotation, and this rise takes place in the main when the excess mass of the wheel is rising in the rotational motion This produces a slowing of the rotation, and the rotational kinetic energy therefore falls off as the arm rises The gain of potential energy of the arm is thus at the expense of the energy of rotation of the wheel

Mr Hinde states that he has found the continued precessional motion unnecessary, that, in fact, he has been able to cause the arm to rise along a vertical rod of square section, when the continuous precessional motion is prevented by a loosely-fitting square ring This I have also verified with my model I find it necessary, however, to provide in this case additional locking action by pressing *down* with my fingers the outer end of the arm This locking action might, of course, be furnished by gravity, and probably was in Mr Hinde's model.

When the necessary locking action is supplied, the explanation given above holds, with only slight modifications. The ring fits loosely, and to-and-fro azimuthal motions take place, corresponding to the alternate acceleration and retardation of the continuous azimuthal motion described above

I may mention that some years ago, Dr J G Gray devised a "climbing gyrostat," which was one of a series of interesting examples of gyrostatic action invented by him, and made in my workshop in the University of Glasgow. This gyrostat was hung by two cords passing over pulleys and carrying weights which only partly counterpoised the weight of the gyrostat The arrangement was such that when the gyrostat oscillated in azimuth, it slowly ascended with, of course, descent of the smaller counterpoising weight.

It is to be observed that in this device the wheel was balanced, and that, though the play of forces was very similar to that in Mr. Hinde's experiment, energy was fed in from outside by the agent maintaining the azimuthal oscillations.

Another example of progressive motion obtained by oscillatory motion of the axis of the spin of a gyrostat is Dr Gray's "walking gyrostat."

I have described some of these devices in a work on 'Gyrostatics,' now in the press

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The Behaviour of Scattering Media in Fully Diffused Light

By H J Channon, F F Renwick, and B V Storr.

(Communicated by Dr. T M Lowry, F R S Received April 10,—Received in revised form October 10, 1917)

Introduction

The present paper is the outcome of a discussion which arose between the authors after the publication of a paper by O Bloch and F F. Renwick,[*] which dealt with the subject of the opacity of diffusing media, chiefly from the practical standpoint, and included also lengthy extracts from the Reports of a Committee of the American Society of Illuminating Engineers[†]

On finding that we were unable to reconcile the experimental data with the theory included in the above paper, we were gradually led into the attempt to find solutions of some of the problems involved in this complex subject, starting only with the indisputable fact that when light falls upon a finite thickness of a scattering medium, part is rejected, part extinguished, and part transmitted In the present communication we have confined ourselves to a study of the action of such media upon light which is assumed to be already completely diffused

Definitions.

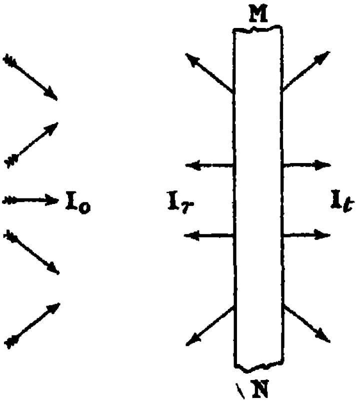
Let MN represent a large sheet of a diffusing medium of uniform thickness and quality, upon which completely diffused light of intensity I_0 falls.

Let the intensity of the light returned in directions opposed to I_0 be I_r, and the intensity of the light which escapes after passage through the sheet

[*] 'Photo Journ,' vol 56, pp. 49-65.

[†] 'Trans Amer Soc Ill Eng.,' vol 10, No 5, pp 353-402 (1915).

be I_t. The fraction I_r/I_0 of the incident light may be called the "rejecting power" or the "rejectance" of the sheet, and will be denoted by R.



The ratio of the light issuing from, to that impinging on the sheet, viz., I_t/I_0, we will call the "transmittance," while the inverse of this, I_0/I_t, we will call the "obstructance" and denote by Ω

We then have by definition the following relationships —

$$I_r/I_0 = R \text{ (rejectance)},$$

$$I_0/I_t = \Omega \text{ (obstructance)}$$

It is convenient also to express the ratio between the unrejected portion of the incident light and the light issuing by the symbol "C" and the term "contendance"

It should be noted that the rejectance (R) includes all light returned in directions opposed to the incident (diffused) light, no matter by what circuitous path within the material it may have travelled It seems therefore undesirable to employ the term "reflection" in describing it* It should also be observed that a medium which neither rejects nor absorbs any of the incident light would have an "obstructance" and a "contendance" both = 1.

The equation $C = I_0(1-R)/I_t$ forms the starting point of our theoretical discussion, in order to eliminate everything not a property of the material, we will write this in the form

$$\Omega = C/(1-R). \tag{1}$$

$$\text{Obstructance} = \frac{\text{contendance}}{\text{selectance}}$$

It is our object to show the changes that take place in the values of these properties with alteration of thickness and to reduce the relationships to their most general forms. Since, by definition, the above equation (1) is independent of the thickness of the sheet under consideration, we can write

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* *Vide* Rayleigh, 'Phil Mag.,' vol. 41, p. 107 (1871)

$$\Omega_t = C_t/(1-R_t), \text{ whatever the value of } t$$

Now suppose two sheets of any diffusing media m and n to be placed together to form a single sheet. Let Ω_m and R_m be the obstructance and rejectance respectively of the first sheet m, and Ω_n and R_n those of the second sheet n through which the light issues.

Then if light of intensity I be transmitted by m and fall upon n, we have an infinite geometrical series of "inter-rejections" which sum up, in the direction of m, to

$$I\,R_n/(1-R_mR_n).$$

In passing out through m this will be reduced to

$$I\,R_n/(1-R_mR_n)\Omega_m,$$

which will be the increment in the rejected light due to the backing up of m by n. But $I = I_0/\Omega_m$, where I_0 = the light intensity incident upon m. Hence the total rejectance of the two sheets together will be

$$R_m+R_n/\Omega_m^2(1-R_mR_n)$$

Hence, generally, the rejectance from m of a composite layer of two diffusing media m and n in contact, whether of similar or dissimilar characters,

$$= R_{m+n} = R_m+R_n/\Omega_m^2(1-R_mR_n) \qquad (2)$$

Similarly the obstructance of the two sheets $m+n$ together

$$\Omega_{m+n} = (1-R_mR_n)\Omega_m\Omega_n. \qquad (3)$$

The second member of this equation is symmetrical with respect to m and n and therefore the obstructance of a composite sheet consisting of two diffusing media is independent of their arrangement.

If the sheets m and n consist of the same material, R_{m+n} will equal R_{n+m} and from (2)

$$R_{m+n}(1-R_mR_n) = R_m+R_n\{1/\Omega_m^2-R_m^2\},$$

the first member of which in these circumstances is also symmetrical with respect to m and n. We may, therefore, interchange m and n throughout and equate the results, thus,

$$R_m+R_n(1/\Omega_m^2-R_m^2) = R_n+R_m(1/\Omega_n^2-R_n^2),$$

or

$$\{1+R_m^2-1/\Omega_m^2\}/R_m = \{1+R_n^2-1/\Omega_n^2\}/R_n,$$

which is, therefore, a constant, independent of the thickness, for any diffusing medium.

Note.—Up to this point we have employed the same mode of reasoning as Stokes followed in his paper "On the Intensity of the Light Reflected from or Transmitted through a Pile of Plates"*

Writing this expression in the form $(1+R_t^2-1/\Omega_t^2)/R_t$ and inserting the values corresponding to $t=\infty$, it follows that the value of Stokes' constant is $(R_\infty+1/R_\infty)$, where R_∞ is the rejectance of an infinitely thick layer

Suppose a sheet of a uniformly lighted diffusing medium of rejectance ρ to be used as source of light, transmitting completely diffused light of intensity I'

Let this light at once encounter another sheet of rejectance R and obstructance Ω, then the light intensity transmitted by the combination $I_1 = I'/(1-\rho R)\Omega$

The ratio $I'/I_1=(1-\rho R)\Omega$ may be called the "relative obstructance" O of the added sheet (*i e*, the obstructance of this sheet relative to ρ).

The inverse of this may be written T and called the relative "transmittance" while the logarithm of O to the base 10 may be denoted by D and called the relative obstructance logarithm of the sheet to diffused light

Measurements made under these conditions have for many years been denoted by the above symbols and it seems, therefore, desirable to retain them but to abandon the incorrect terms, opacity for O, transparency for T, and density for D, usually given to them, in favour of those suggested

In the case of measurements of black scattering media like photographic negatives, the value of R is usually so small that it may be permissible in ordinary practice to assume that it is zero and to regard the measured values as representing the true constants of the silver deposits, ignoring the complication due to the properties of the diffusing medium in the photometer

We have seen that, generally, the relative obstructance of a sheet of diffusing medium *m*,

$$= O_m = (1-\rho R_m)\,\Omega_m, \tag{4}$$

where ρ is the rejectance of the photometer diffusing medium *p*.

The combined rejectance, on the side *m*, of *m* and *p* together will be (from equation 2)

$$R_{m+p} = R_m + \rho/(1-\rho R_m)\,\Omega_m^2$$

Now if I_1 be the light intensity issuing through the first specimen *m*, that leaving a second superposed specimen *n* will be, by the usual process of reasoning,

$$I_2 = I_1/(1-R_n R_{m+p})\,\Omega_n.$$

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* 'Roy Soc. Proc.,' vol. 11, p. 545 (1860–62)

But for the first specimen m

$$I_1 = I'/(1-\rho R_m)\,\Omega_m,$$

where I′ is the light intensity emitted by the bare photometer diffuser

Hence $$O_{m+n} = \Omega_n O_m (1-R_m R_n) - \rho R_n \Omega_n/\Omega_m \qquad (5)$$

In the special case in which $R_n = R_m = R_1$ and $O_m = O_n = O_1$, we have

$$O_2 = O_1\Omega_1(1-R_1^2) - \rho R_1,$$

and if also $R_1 = \rho$ then

$$O_2 \text{(rel. to } \rho) = (1-\rho^2)^2\Omega_1^2 - \rho^2$$

But since, in this case, $O_1 = (1-\rho^2)\,\Omega_1$, it follows that, relative to ρ,

$$O_1^2 - O_2 = \rho^2 \qquad (6)$$

Hence if of three similar specimens we measure the relative obstructances of one specimen and of two specimens together, using completely diffused light emitted by the third specimen, we are in a position to derive at once the rejectance ρ of one specimen and therefrom, by means of the equations $O_1 = (1+\rho)C_1$ and $O_1 = (1-\rho^2)\Omega_1$ to deduce C_1 and Ω_1 also

This result is obviously one of considerable value in practical photometry

Some General Relationships Existing Between Values of Ω, etc

1 Proof that

$$\Omega_{n+2} + \Omega_n = \Omega_{n+1} \times \text{constant},$$

where n, $n+1$, etc, refer to the number of similar layers of which a sheet of medium may be imagined to consist

From equation 3 (p 224),

$$\Omega_{n+2} = (1-R_1R_{n+1})\,\Omega_1\Omega_{n+1}, \qquad (a)$$

$$\Omega_{n+1} = (1-R_1R_n)\,\Omega_1\Omega_n \qquad (b)$$

From equation 2 (p 224),

$$R_{n+1} = R_1 + R_n/\Omega_1^2(1-R_1R_n),$$

(the layers being all alike, R_{n+1} will be the same as R_{1+n})

Hence, in (a)

$$(1-R_1R_{n+1}) = 1-R_1^2 - R_1R_n/\Omega_1^2(1-R_1R_n) \qquad (c)$$

From (b)

$$R_1R_n/(1-R_1R_n) = (\Omega_1\Omega_n - \Omega_{n+1})/\Omega_{n+1} \qquad (d)$$

Therefore, from (c) and (d),

$$(1-R_1R_{n+1}) = \Omega_{n+2}/\Omega_1\Omega_{n+1} = 1-R_1^2 - (\Omega_1\Omega_n - \Omega_{n+1})/\Omega_1^2\Omega_{n+1}.$$

Hence $$(1-R_1^2)\,\Omega_1^2\Omega_{n+1} - \Omega_1\Omega_n + \Omega_{n+1} = \Omega_1\Omega_{n+2},$$

or $$\Omega_{n+2} + \Omega_n = \{(1-R_1^2)\,\Omega_1 + 1/\Omega_1\}\,\Omega_{n+1} \qquad (7)$$

Since R_1 and Ω_1 are the rejectance and obstructance of a single layer, the expression $[(1-R_1^2)\Omega_1+1/\Omega_1]$ is a constant Consequently, we may write

$$\Omega_{n+2}+\Omega_n = K\Omega_{n+1}$$

From this, by substitution, it is easy to prove also that

$$C_{n+2}+C_n = KC_{n+1} \quad \text{and} \quad O_{n+2}+O_n = KO_{n+1},$$

where K has the value already found above Writing these in the form

$$(\Omega_{n+1}-\Omega_n)-(\Omega_n-\Omega_{n-1}) = (K-2)\Omega_n,$$

it is evident that the values of Ω, O, and C for 0, 1, 2, 3, etc, units of thickness form a series 1, Ω_1, $K\Omega_1-1$, $K^2\Omega_1-K-\Omega_1$, etc, whose differences of the second order constitute a new series identical with the original series multiplied by the factor $K-2$

This property of a finite series is consistent only with the conclusion that the relationships between Ω, C, and O, and the variable t, are either simple exponential relationships or the sums or differences of two exponential expressions having the same indices, one positive, the other negative

It is evident that the above series for 0, 1, 2, 3, etc, thicknesses is not a simple exponential series, and consequently the formulæ required must all be of the general form

$$\Omega_t\,(\text{or } C_t \text{ or } O_t) = Pe^{\lambda t}+Qe^{-\lambda t}, \tag{8}$$

which is the integral of $d^2\Omega/dt^2 = K\Omega$, P, Q and λ being constants

Since all three series are identical except for the interchange of Ω_1, C_1, and O_1 it is clear that the evaluation of the constants for Ω will give those of the other two by simple substitution of C_1 and O_1 for Ω_1

Confirmation of the above general equation is to be found in a paper by Prof. A Schuster* His solution for the emergent radiation R through any thickness t of a self-radiant foggy atmosphere is as follows —

$$R = 2\alpha\,\frac{[(1+\alpha)e^{\alpha(k+s)t}+(1-\alpha)e^{-\alpha(k+s)t}]\,E+2(S-E)}{(1+\alpha)^2e^{\alpha(k+s)t}-(1-\alpha)^2e^{-\alpha(k+s)t}},$$

where E = total energy of radiation sent out by unit surface of a completely black surface,

S = total energy of radiation incident on unit surface of plane layer of foggy gas,

k = coefficient of absorption, s = coefficient of scattering,

$\alpha = \sqrt{[k/(k+s)]}$, t = thickness.

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* "Radiation through a Foggy Atmosphere," 'Astrophys. Journ,' vol 21, p 1 (1905)

In our case E may be taken as zero, for we are not considering self-luminous or fluorescent media, hence Schuster's equation simplifies to

$$S/R = \Omega_t = [(1+\alpha)^2 e^{\alpha(k+s)t} - (1-\alpha)^2 e^{-\alpha(k+s)t}]/4\alpha,$$

and by writing

$$(1+\alpha)^2/4\alpha = P, \qquad -(1-\alpha)^2/4\alpha = Q,$$

and

$$\alpha(k+s) = \lambda,$$

Schuster's equation is the same as our general expression

We are now in a position to evaluate the constants in the above general equation from the knowledge we possess of the values of the first few terms for $t = 0$, 1 and 2, namely, 1, Ω_1, and $K\Omega_1 - 1$

Putting $t = 0$,

$$P+Q = 1, \qquad Q = 1-P.$$

Putting $t = 1$,

$$Pe^{\lambda} + (1-P)e^{-\lambda} = \Omega_1 \qquad \text{(or } C_1 \text{ or } O_1 \text{ resp.)},$$

$$P = (\Omega_1 - e^{-\lambda})/2\sinh\lambda, \qquad Q = (e^{\lambda} - \Omega_1)/2\sinh\lambda.$$

Putting $t = 2$,

$$(\Omega_1 - e^{-\lambda})e^{2\lambda}/2\sinh\lambda + (e^{\lambda} - \Omega_1)e^{-2\lambda}/2\sinh\lambda = K\Omega_1 - 1,$$
$$= 2\Omega_1\cosh\lambda - 1,$$

therefore

$$K = 2\cosh\lambda.$$

Hence the full expressions for Ω_t, C_t, and O_t are as follows —

$$\Omega_t = (\Omega_1 - e^{-\lambda})e^{\lambda t}/2\sinh\lambda + (e^{\lambda} - \Omega_1)e^{-\lambda t}/2\sinh\lambda, \tag{9}$$

$$C_t = (C_1 - e^{-\lambda})e^{\lambda t}/2\sinh\lambda + (e^{\lambda} - O_1)e^{-\lambda t}/2\sinh\lambda, \tag{10}$$

$$O_t = (O_1 - e^{-\lambda})e^{\lambda t}/2\sinh\lambda + (e^{\lambda} - O_1)e^{-\lambda t}/2\sinh\lambda. \tag{11}$$

$$\Omega_1 = C_1/(1-R_1), \qquad O_1 = (1-\rho R_1)\Omega_1 = (1-\rho R_1)C_1/(1-R_1),$$

where ρ is the rejectance of the photometer diffuser, Ω_1, C_1, and R_1 being the obstructance, contendance, and rejectance respectively of unit thickness of the medium, t being expressed in the same units

Any of the above may be expressed in a number of different forms, of which the following should be mentioned:—

$$\Omega_t = [\Omega_1\sinh\lambda t - \sinh\lambda(t-1)]/\sinh\lambda, \tag{12}$$

(with similar forms for C_t and O_t) as being the simplest for calculating numerical values

The Change of R with t.

Since $\Omega_t = C_t/(1-R_t)$, $R_t = 1 - C_t/\Omega_t$.

Hence
$$R_t = 1 - \frac{(C_1 - e^{-\lambda})\,e^{\lambda t} + (e^{\lambda} - C_1)\,e^{-\lambda t}}{(\Omega_1 - e^{-\lambda})\,e^{\lambda t} + (e^{\lambda} - \Omega_1)\,e^{-\lambda t}}$$

$$= \frac{(\Omega_1 - C_1)\{e^{\lambda t} - e^{-\lambda t}\}}{(\Omega_1 - e^{-\lambda})\,e^{\lambda t} + (e^{\lambda} - \Omega_1)\,e^{-\lambda t}}$$

$$= \frac{2\,\Omega_1 R_1 \sinh \lambda t}{(\Omega_1 - e^{-\lambda})\,e^{\lambda t} + (e^{\lambda} - \Omega_1)\,e^{-\lambda t}} \tag{13}$$

$$= \frac{\Omega_1 R_1 \sinh \lambda t}{\Omega_1 \sinh \lambda t - (1 - \rho R_1) \sinh \lambda (t - 1)} \tag{14}$$

If we rewrite equation (13) thus

$$R_t = R_1 \Omega_1 \frac{1 - e^{-2\lambda t}}{(\Omega_1 - e^{-\lambda}) + (e^{\lambda} - \Omega_1)\,e^{-2\lambda t}}$$

it becomes evident that when t is infinitely great

$$R_t = \Omega_1 R_1/(\Omega_1 - e^{-\lambda}) = R_\infty$$

the maximum possible rejectance of the material with increasing thickness

This constant enables us to derive a number of simpler expressions for R_t, and may itself be expressed in many different forms, of which

$$R_\infty = \Omega_1 R_1/[\Omega_1 - (1 - \rho R_1)\,e^{-\lambda}] \tag{15}$$

is useful for its practical application, while

$$R_t = 2 \sinh \lambda t/(e^{\lambda t}/R_\infty - R_\infty\,e^{-\lambda t}),$$

$$\text{or, if} \quad R_\infty = e^{\beta}, \qquad R_t = \sinh \lambda t/\sinh(\lambda t - \beta) \tag{16}$$

is a convenient form for calculating a number of values of R_t

Lastly, it appears desirable to give the relations existing between Schuster's constants α, k, and s, and those we employ They are expressed as follows —

$$\Omega_1 = (1+\alpha)^2\,e^{\alpha(k+s)}/4\alpha - (1-\alpha)^2\,e^{-\alpha(k+s)}/4\alpha,$$

$$C_1 = \tfrac{1}{2}(1+\alpha)\,e^{\alpha(k+s)} + \tfrac{1}{2}(1-\alpha)\,e^{-\alpha(k+s)},$$

$$R_1 = \frac{(1-\alpha^2)\,2 \sinh \alpha(k+s)}{(1+\alpha)^2\,e^{\alpha(k+s)} - (1-\alpha)^2\,e^{-\alpha(k+s)}},$$

$$R_\infty = (1-\alpha)/(1+\alpha) \tag{17}$$

From equation (17) it is easy to calculate numerical values for α, and therefrom to derive k and s by means of $\lambda = k/\alpha$ and $\lambda = \alpha(k+s)$ when R_∞ is known.

Considerations affecting the Practical Problem of Measurement

In order to determine the actual values of those properties of scattering media which have been considered in the foregoing theoretical discussion, it is necessary to employ a sufficiently large source of completely diffused light and specimens of the materials to be studied of sufficient area to avoid the serious errors which may arise with small specimens from loss of light at the edges

The only convenient means available for getting uniform diffused light of constant intensity is to employ a good diffusing medium illuminated by a constant light source, and hence it is necessary in practice to measure values of O_t or D_t relative to a diffusing medium of known rejectance ρ A photometer, therefore, to be suitable for this class of work should be provided with at least three exactly similar good diffusing screens, one of them to serve as the permanent diffusing screen of the instrument, and two others to be measured as specimens for determining the value of ρ in accordance with the equation $O_1^2-O_2=\rho^2$, already derived

When other specimens of unknown properties are now measured in contact with the same diffusing screen, it is possible, from the results obtained and the value of ρ for the photometer, to deduce the desired values for any new material

For this method it is necessary, as a minimum, to obtain by measurement the values of the relative obstructances for specimens both one unit and two units thick The rejectance R_1 of unit thickness of the new material can then be derived from equation (5) in the form —

$$R_1^2(O_1^2-\rho^2)-R_1(O_2-1)\rho-(O_1^2-O_2)=0 \tag{18}$$

The values of C_1 and Ω_1 then follow from

$$O_1=(1-\rho R_1)C_1/(1-R_1)=(1-\rho R_1)\Omega_1,$$

while

$$K=2\cosh\lambda=(O_2+1)/O_1$$

From these data any desired values can be calculated with the aid of the equations given in the first part of this paper

It is necessary to utter a warning concerning the extreme care necessary for determining the values of ρ and R_1 in the above manner The results depend on the comparatively small differences between two much greater values, O_1^2 and O_2, consequently any errors in the values of O_1 and O_2 will be greatly increased in those of ρ and R_1 deduced from them

If two photometer diffusing screens of widely different and accurately known rejectances ρ and ρ' are available, a simpler solution of the problem offers itself, having, moreover, the advantage of requiring only one specimen of the material In these circumstances, if O and O' are the obstructances of the specimen relative to ρ and ρ' respectively, then the rejectance R of the specimen is given by

$$R=(O-O')/(\rho'O-\rho O') \tag{19}$$

The method of measurement between two opal glasses, prescribed by Nutting,[*] which is supposed to give the absolute or "total" transparency to diffused light, can be shown in accordance with our theoretical discussion of the subject to give in reality the following highly complex function of the properties of both diffusers and specimen —

$$\frac{\Omega_{m+p}}{\Omega_{m+n+p}} = \frac{\Omega_n(1-R_mR_p)}{\Omega_n^2(1-R_mR_n)(1-R_nR_p)(1-R_mR_p)-R_mR_p},$$

where n refers to the specimen between the two diffusing plates m and p.

Such a result evidently bears no simple relation to the obstructance of the specimen, nor could its value be deduced even if R_m and R_p were known

Experimental Confirmation of the Theory.

The paper already mentioned, published by one of us[†] in collaboration with O Bloch, contains measurements of the relative obstructance logarithms of specimens of opal glass and various other granular media (suspensions in gelatine) These results led to the discovery of an empirical rule connecting relative obstructance with thickness (or concentration) of the medium, viz —

$$D = at^b,$$

where a and b are constant for the medium and the given conditions of measurement This formula was found to represent very closely the observed relationship, the agreement falling well within the limits of experimental error in most cases.

No physical explanation of this rule was apparent, so that it was deemed advisable to make further measurements to see if any departure from it could be established by exercising every care, and especially to test practically our theoretical deductions. A sheet of thin ordinary opal glass of good uniform quality was cut up into six circles, one being of 5 cm. diameter, the rest of 3½ cm. diameter All of them were ground to a fine matt surface on both sides and brought to approximately equal and uniform thickness, except one (*f*) which was made specially thin. Measurements with a micrometer gauge showed that the extreme variation in thickness of any one specimen did not exceed 0·01 mm Before measurement, all were thoroughly cleaned by boiling in strong mixed chromic and hydrochloric acids.

The following are the thicknesses of the specimens as the mean of about 10 measurements of each :—5 cm circle, used as diffuser in the photometer, 0 969 mm. Specimen *a*, 0·962 mm ; *b*, 0·987 mm., *c*, 0 989 mm

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[*] 'Amer Ill. Eng. Soc. Trans.,' vol 10, p. 366 *et seq.*

[†] 'Photo Journ.,' vol 56, pp. 49–65.

d, 0·986 mm, *e*, 0·952 mm, *f*, 0·655 mm. The photometer used was of a simple type, operating on the principle of the law of inverse squares of distances of the illuminant. The illuminant was a fixed incandescent electric bulb having a compact gridiron filament, set at the back of the photometer box, which illuminated a piece of opal filling one half of the field of a Lummer-Brodhun cube, and also emitted a beam along the photometer bench. A travelling carriage on the bench carried two mirrors, set at 90° to one another, by which the light was returned on a parallel path and illuminated the 5 cm diameter opal disc already referred to, which was set in the hinged side of the photometer box close to the Lummer-Brodhun cube. The hinged side of the photometer box was double-walled, and so constructed that the 3½ cm specimens placed between the double walls were in intimate contact with the 5 cm disc of opal, but light from the central spot alone, about ¼ inch diameter, entered the photometer box, and was viewed in the Lummer-Brodhun cube.

The scale of the photometer bench, which was calibrated to read direct in relative obstructance logarithms, was carefully checked, and its setting relative to the diffusing opal was found correct by repeated measurements of the same specimen with several zero readings at widely different parts of the scale as well as by direct length measurements. No appreciable difference in the readings could be observed whether the surrounding, ¾ cm wide, annulus of the instrument opal was masked or not. (Bloch and Renwick found that this makes a difference if the specimens are too small.)

A number of sets of observations upon these specimens were made both with simple dry contact and also with the piles of specimens and the diffuser in optical contact throughout by means of films of cedar-wood oil. Very similar results were obtained in all repetitions of the same conditions, so that it is only necessary to give two sets of observations here by way of illustration.

In the first set the conditions were those of dry contact throughout the pile of specimens, and therefore surface, as well as body, reflection effects play their part.

Table I

Specimen added	t = total thickness	D observed	D calculated	Obs − calc Δ.
c	0·989	0·278	0·280	−0·002
e	1·941	0·480(5)	0·480	+0·0005
d	2·927	0·669	0·667	+0·002
a	3·889	0·886	0·887	−0·001
b	4·876	1·006	1·003	+0·003
e only	0·952	0·272	0·272	0·000

Plotting logs D against logs t, the values all fall well on a straight line, represented by the equation

$$\log D = 0{\cdot}80 \log t + \bar{1}{\cdot}451\,,$$

hence

$$D = 0{\cdot}282(5)\, t^{0{\cdot}80}$$

The values in the fourth column are calculated from this equation, and the (last) column of differences shows how accurately it represents the observations. These results and several other similar sets of readings show that the empirical rule discovered by Bloch and Renwick can be relied upon well within the limits of experimental error in this case, and we feel, therefore, justified in using it as an interpolation formula for calculating the relative obstructance logarithms of any desired intermediate thicknesses.

From the above equation we now calculate the relative obstructance values for one and for two specimens of the same thickness as the 5 cm. diameter instrument opal (viz., 0·969 mm.)

The results are:—

One specimen, 0·969 mm. thick, Rel. D = 0·275 (4), O_1 = 1·886
Two specimens, 0·969 mm. thick, Rel. D = 0·479 (5), O_2 = 3·016

Assuming that the instrument opal does completely diffuse the light reaching the specimens, we next calculate the rejectance of the diffuser towards the specimens for dry contact from the equation

$$O_1^2 - O_2 = \rho^2, \text{ i.e. } 1{\cdot}886^2 - 3{\cdot}016.$$

This gives the value of $\rho = 0{\cdot}734$.

From the same empirical formula we calculated the values in Table II for a set of specimens all exactly 1 mm. thick, in contact with the photometer opal (0·969 mm. thick). A trifling error is introduced in making these calculations when dealing with dry contact, by reason of the fact that part of the rejectance and obstructance is due to the air-opal surfaces, and this part of the whole will depend on the number of surfaces, not on the thickness of the specimens. Owing, however, to the fact that our measured specimens were all approximately equal to 1 mm. in thickness, the total correction for thickness is very small, and therefore the neglect of this trifling error is quite justifiable here. The effects of air-surface reflections among a pile of dry specimens become noticeable, however, if the specimens vary considerably in thickness. We then find that the series of observations do not fall well on a straight line when logs D are plotted against logs t, whereas they do so if these reflections are eliminated by uniting the specimens with films of cedar-wood oil (see the second set of observations recorded later).

Table II

Thickness	Log D	Log O_t	O_t
mm			
1	$\bar{1}$·451	0·282 (5)	1·916
2	$\bar{1}$·6918	0·492	3·105
3	$\bar{1}$·8327	0·680	
4	$\bar{1}$·9827	0·856 (5)	
5	0·0102	1·023	
6	0·0786	1·185	
7	0·1271	1·340	

Using the values of O given in Table II for 1 and 2 mm. specimens, we deduce the value of R_1 for a 1 mm. thick specimen from equation (18)

$$R_1^2(O_1^2-\rho^2)-R_1(O_2-1)\rho-(O_1^2-O_2) = 0,$$

taking $\rho = 0·734$, the result is $R_1 = 0·738$.

From $K = (O_2+1)/O_1 = 2\cosh\lambda$ we have $K = 2·142$, $\cosh\lambda = 1·071$, $\sinh\lambda = 0·3838$, $e^{-\lambda} = 0·6875$, and from equation (15), $R_\infty = 0·883$.

We are now in a position to test our complete theory by calculating values for the remainder of the specimens included in Table II from formula (12) for O_t.

The formula is

$$O_t = [O_1\sinh\lambda t-\sinh\lambda(t-1)]/\sinh\lambda.$$

The next table (Table III) is calculated from these data.

Table III

t	Log sinh λt	O_1 sinh λt	Sinh λ $(t-1)$	Log O_t = D_t
3	0·1392	2·640	0·822	0·675
4	0·3282	4·081	1·378	0·848
5	0·5029	6·101	2·129	1·015
6	0·6713	8·991	3·184	1·180
7	0·8367	13·16	4·691	1·344

If the values of log O_t in this third Table are compared with those in Table II (which are based directly upon observed values) it will be seen that the agreement is remarkably good; in no case is the difference so great as 0·01.

The agreement between theory and observation was further checked by plotting the logs of D_t taken from Table III against logs t, including also the assumed values for 1 and 2 mm. specimens. The equation to the line passing best through the plotted values was found to be identical with that found for the actual observations, viz —

$$\log D = 0{\cdot}80 \log t + \bar{1}{\cdot}451$$

The next illustration is one of a number of sets of observations made upon the same specimens with all surfaces in optical contact by means of films of cedar-wood oil, so as practically to eliminate the air-opal surface reflections

Table IV contains a summary of the observed values, and a comparison of these with those calculated from the empirical formula

$$\log D = 0{\cdot}86 \log t + 1{\cdot}431$$

Table IV

Added specimen	Total thickness	Log O_t = D observed	Calc D	D obs − D calc Δ
	mm			
f	0·655	0·185	0·187 (5)	−0·0025
e alone	0·952	0·252	0·258 (5)	−0·0065
c alone	0·989	0·270	0·267	+0·008
c + *e*	1·941	0·477	0·477	0·000
d	2·927	0·683	0·679 (5)	+0·003 (5)
a	3·889	0·869	0·867 (5)	+0·001 (5)
b	4·876	1·047	1·054	−0·007
j	5·531	1·174	1·174	0·000

Once again the agreement between the observations and the empirical rule is extremely good.

The following values of O were calculated as before for one and two specimens, 0·969 mm thick —

$$O_{0{\cdot}969} = 1{\cdot}830 \qquad O_{1{\cdot}938} = 2{\cdot}997,$$

whence (for oil contact) $\rho^2 = 1{\cdot}830^2 - 2{\cdot}997$, $\rho = 0{\cdot}594$.

The next Table, No. V, gives the values for thicknesses, 1 up to 7 mm, calculated from the same empirical equation.—

$$\log D = 0{\cdot}86 \log t + 1{\cdot}431.$$

Table V.

t	Log D	D	O
mm			
1	$\bar{1}{\cdot}431$	0·270	1·861
2	$\bar{1}{\cdot}6899$	0·490	3·088
3	$\bar{1}{\cdot}8413$	0·694	
4	$\bar{1}{\cdot}9488$	0·889	
5	0·0321	1·076	
6	0·1002	1·260	
7	0·1578	1·438	

As in the first example, we evaluate from the known values of ρ for oil contact, and of O for 1 and 2 mm thickness, the constants of the theoretical equation for O_t thus —

$$R_1^2(1{\cdot}861^2-0{\cdot}594^2)-R_1(3{\cdot}088-1)\,0{\cdot}594-(1{\cdot}861^2-3{\cdot}088) = 0,$$

$$R_1 = 0{\cdot}5995,$$

$$K = 2\cosh\lambda = (O_2+1)/O_1 = 2{\cdot}196, \qquad \epsilon^{-\lambda} = 0{\cdot}6445,$$

$$\sinh\lambda = 0{\cdot}4534, \qquad \lambda = 0{\cdot}4392$$

Also, from equation (15),

$$R_\infty = O_1R_1/[O_1-(1-\rho R_1)\epsilon^{-\lambda}] = 0{\cdot}7716$$

which leads to the following values for Schuster's constants —

$$\alpha = 0{\cdot}1289, \qquad \kappa = 3{\cdot}350 \qquad k = 0{\cdot}05662$$

The difference between R_1 for oil and air contact shows that about 14 per cent is rejected by the first air surface, and therefore the maximum total rejectance of a thick solid block of this opal = 0·91

The next Table (No. VI) is calculated from the above data on the formula

$$O_t = [O_1\sinh\lambda t - \sinh\lambda(t-1)]/\sinh\lambda$$

Table VI

t	Log sinh λt	O_1 sinh λt	Sinh $\lambda\,(t-1)$	Log O_t = D_t
3	0·2386	3·223	0·995	0·692
4	0·4484	5·225	1·732	0·887
5	0·6468	8·260	2·808	1·080
6	0·8407	12·890	4·434	1·271
7	1·0328	20·05	6·929	1·462

Comparing these values of log O_t with the values in Table V (based on the experimental observations), it is seen that, excluding the 7 mm thickness, which is beyond the range of our measurements, the agreement is remarkably close throughout, and also, how good an approximation the empirical rule is to the full theoretically deduced equation is strongly brought out

These and other similar sets of observations of the most careful kind appear to us amply to demonstrate the soundness of our theoretical conclusions and to confirm Bloch and Renwick's observations on the practical utility of the empirical formula if used within certain limits We have endeavoured to discover how closely the theoretical laws as here set out can be approximated to by use of the empirical rule, by comparing the results for a number of imaginary cases It is clear that any of our formulæ for Ω_t, C_t and O_t become identical with the well-known ordinary law of absorption in non-scattering media if Ω_1, C_1 and O_1 respectively $= e^{\lambda}$

But if R_1 is not zero and $O_1 = e^{\lambda}$, $(1-\rho R_1)\Omega_1 = e^{\lambda}$ and consequently $\rho = (\Omega_1 - e^{\lambda})/\Omega_1 R_1$, which is only another expression for R_∞

It is clear, therefore, that if by chance the rejectance of the photometer diffuser is equal to the maximum possible rejectance of the material under measurement, the results will follow the ordinary law of absorption for non-scattering media and the empirical rule $D = at^b$ will be exactly followed with $b = 1$

Although these precise conditions are not often realised, practical results and a study of imaginary cases indicate that the empirical rule holds very well for five and sometimes more units of thickness when a photometer diffuser of high rejectance such as white solid opal glass about 1 or more millimetres thick is used, whereas it fails to hold if R_1 is much higher than ρ

In conclusion we desire to express our great indebtedness to Prof A. Schuster and to Dr H S Allen, of King's College, London, for their valuable advice and suggestions in connection with the drafting of this paper, and to Dr T. M Lowry, F.R S, for so kindly bringing our work before the Royal Society

The Rankine Trochoidal Wave.

By Sir George Greenhill, F R S

(Received June 19, 1917)

Rankine's Wave Theory of 1862 is given in the 'Philosophical Transactions,' vol 153, p 127, "The Exact Form of Waves at the Surface of Deep Water." The theory finds favour with the naval architect from its simple geometrical structure, intelligible to a student of elementary mathematics.

But this student does not like to be reminded that the waves at sea are not trochoidal, as this would require a supernatural state of internal commotion in the water, involving a distribution of molecular rotation. Sea waves are not permanent, but in a state of perpetual growth and decay, in a procession of group motion

The Rankine theory will serve, however, as a first introduction to the structure of wave motion in water, so Rankine's treatment is reproduced here, with some simplification in his geometry But, so far, the Rankine trochoidal wave has defied any effort of generalisation, as to a kindred state of wave motion in shallow water, or to the stationary wave in reflexion at a wall. The equation of continuity breaks down, and cannot be satisfied. cavitation takes place, or else overcrowding of the particles of water

At the recent meeting of the Institution of Naval Architects, March, 1917, I have shown that, if thin parallel vertical walls are introduced into the water, perpendicular to the wave crest, without disturbing the wave motion, dividing up the water into compartments, these water compartments may be sheared over each other in an arbitrary manner, giving the wave front an échelon appearance, like a line of advancing infantry, and the wave motion will continue in each compartment

The motion may also be sheared in these compartments by inclining the walls at the same angle with the vertical, like a row of books sloping over on a shelf The circular orbit of a particle in the Rankine wave motion will then slope over at this angle, and his waves can exist between two parallel walls, either vertical or else sloping over at any assigned angle.

When the walls are brought close together and the shear steps are small, we can pass from discrete steps to continuous shear, uniform or variable; the wave front can then advance at any angle with the plane of the water motion, or in any curve

With a straight crest, the vertical section of the surface by a plane parallel to the shear is the original trochoid, and by a perpendicular plane is a sinusoid, by another plane a foreshortened trochoid

These curves can be shown off as various aspects or projections of a revolving helix of wire, which ought to have its axis horizontal to imitate the wave motion But it is more convenient in the experiment to let the helix hang vertically by a thread and revolve, and to treat the axis as horizontal in imagination.

Any projection by parallel lines of the helix on the floor will be a trochoid. a cycloid, if a line of projection can touch the helix

A projection by horizontal lines on a vertical wall will be a sinusoid, any other projection in general will be an elongated or foreshortened trochoid

In this way it is possible to break new ground in the Rankine trochoidal wave, and extend the treatment in some fresh directions But here a delicate question arises, requiring examination The continuity of the motion is preserved in the compartments when sheared, and also the free surface condition of constant pressure But the continuity of the pressure in crossing a wall must be examined as this may break down, and some new variable field of force must be introduced in addition to the constant field of vertical gravity, if the continuity of pressure is to be preserved when the walls or diaphragms are suppressed

Reduced to a standing wave by a reversal of the wave velocity, *k* or U, or else by taking a moving origin, as in a ship moving with the waves, the motion is treated as steady motion, and the equation of continuity and of pressure is somewhat easier to investigate This has been carried out by Rankine in his paper, 'Phil Trans,' 1863, and we are at liberty to apply to this steady motion a constant, cross-current velocity, V, parallel to the wave crest The effect is to make a stream line change from a trochoid to an elongated projection of the trochoid, on a vertical plane at an angle $\tan^{-1}(V/U)$ with the wave crest.

Advancing waves are called "rollers" and the standing wave may be called a "swell" (houle) in the language of the sailor Any intermediate state between rollers and a swell will be encountered by a steamer moving straight across the crest

To an observer on board the wave motion appears a swell if the steamer advances at the same rate as the rollers, and the relative motion appears steady, as in the trochoidal wave But at any other velocity the motion appears to be a mixture of the roller and the swell

The steamer may be moving faster or slower than the rollers, or in the opposite direction, head to sea A sailing ship before the wind, and moving through the water slower than the rollers, will be overtaken by the wave crest, and runs a risk of being "pooped"

When the course of the ship is parallel to the wave crest of the rollers, a "beam sea" is said to be encountered The circular orbit of the Rankine roller is then drawn out into a helix in the motion relative to the ship And this motion is added vectorially to the relative motion for a course perpendicular to the wave crest, when a course is steered at any other angle and given velocity In this way the trochoidal motion of Rankine can be extended to motion in three dimensions

Returning again to the rolling waves, and the circular orbit of a particle the effect of this cross current is to change the circle into a helix of stationary form advancing in rotation by passing through itself, as a screw through a nut or board, with no apparent alteration of the form of the wave surface

The question arises further, whether it is legitimate to shear the Rankine motion in vertical planes parallel to the wave crest In the previous states of shear the water was supposed divided up, by a number of thin vertical planes at close intervals, into compartments in which the motion can exist independently of the neighbours on each side, any pressure difference being taken up on the separating wall

In the Rankine theory of the advancing waves, or rollers in the sailor's name, the orbit of a particle is a circle But with respect to a vessel steaming with the waves, the wave motion will be stationary on the side relatively to the hull, while the water streams past, in trochoidal stream-lines The relative motion is reduced to a state of steady motion with respect to an origin on the vessel, as in a standing wave or swell To an observer ashore the waves appear as rollers

It is easier to examine the continuity and dynamical conditions in the steady motion of the swell, and so, following Rankine, a stream-line LBM is considered in fig 1, a trochoid generated by a point B fixed on a circle, centre C, and circumference the wave-length $L = 2\pi R$, rolling at A on the underside of a horizontal line OA with the wave velocity k, and angular velocity (A V) $n = k/R = 2\pi k/L$

The velocity of B is then n AB perpendicular to AB, and with $CB = r$, $FCB = \theta$, the co-ordinates of B are

$$x = R\theta + r\sin\theta, \qquad z = R + r\cos\theta, \qquad \theta = nt, \qquad R\theta = kt, \tag{1}$$

and its component velocities

$$u = k + nr\cos\theta, \qquad w = -nr\sin\theta \tag{2}$$

Join FB, and draw the next trochoid lbm a little lower down, produced by rolling an equal circle, centre c and cb parallel to CB, on a line through a at additional depth Aa = Cc

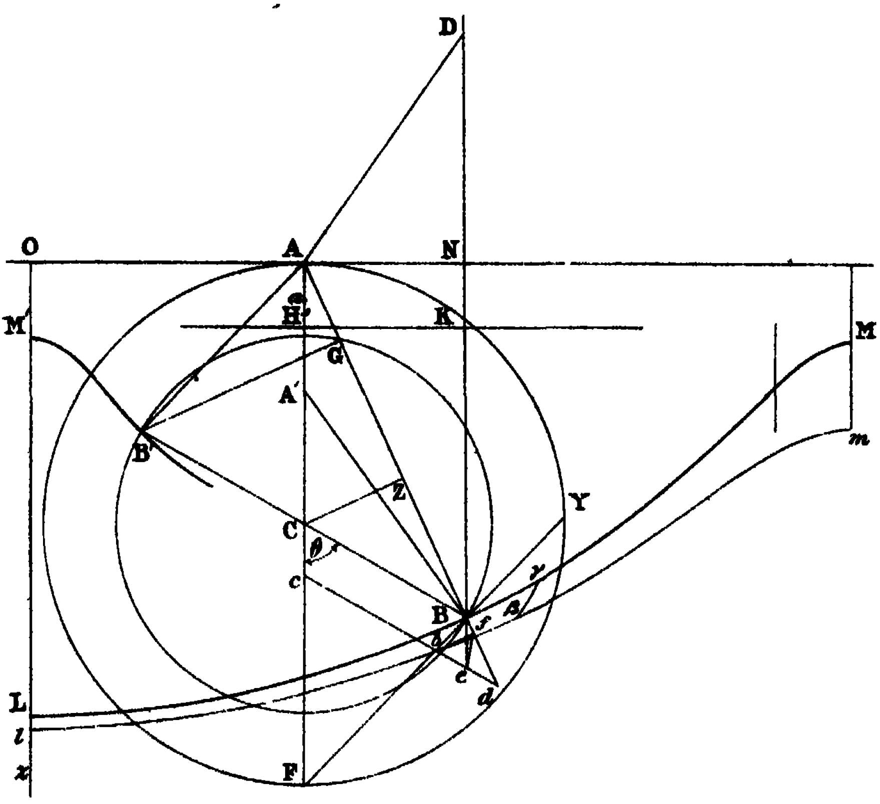

FIG 1

Then, cb crossing AB, NB in d, e, the infinitesimal figure B$bedf$B is a small reduced copy of AB′CBGA, so that $be = ed = ef$, and bfd is a right angle, and bf may be taken as a tangent element of the trochoid through b

Then

$$be/\mathrm{CB} = \mathrm{B}e/\mathrm{AC} \tag{3}$$

and with $\mathrm{B}e = \mathrm{C}c = \mathrm{A}a = dc$, and $cb = r'$, $be = r - r' = -dr$,

$$(r-r')/r = -dr/r = dc/\mathrm{R}, \tag{4}$$

so that r diminishes in G.P with the depth c increasing in A.P., and in accordance with the Exponential Theorem,

$$r = \mathrm{R}e^{-mc}, \qquad m\mathrm{R} = 1 \tag{5}$$

The flow across Bb due to the velocity $q = n\,.\,\mathrm{AB}$ perpendicular to AB is

$$n\,.\,\mathrm{AB}\ \ \mathrm{B}f = n\ \ \mathrm{AB}\ \ \mathrm{AG}\ \ \mathrm{B}e/\mathrm{A}c = n(\mathrm{R}^2-r^2)(dc/\mathrm{R}), \tag{6}$$

the same for all sections such as Bb, so that the space between the trochoids can flow full bore and the condition of continuity is assured

The arc $B\gamma = AB \, . \, d\theta$ for an advance $d\theta$ of the phase angle θ, so that an element of area $B\gamma\beta b$ between the two trochoids is

$$B\gamma \, . \, Bf = AB \, . \, Bf \, . \, d\theta = \left(R - \frac{r^2}{R}\right) dc \, d\theta, \tag{7}$$

and the area of a half wave-length between the trochoids LBM, lbm is

$$\pi \left(R - \frac{r^2}{R}\right) dc \tag{8}$$

This is the area of a horizontal stratum of the same half wave-length $\frac{1}{2}L = \pi R$ and of depth dc_0, if

$$\pi R \, dc_0 = \pi \left(R - \frac{r^2}{R}\right) dc \tag{9}$$

$$\frac{dc_0}{dc} = 1 - \frac{r^2}{R} = 1 - e^{-2mc}, \tag{10}$$

$$c_0 = c + \tfrac{1}{2} R e^{-2mc} = c + (r^2/2R), \tag{11}$$

and $c_0 - c = r^2/2R$ is the rise in height of the central line through C of the trochoid LBM over the height when the stream LBM is flattened into a horizontal line in still water The sea "rises" above its former mean level by this amount at the surface

Thus $c_0 - c$ may be taken as the rise on the side of the mean water line of a vessel, several wave-lengths long, moored in the current of these standing trochoidal waves And the flow between any two trochoid stream-lines a finite distance apart is k times the depth of the stratum when at rest

The curve Bb prolonged is shown in Froude's figure of Rankine's memoir, and is identified, by the property of the constancy of the subtangent CF, as a logarithmic curve, referred to oblique axes, one vertical, the other inclined at the angle θ. This may be called the curve of a bulrush, waving in the water The curve is recognised also in the trajectory of a sphere in still water or air when the resistance varies as the velocity, and then C descends with the terminal velocity k, and $CB = Re^{-kt}$

In accordance with roulette theory, the circle of inflexions of the trochoid is on the diameter AC, and the radius of curvature,

$$\rho = AB^2/BZ, \qquad q^2/g\rho = BZ/AC, \quad \text{with} \quad g/n^2 = AC, \tag{12}$$

implying $k^2 = gR = gL/2\pi$,

$$(q^2/g\rho) + \cos\psi = BZ/AC + AZ/AC = AB/AC. \tag{13}$$

Then if dp is the increase of pressure along Bf to the next trochoid

$$dp/Bf = \text{force along } Bf = w \, . \, AB/AC, \tag{14}$$

the same as in the circular orbit of the roller wave, with stationary centre C,

$$\frac{dp}{w} = \frac{\mathrm{AB}\ \mathrm{B}f}{\mathrm{AC}} = \left(1 - \frac{r^2}{p^2}\right) dr = dc_0, \tag{15}$$

so that the pressure gradient and the pressure are the same as in still water

We are using Rankine's gravitation units here of the engineer, in the F.P S system (foot-pound-second) pressure p is measured in lb/ft^2, density w in lb/ft^3

Take A′ the inverse point of A in the circle BC, and draw HK horizontal, bisecting AA′ at right angles With BD = 2BK, the triangles ABC, BA′C, ADB, are similar

$$\mathrm{AC}/\mathrm{AB} = \mathrm{CB}/\mathrm{A'B} = \mathrm{AB}/\mathrm{BD}, \quad \mathrm{AB}^2 = \mathrm{AC}\,.\,\mathrm{BD} = 2\mathrm{AC}\ \ \mathrm{BK}, \tag{16}$$

$$q^2 = n^2\ \ \mathrm{AB}^2 = g\ \ \mathrm{AB}^2/\mathrm{AC} = 2g\ \ \mathrm{BK}, \tag{17}$$

so that the particle B moves along the trochoid as if on a smooth switchback railway, with velocity due to the level of HK.

In the liquid stream between LBM and *lbm*, the thickness is inversely as the velocity, n AB

If R is the pressure at B of the particle, of weight W,

$$\mathrm{R}/\mathrm{W} = \cos\psi - q^2/g\rho = \mathrm{AZ}/\mathrm{AC} - \mathrm{BZ}/\mathrm{AC} = \mathrm{AB}/\mathrm{AC}, \tag{18}$$

so that the pressure per unit length of the stream

$$dp = \mathrm{R}/\mathrm{B}\gamma = \mathrm{W}/\mathrm{B}\gamma(\mathrm{AB}/\mathrm{AC}) = w\ \ \mathrm{B}f(\mathrm{AB}/\mathrm{AC}), \tag{19}$$

$$\frac{dp}{w} = \frac{\mathrm{AC}^2 - \mathrm{BC}^2}{\mathrm{AC}^2}\,\mathrm{BC} = \left(1 - \frac{r^2}{\mathrm{R}^2}\right) dr = dc_0, \tag{20}$$

as before

The liquid particles are crowded together, and bunch up, like traffic on the road, near L, where the speed is small, and then tail out towards M as the speed increases.

Treating Bβ as a horizontal step dx from B to the next trochoid at β, the difference of pressure at B and β is

$$dp = w\ \ dc_0 = w\ \ \mathrm{AB}\ \ \mathrm{B}f/\mathrm{AC}, \tag{21}$$

$$dx = \mathrm{B}\beta = \mathrm{B}f/\sin\psi, \tag{22}$$

$$\frac{dp}{wdx} = \frac{\mathrm{AB}\sin\psi}{\mathrm{AC}} = \frac{\mathrm{AN}}{\mathrm{AC}} = \frac{r}{\mathrm{R}}\sin\theta = e^{-mc}\sin\theta \tag{23}$$

With thin compartments of breadth dy, sheared at angle S, $\tan \mathrm{S} = dx/dy$, and dp the pressure difference on the sides of a wall,

$$dp/w\,dy = dp/w\,dx\,.\,\tan\beta = e^{-mc}\sin\theta\tan \mathrm{S}, \tag{24}$$

requiring the introduction of a field of force of this intensity perpendicular to the wall.

The échelon waves cannot exist, then, with the wave front slewed through this angle S without the introduction of this horizontal field of force perpendicular to the vertical plane of the circular orbits or trochoidal stream-lines. So also a vertical shear of the compartments would require an additional field of

$$Y = [1+(r/R)\cos\theta]\tan S \tag{25}$$

But if the walls of the compartments lean over at an angle α with the vertical, the trochoidal stream-lines can still remain comparable with each other in the steady motion of the standing wave, and in the rollers the circular orbits are inclined at α to the vertical.

This motion can persist when the walls are removed, without a need of an additional field of force, provided g is replaced by $g\cos\alpha$

In the associated state of wave motion of small displacement, the velocity function ϕ is changed from

$$\phi = Ae^{-mz}\cos(mx+nt) \quad \text{to} \quad Ae^{-mz\cos\alpha - my\sin\alpha}\cos(mx+nt), \tag{26}$$

with wave crest perpendicular to a wall sloping at angle α, and with circular orbits parallel to the wall

So it is worth while examining if it is possible to cut down the original free surface of the Rankine wave motion between the slanting walls, to a mean slope such that it can still remain a surface of constant pressure, without interfering with the continuity of the pressure in the interior

Consider the liquid motion between two vertical walls parallel to the wave crest of the standing trochoidal waves in steady streaming motion, and the trochoidal path of a water particle when a horizontal field of gravity, $g\tan\alpha$, is introduced perpendicular to the walls in addition to the vertical field g, which we assume will not interfere with the motion

Cover the free surface with a fixed cylinder moulded to the surface of the running water in stationary waves, capable of taking a variation of pressure

The liquid motion is altered by the new field, $g\tan\alpha$, and continuity will be preserved, but a surface of equal pressure will change in the interior of the water, and if a new surface of equal pressure is composed of trochoidal stream-lines, the water above it may be removed, and a new state of motion is obtained in a field of gravity $g\sec\alpha$, and the plane of a trochoidal path is inclined at an angle α with the resultant gravity

Suppose, then, the original g is reduced to $g\cos\alpha$, and a new field $g\sin\alpha$ is introduced perpendicular to the walls, this is equivalent to a field g at an angle α with the walls and Bernoulli's equation in (40) changes with absolute units into

$$\frac{p}{\rho} - gz\cos\alpha + gy\sin\alpha + \tfrac{1}{2}q^2 = gH, \tag{27}$$

where H is constant along a stream-line,

$$\frac{p}{\rho}-gc\cos\alpha-gr\cos\alpha\cos\theta+gy\sin\alpha+\tfrac{1}{2}n^2(R^2+2Rr\cos\theta+r^2) = H \qquad (28)$$

Then p is constant along the stream-line, if $n^2R = g\cos\alpha$, and then

$$y\sin\alpha-c\cos\alpha+\tfrac{1}{2}\frac{r^2}{R}\cos\alpha = \text{a constant} \qquad (29)$$

along the profile of a vertical section parallel to the wave crest and perpendicular to the walls, a line connecting points of equal pressure

A surface of equal pressure is built up of these profiles, and since $r = Re^{-mc}$, $mR = 1$, to satisfy the continuity, the profile is the exponential curve

$$y\tan\alpha-c+\tfrac{1}{2}Re^{-2mc} = \text{a constant}, \qquad (30)$$

having an asymptote parallel to $y\tan\alpha = c$, the new horizontal

Thus the water may be drawn off above one of these surfaces of equal pressure, and, by tilting the walls to an angle α with the vertical, the new field of g may be made vertical A new system of stationary trochoidal waves is then realised in which the plane of a trochoidal stream-line (or of the circular orbit of the rollers) is parallel to the sloping walls, the wave amplitude dying out in G P as it recedes from the upper side of the sloping wall

The waves diminish in height towards the under side of a sloping wall, and so illustrate the form of a wave under the sloping bow of a steamer

These new stationary trochoidal waves may be changed into moving rollers by reversing the general stream of the water in the steady motion, and then the velocity of advance will be

$$k' = k\sqrt{(\cos\alpha)} = \sqrt{(gR\cos\alpha)} = \sqrt{\left(\frac{gL}{2\pi}\cos\alpha\right)} \qquad (31)$$

The curve on the wall or sloping shore of the surface of the water will be a trochoid, stationary, or advancing with velocity k', the wave crests reaching out to the offing at sea with diminishing amplitude, and tending to the small motion considered previously in (26) of waves along a sloping beach.

The analytical expression of the Rankine moving wave or roller may be stated and generalised, in the Lagrangian form, with Ox in the surface perpendicular to the wave crest, and Oz vertically downward,

$$x = a+r\sin\theta, \qquad z = c+s\cos\theta, \qquad \theta = ma+nt, \qquad (32)$$

where r, s are supposed functions of the depth c, determined from the condition of the equation of continuity. Afterwards the pressure equation can be investigated, and the condition for a free surface, if possible

Then by differentiation

$$dx/da = 1 + mr \cos\theta, \qquad dz/da = -ms \sin\theta, \tag{33}$$

$$dx/dc = (dr/dc) \sin\theta \qquad dz/dc = 1 + (ds/dc) \cos\theta, \tag{34}$$

$$\frac{d(x,z)}{d(a,c)} = (1 + mr \cos\theta)\left(1 + \frac{ds}{dc} \cos\theta\right) + ms \frac{dr}{dc} \sin^2\theta$$

$$= 1 + \left(mr + \frac{ds}{dc}\right) \cos\theta + mr \frac{ds}{dc} \cos^2\theta + ms \frac{dr}{dc} \sin^2\theta, \tag{35}$$

and this must be independent of θ for Lagrange's equation of continuity to be satisfied Thus

$$r\, ds/dc = s\, dr/dc, \quad 1/r \;\; dr/dc - 1/s \;\; ds/dc = 0, \quad r/s = e, \text{ a constant}, \tag{36}$$

$$mr + ds/dc = 0, \qquad 1/s \;\; ds/dc = -em, \qquad s = \mathrm{R}e^{-emc}, \tag{37}$$

and the particle orbit is an ellipse

In the associated steady motion swell

$$u = \mathrm{U} + nr \cos\theta, \qquad w = -ns \sin\theta, \qquad \theta = ma, \tag{38}$$

$$q^2 = u^2 + w^2 = \mathrm{U}^2 + 2\mathrm{U}nr \cos\theta + n^2(r^2 \cos^2\theta + s^2 \sin^2\theta) \tag{39}$$

and in Bernoulli's equation and a gravity field

$$\mathrm{H} = p/w - z + q^2/2g \tag{40}$$

is constant along a stream-line Here all the terms are of the dimensions of a length.

But this condition cannot be satisfied unless $r = s$, $e = 1$, and the orbit of the roller a circle, and we arrive at the Rankine motion, and then

$$\mathrm{H} = p/w - c - r \cos\theta + \mathrm{U}^2/2g + \mathrm{U}nr/g \;\; \cos\theta + n^2r^2/2g, \tag{41}$$

is constant along a stream-line, if

$$\mathrm{U}n = g, \qquad n = \mathrm{U}m, \qquad \mathrm{U}^2 = gm = g\mathrm{L}/2\pi \tag{42}$$

Otherwise, if r and s are unequal, an additional field of force must be introduced to make a free surface possible

Reducing Rankine's wave to a stationary wave by a combination with a reflected wave

$$x = a + r \sin ma \cos nt, \quad z = c + r \cos ma \cos nt, \quad dr/dc = -mr \tag{43}$$

and a liquid particle oscillates in a straight line; but

$$d(x,z)/d(a,c) = 1 - m^2r^2 \cos^2 nt, \tag{44}$$

and the equation of continuity is not satisfied; so it is useless to proceed further, unless r^2 is treated as negligible.

So also, if we tried to generalise the equations of wave motion of small displacement in shallow water of depth h, with

$$x = a + \mathrm{A\,ch}\, m(h-c) \sin(ma+nt), \quad z = c + \mathrm{B\,sh}\, m(h-c) \cos(ma+nt), \tag{45}$$

the equation of continuity cannot be satisfied, unless we suppose A = B, and AB insensible, and then we arrive at the ordinary theory for waves of small displacement in shallow water, and exact equations cannot be maintained

In the usual theory of hydrodynamics of the straight crested wave of small displacement in a canal, where the velocity function $\phi = \mathrm{P} \cos(mx+nt)$ and P is a function of y and z, satisfying the condition of continuity

$$(d^2\mathrm{P}/dy^2) + (d^2\mathrm{P}/dz^2) = m^2\mathrm{P}, \tag{46}$$

and the conditions at the boundary The lines of flow are given by

$$dx/\mathrm{P} = dy/\mathrm{Q} = dz/\mathrm{R}, \qquad \mathrm{P, Q, R} = d\phi/dx,\ d\phi/dy,\ d\phi/dz, \tag{47}$$

and if two solutions can be obtained of these differential equations, in the form $u = a$, $v = b$, then $\mathrm{F}(u, v) = 0$ will give a bounding surface which can hold the motion, and this is the solution of the partial differential equation

$$\mathrm{P}(dz/dx) + \mathrm{Q}(dz/dy) = \mathrm{R} \tag{48}$$

Any of the solutions obtained hitherto in hydrodynamics for wave motion can be sheared continuously in a vertical plane perpendicular to the straight wave crest at an angle S, constant or variable, and the wave crest may be made to take any curve, and the shape of the containing vessel may be determined to hold the new motion

But if continuity in the pressure is to be preserved, a field of horizontal force must be introduced

$$\mathrm{Y} = \frac{1}{w}\frac{dp}{dy} = \frac{1}{w}\frac{dp}{dx} \tan \mathrm{S}, \tag{49}$$

and in the pressure equation, in gravitation units,

$$\frac{p}{w} - z + \frac{d\phi}{g\,dt} = \text{constant}, \tag{50}$$

so that in the shear, with z unaltered,

$$\frac{1}{w}\frac{dp}{dx} = -\frac{d^2\phi}{g\,dx\,dt} = \frac{mn}{g}.\phi \tag{51}$$

Curl of a Vector and Molecular Rotation, Circulation, and Rotation.

Interpreted in uniplanar motion and a vector velocity q, with components u, v, the circulation is defined as the line integral of the tangential component of the velocity, divided by twice the area, this is called the curl or rotation, by analogy with a rigid body, and of a liquid moving bodily as if solid.

Thus if l, m denote the direction cosines of the outward drawn normal of a closed curve $l = dy/ds$, $m = -dx/ds$, and the circulation

$$C = \int (lv - mu)\, ds \qquad (52)$$

while twice the area

$$2A = \int (lx - my)\, ds. \qquad (53)$$

Draw the line $NP_1P_2P_3P_4$ in fig 2 parallel to Ox, entering the closed

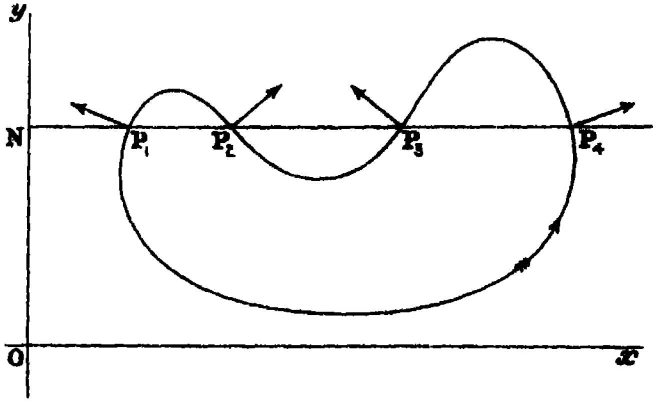

Fig 2.

curve at P_1, P_3, . and leaving it at P_2, P_4, . , then

$$lv\, ds = -v dy \text{ at entry}, \quad +v dy \text{ at exit},$$

$$= \int \frac{dv}{dx}\, dx\, dy \qquad (54)$$

and similarly

$$mu\, ds = \int \frac{du}{dy}\, dx\, dy. \qquad (55)$$

so that

$$C = \iint \left(\frac{dv}{dx} - \frac{du}{dy}\right) dx\, dy, \qquad A = \iint dx\, dy \qquad (56)$$

For a small circuit of an area element $dx\, dy$, enclosing a point P, the rotation becomes

$$\tfrac{1}{2}(dv/dx - du/dy) \qquad (57)$$

For a bodily rotation about O, with A V ω, $v = \omega x$, $u = -\omega y$,

$$C = \omega \int (lx - my)\, ds = 2\omega A \qquad (58)$$

Consider the circulation in the trochoidal motion round the element of area $Bb\beta\gamma$ in fig 1

The circulation over $B\gamma$ is

$$n \cdot AB \cdot B\gamma = n \cdot AB^2\, d\theta = n(R^2 - 2Rr\cos\theta + r^2)\, d\theta \qquad (59)$$

and the counter circulation over $b\beta$ is

$$n(\mathrm{R}^2 - 2\mathrm{R}r'\cos\theta + r'^2)\,d\theta, \tag{60}$$

the difference being

$$-2n\mathrm{R}(r - r')\cos\theta\,d\theta + n(r^2 - r'^2)\,d\theta,$$

$$= -2nr\cos\theta\,dc\,d\theta + n(r+r')\frac{r}{\mathrm{R}}\,dc\,d\theta,$$

$$= -2nr\,d(\sin\theta)\,dc + n(r + r')\frac{r}{\mathrm{R}}\,dc\,d\theta \tag{61}$$

The counter circulation over $b\mathrm{B}$ is

$$n\ \mathrm{AY}\ b\mathrm{B} = n\ \mathrm{AB}\ bf = n\ \mathrm{AB}\ \mathrm{GB}'\,.\,\mathrm{B}c/\mathrm{AC},$$

$$= 2n\ \Delta\,\mathrm{ABB}'\,dc/\mathrm{R} = 2n\ \Delta\,\mathrm{ABF}\,dc/\mathrm{R}$$

$$= 2n\mathrm{R}r\sin\theta\,dc/\mathrm{R} = 2nr\sin\theta\,dc, \tag{62}$$

and the circulation over $b\mathrm{B}$ and $\beta\gamma$ is

$$2nr\,d(\sin\theta)\,dc \tag{63}$$

Thus the total circulation round $\mathrm{B}\gamma\beta b$ is

$$n(r + r')r/\mathrm{R}\ \ dc\,d\theta \tag{64}$$

Dividing by twice the area, the rotation

$$\omega = n\frac{(r+r')r/\mathrm{R}\ \ dc\,d\theta}{2(\mathrm{R} - r^2/\mathrm{R})\,dc\,d\theta} = \frac{n(r+r')r}{2(\mathrm{R}^2 - r^2)} \rightarrow \frac{nr^2}{\mathrm{R}^2 - r^2} \quad . \tag{65}$$

Such liquid molecular rotation could not have been set up from rest in a perfect fluid under the action of natural forces and the pressure of the surrounding liquid, on the principle of normality. A cylindrical or spherical element of the liquid for instance, if solidified, could not acquire a rotation, or lose it As stated by Aristotle in 'De Cœlo,' *ἥκιστα δὲ κινητικὸν ἡ σφαῖρα διὰ τὸ μηδὲν ἔχειν ὄργανον πρὸς τὴν κίνησιν*

Revolving Fluid in the Atmosphere.

By John Aitken, F R S

(Received December 14, 1917)

In the 'Proceedings' of this Society (Series A, vol 94, No A 656, November, 1917), there appears a paper under the above title by Sir Napier Shaw In that paper reference is made to my experiments on cyclonic motion As no objections are offered to any of the points raised by me, there does not seem to be any necessity for my making a reply, unless it be that, having used the language of cyclonic and anticyclonic circulation, I am called upon to support that theory, or to acknowledge that Sir Napier Shaw's holds the field

Before accepting this new theory, one naturally enquires as to the foundations on which it is built, but first we had better to try to understand why the writer of the paper is dissatisfied with the cyclonic theory He tells us that we have not been interpreting correctly the charts of winds and isobars formerly issued by the Meteorological Office, and says that, "if the motion of the air in a cyclone had really been motion of a revolving fluid symmetrical with regard to a vertical axis it would not have appeared in circular form on the map consequently the appearance of uniform and symmetrical instantaneous motion in a cyclone is in itself proof that we have not in that case symmetrical motion about a centre of a mass which travels as a whole"

Having thus stated that the charts of isobars and winds are frauds, as they are not what they seem to be, he proceeds to discuss what the winds and isobars ought to be in an area over which a cyclone is passing. For this purpose he selects conditions which are only to be found in the very small disturbances known as secondary cyclones, and at p 41 he gives a series of figures showing what he thinks the winds ought to be when a cyclone is stationary, and when it is travelling at different relative velocities.

Let us now consider in detail those figures which we are told represent what the winds ought to be in moving cyclones The problem here illustrated is, with slight variations, the old one of the motion in the wheel of a carriage while moving along a road Some people have difficulty in seeing that the wheel is rotating in the way it rotates when the axle is raised and the wheel is free to revolve The fact that the part of the wheel touching the ground has no motion seems to confuse our ideas, and some call it a rolling movement and not a true revolving one. To help to clear our ideas we can imagine the following experiment Mount in the carriage, on a vertical axis, another wheel of the same diameter as the one resting on the ground, and gear the two to revolve in the same direction and at the same speed Sitting in the carriage while it is in motion, we shall see that the horizontal wheel is spinning round all right, and that its centrifugal force is equal all round It has no "dead point" like the wheel on the road If we now steer the carriage to the side of the road where there is a wall we shall find that the part of the horizontal wheel next the wall has no motion relatively to it It is as "dead" at that point as the part of the other wheel touching the ground It is easy to see, however, that the seeming deadness here is due to the backward motion of the rim of the wheel being just equal to the forward movement of the carriage

The movements in the cyclone illustrated by the figures referred to are of the same kind It is the movement of the air in the cyclone relatively to the earth which is shown by the lines drawn through the circles, that is, of the winds in cyclones when stationary, and when moving at different velocities We might say that the cyclone rolled its way through the air At some point to the left of the centre where the circular motion is equal to the translational one, there will be no wind, and the cyclone will, so to speak, roll along a line at that distance from the centre of its track In the figures referred to this "dead" or calm part will be above the centre of the circle representing the cyclone in fig 2, and in fig 3 it will be at the very top of the circle

At p 44 of the paper, the writer, after discussing the meaning of the lines in the four figures, says "Let us, therefore, now regard figs 2 and 4 as diagrams of isobars of that part of a map which contains a mass of revolving fluid carried along in a main current" I feel quite unable to accept this conclusion These lines represent the paths of the air relatively to the earth Now, the centrifugal force which produces the isobars is determined by the rotation of the air relatively to the axis We cannot help coming to the conclusion, therefore, that the isobars will be circular as in fig 1, where there is no translation, and remain circular while travelling The centrifugal conditions in a horizontally spinning wheel such as in the imaginary carriage previously described do not change when the carriage is put in motion, and one would expect that the isobars represented in fig 1 would continue while it was travelling

The conditions selected in the paper for illustration of a moving mass of revolving fluid are very exceptional, and are only found in secondary cyclones, in which the revolving mass of air is formed in a body of air all travelling in the same direction, while in ordinary cyclones the air blows from different directions on opposite sides, the result being that the latter do not travel at anything like the speed of the former, but only at about one-third to one-half of it, so that figs. 2, 3 and 4, p 41, give us no information as to what the winds should be in ordinary cyclones, as these travel, not in a general current, but, as has been shown,* in the direction of the strongest winds entering them, and hence the vagaries in their wanderings. Though they generally get the strongest winds on their southerly side and move eastwards, yet they sometimes get them on other sides, and they may slow down or stop when the winds from all directions become nearly equal, or go back on their track, if the strongest winds are on the north side. Secondaries, on the contrary, have to travel in the main current, and, though they may disappear, they cannot retrace their steps. In ordinary cyclones, therefore, we have to deal with totally different and much more complicated conditions, but I think it will be found that the winds and isobars in the weather charts really represent what they look like.

We will now confine our attention to these ordinary cyclones, which the writer of the paper says are evidently not revolving masses of air, since their winds do not blow in the paths they ought to follow if the revolution is accompanied by translation. I suppose it was necessary to make some assumption for the purpose of working out the mathematics of the subject, but it was hard on the cyclonic theory to treat the cyclone as an eddy, with a closed lower end. If that had been its structure, then there would have been some justification for saying that if the winds were circular it was evident that their motion was not a combination of rotational and translational motions. Now, according to the cyclonic theory the cyclone is not simply an eddy in which the air moves in circular paths, but is a system in constant flux; fresh air is flowing into it all round. But this incoming air does not belong to the cyclone, and does not possess the same motion as the air in the cyclone. It is new matter coming in which will ultimately take up cyclonic motion, but while approaching the centre cannot be considered to have true cyclonic and translational motion. As the isobars in the weather charts are the result of observation, I presume we may treat them as facts. Such being the case, are not the winds in cyclones just those which we would expect under the conditions represented by them? A travelling cyclone will pick up its air in very much the same way as it does when it is stationary. That it does so we can see by observing what is taking place in small artificial cyclones and in eddies in water. In these we can see that the lines of inflow at the bottom are alike when they are stationary and when travelling. If the cyclone were a closed system, it would, in pushing its way through the atmosphere, cause the stream-lines of the surrounding air to be quite different from what they are. If we are to find winds whose direction of motion relatively to the earth is a compound of their rotational and translational velocities, then we must look for them at higher elevations, where the air has acquired a true cyclonic motion.

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* 'Phil Trans. Roy Soc., Edin.,' vol. 40, Part I (No. 7).

If the above explanation be correct, then some evidence might be obtained from cloud observations. The path of a cloud moving under these conditions, as seen from the earth, is a curtate cycloid similar to that shown here in fig. 1. In this curve the rate of advance of the centre is two-thirds the velocity of revolution. The arrow-heads in the line of advance of the centre of revolution, and on the curved lines, show the direction of motion. The distances between the black dots on the lines give the relative velocities at the different parts. This curved line would represent the track of the shadow of a cloud on the earth under a vertical sun. It should be noted that this curve gives the successive motions of the air in a cyclone, while the curves and arrows in the figures in the paper referred to give the instantaneous conditions. An examination of fig. 1, here given, shows that the clouds on the right-hand side of the cyclone will travel quickly and change their direction slowly, while those on the left-hand side will travel slowly and move in more curved paths.

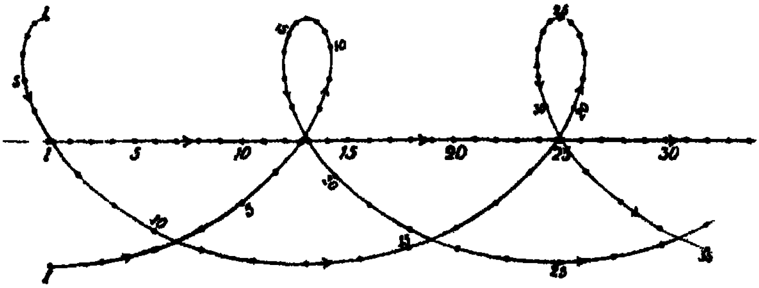

Fig. 1

While on this subject, it may be pointed out that, if we take an ordinary spiral spring and place it vertically, it gives a fair illustration of the circular and upward movements of the air in a stationary cyclone, and, if we distort this spring by drawing the top end to one side in a direction at right angles to its length, we get a series of curves similar to those drawn in fig. 1. Fig. 2 is from a photograph of a piece of brass wire, which was first wound round a cylinder to make it spiral, and its two ends then simply drawn apart.

I do not know if there is any observational evidence of a quicker and more uniform direction of the clouds over the right-hand side of a cyclone than over the left, but it appears to me that the writer of the paper must look to the direction of motion in the higher winds for evidence of the combined circular and translational movements, and not to the winds at the surface of the earth We cannot expect, however, to find a close resemblance between the motion of clouds and paths such as are shown in fig 1, because a cyclone is not a closed eddy, and, as there is an intake of air at the bottom, so there must be an outflow somewhere, and this will affect the motion of the clouds

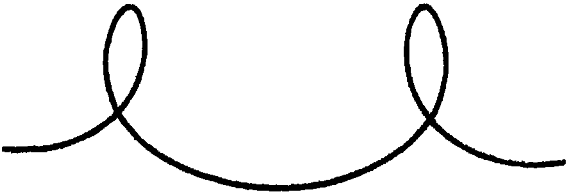

Fig 2

The writer of the paper has selected a small secondary cyclone for his illustration, and, as these cyclones move in a uniform general current, seems to keep to this idea for all cyclones, because further on he objects to Egnell's law, that the velocities of the air at any height are inversely proportional to the densities of the air at those heights He ridicules this statement, and gives figures to show that, if it is true, the top of a column of revolving fluid 10 kilom high would be 2000 miles away from its foot after a day Let us turn to nature We do not find any such supposed upper current The clouds in the upper currents move in very much the same directions as the lower winds, though veering a little in advance in veering winds There is no evidence of a general current sweeping across the cyclonic area, therefore there is no chance of their tops being swept away

I think it will be admitted that it is unfair to the theory of cyclonic circulation to perpetuate the commonly received statement—repeated in the paper referred to—that small revolving storms, such as the eddy of dry leaves, the "dust-devil" waterspouts, whirlwinds, and tropical revolving storms, all have their analogue in the vortex-ring They bear a likeness to it, but they are fundamentally different The vortex-ring is a closed system, in which the same air keeps circling Its rotation and its travel through the air are derived from the impetus it receives in the smoke-box, but it is a dying system from the moment of its birth, whereas eddies and small revolving storms are open systems, and can be fed and grow. The air composing them is constantly changing, and their travel is determined by the motion of the air entering them They are only partially closed at the bottom, where the air enters, because the air there has a less velocity, owing to its friction on the surface of the earth, and they are partially closed at the top by the ascent of the lighter air, and by the change of the velocity of the outflowing air So far as can be seen, they do not require the elaborate capping suggested by the writer of the paper It seems very strange that this erroneous idea of cyclones and eddies being closed systems should have persisted, because we are constantly seeing in our everyday experience evidences of the indraught at the bottom One cannot even stir one's tea without demonstrating this. Tea-leaves are heavier than water, and, if they were rotating at the same velocity as the tea higher up, would, by their centrifugal force, be driven to the circumference of the cup. But we always find them at the centre of it, because the tea and the leaves, by their friction on the bottom, have part of their velocity destroyed, their centrifugal force is therefore less than the tea higher up, with the result that the tea and the leaves on the bottom are drawn by the bottom current entering the centre of revolution Rivers would also seem to do some of their excavating by means of their eddies, as these draw in solid matter at the bottom, and transport and mix it with the higher water, so assisting the stream to carry it away

I shall now return to the somewhat exceptional conditions represented by the charts of isobars and winds selected by the writer of the paper to illustrate his ideas of a revolving fluid in the atmosphere, as shown in his fig. 5. Most of us will admit it is a cyclone, but it is a very different one from the ordinary cyclone It is formed in a current all flowing in the same direction on both sides of its centre, and not, as usual, surrounded by winds from different directions The extreme violence of the winds accompanying the passage of this cyclone was due to the great velocity of translation of the centre, so that on the south side of the cyclone the rotational velocity was added to the translational It is probable that the calmest area in the cyclone represented in fig. 5 was on the northern side of the centre, where the circular motion was equal and opposite to the translation velocity, and not near the centre as in ordinary cyclones

Again, it may be noted that the speed of translation of this cyclone was quite extraordinary, being two or three times that of an ordinary one. This selected example of cyclonic motion represents conditions only to be found in secondaries, and helps but little in explaining ordinary cyclones.

I should like to make a few remarks on these small secondary and violent storms. Their origin seems to be somewhat obscure What has disturbed the even flow of the main current and caused the eddy? There seems to be a possibility that rain may be the cause, not the well-known liberation of heat due to the condensation of water vapour, but purely its mechanical action. Long ago, when making observations on the dust particles in the air of Paris—where I was kindly given permission by M Eiffel to use his tower for this purpose, long before it was opened to the public—while observing at the top of the tower, the number of particles was found to vary greatly, as might be expected at so great an elevation over a highly polluted area. The highest number observed was over 100,000 per cubic centimetre, frequently it was half that number, or even less But about mid-day a cloud came up, and it began to rain After a short time, the number fell to 226 per cubic centimetre, which is just about the figure for pure air anywhere on the Continent This change in the number had evidently been brought about by the rain, not by its washing out the dust, as that is a slow process, but by the mechanical action of the falling drops, which by their friction had caused a downward component in the wind, which brought the purer air of a greater elevation down to the top of the tower The effect of this downward current of the upper air, induced by the drag of the rain, can frequently be seen at low levels On the approach of rain, especially if it be heavy, a change in the wind can be frequently observed, the wind in front being accelerated, that behind retarded, while the direction of the winds on the sides is changed. This downward current will obviously give rise to a want of uniformity in the mass of moving air, and it seems possible that it may form an obstruction to it, and so give rise to an eddy.

Looking for experimental illustration of this idea, I took a large circular flat-bottomed vessel and filled it with water in which a little fine sawdust was mixed to show its movements After putting the water in circular motion, and allowing it time to settle into steady flow so as to provide a steady stream of water, a small quantity of sand—to act like rain on the air—was allowed to fall into the water, and while pouring it out the hand was kept as nearly as possible following the current, so as to cause the sand to fall into the same part of the moving water. When this was done a quite well formed eddy or cyclone was observed, which travelled round in the vessel at the distance from the centre at which the sand was dropped in It was, in miniature, like a secondary cyclone moving in the current of the large cyclone revolving in the vessel Only a small quantity of sand is required to produce this result, 1 or 2 grm being sufficient.

The writer of the paper referred to, when describing the conditions in the small secondary cyclone shown in his chart, fig. 5, states that there had been heavy rain in the line of the cyclone before it reached the place marked in the chart He also gives in fig. 7 the isobars surrounding another small cyclone—so small that it had no effect on the isobars of the main cyclone. The diameter of the area of damage was only 150 to 300 yards. The barometer was hardly affected at a distance of two or three miles, though close to the centre of the track the barometer made a sudden fall and recovery of 10 mb., which took place in less than a minute This destructive storm was evidently a secondary, and was formed in conditions similar to those of the one above referred to. The weekly weather reports of the Meteorological Office do not give sufficient detailed information as to the rainfall accompanying this storm but as Mr Billett, in his memoir of the storm, quoted in the paper referred to, states that on the day of the storm "a severe thunderstorm swept the West of England and Wales from south of Devon to Cheshire and developed locally into a tornado of exceptional violence," the presumption is that, as there had been thunder, there had also been rain along the path of this cyclone* It would not be advisable, however, to draw the conclusion that the rainfall was the primary cause of these two secondary cyclones, without further information as to similar cases, and clearer ideas as to the details of how the changes produced by the rainfall current cause the cyclone

Sir Napier Shaw, in comparing ordinary cyclones, in which the height is only a fraction of their diameter, with small cyclones and tornadoes, in which the height is a number of times their diameter, likens them to "a penny and a pin" This simile, like most others that at once catch the imagination, very much overdoes the comparison If we are to compare the two forms of motion to a penny and a pin, we should remember that we must either thicken the penny to the length of the pin, or we must cut down the pin to a length of a little over 1 mm to make it equal to the thickness of the penny; the contrast is now truer, though not so striking

I should like to say a few words about the revolving fluid referred to in the last paragraph of p 36 of the paper The writer says this was not a cyclonic depression such as those experienced in the British Isles He tells us it was first identified on August 23, 1891, in the mid-Atlantic He says it was "a tropical revolving storm, an acknowledged eddy" it first journeyed westwards to the coast of Florida, then turned eastwards over the Gulf Stream as an intense cyclonic depression, and continued to move eastwards, to the English Channel, ultimately losing itself over the Mediterranean on September 9 This revolving storm, we are told, began as an eddy—or pin, to use the writer's simile—and developed into a penny Now, if the pin and the penny forms of revolving fluids were two quite distinct kinds of motion, would it be possible for the pin to develop into the penny? We can hardly suppose that there was sufficient energy in the eddy to enable it to expand and to keep it rotating and travelling for these 36 days while it moved from the mid-Atlantic to Florida, and from there eastwards to the Mediterranean Does it not seem far more likely that the eddy, when it got into the warm moist air as it approached the coast of Florida, would, by drawing it in soon get fed and grow to the size of an ordinary cyclone, and that it would keep up its energy by the supplies it received while passing over the Gulf Stream on its way to the Channel?

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* *February*, 1918 Since the above was written I have, owing to the kindness of Sir Napier Shaw, read Mr Billett's report on the above storm, and find that about 4 P M heavy rain fell in Devonshire in the line of the track of the storm. One observer near Exeter wrote "I do not think I ever saw such rain out of the tropics" Another, writing from near mid-Devon, says, "At 4.5 P.M a perfect deluge of rain fell' . "This storm only lasted 10 minutes." This seems to have been the site and origin of the tornado which moved northwards through Wales less than two hours later. Both the heaviness of the rainfall, and its local distribution, were favourable for the formation of eddy or tornado motion in the manner above suggested.—J A

There is another point to which I may refer bearing on the above change of "the pin into the penny," or eddy into cyclone, as I think it helps us to understand them better It is the effect of the pressure gradient on the form of circulation. If we make the experiment I have described in a previous paper* with an inverted cyclone in water, giving the water a certain rate of rotation, and keeping the level of the outflow only slightly under the level of the water in the vessel, so as to give a small "head" and slow outflow, we shall find, if we watch the lines of outflow in the cyclone in the water approaching the outlet under these conditions, that the water comes from a wide area all round the outlet, and slowly curves round, the curving becoming quicker as the outlet is approached. When no previous circular motion is given to the water, the lines of flow are radial and come from all directions, but with an initial circular motion and a slow outflow the lines of flow are now the radial lines slightly curved, but still coming from a distance all round When the rate of outflow is increased, however, and the gradient deepened, the centrifugal force of the inflowing water puts a barrier against this inflow from the sides and causes the inflow to pass through the end of the cyclone, and in so doing lengthens the tube of resisting water and causes it to grow till it reaches the bottom of the vessel, the cyclone having now the appearance of a water tube of small diameter By raising or lowering the outlet end of the syphon tube we can cause the cyclone to expand or contract at pleasure—that is, change it from the penny to the pin We can deduce the same conclusion from the form of the cyclone made with a steep gradient, as it can be seen that there is a tendency for it to widen at its lower end, owing to the gradient there being less steep than at the upper With a slight gradient the lines of flow are well represented by the balance spring of a watch distorted perpendicularly to its plane, while with a steep gradient the circulation is represented by an ordinary spiral spring

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* 'Roy Soc. Proc. Edin.,' vol 36, Part I (No. 7).

The bearing of these observations on the cyclones in the atmosphere is very evident With slight gradients the cyclones remain large, with wide range of indraught and moderate winds But if the barometric gradient becomes steep, the cyclone draws itself into a smaller area and expends its energy on a smaller amount of air, so increasing the violence of its winds and powers of destruction It would appear that we have in our latitudes the first form, while tornadoes and other violent cyclones belong to the latter type, but there is no essential difference between them The only difference seems to be in effects of the steepness of their barometric gradients That these should differ is quite what might be expected, as the supplies of driving power are much greater and more concentrated in the tropics than in the temperate zones

In conclusion, let me say that though this paper will appear to many to be written with more confidence and finality than the subject admits of, yet I am quite conscious that there are many difficulties in the cyclonic theory of atmospheric circulation and since much remains to be revealed regarding cyclonic and eddy motion, the solution of these difficulties will probably have to await the extension of our knowledge in this direction

Ultra-violet Transparency of the Lower Atmosphere, and its Relative Poverty in Ozone.

By the Hon. R. J. STRUTT, F.R.S.

(Received December 19, 1917.)

[PLATE 2.]

Many years ago it was suggested by Hartley[*] that the limit of the solar spectrum towards the ultra-violet was attributable to absorption by atmospheric ozone, which, as he showed, would give rise to a general absorption beginning at about the place where the solar spectrum ends.

In a recent paper by Prof. A. Fowler and myself,[†] the evidence for this view was very much strengthened. For it was shown that just on the limits of extinction the solar spectrum shows a series of narrow absorption bands which are eventually merged in the general absorption, and these narrow bands are precisely reproduced in the absorption spectrum of ozone. For my own part, I do not feel any doubt that ozone in the atmosphere is the effective cause limiting the solar spectrum.

The work just cited naturally led to an investigation of the absorption by the atmosphere of ultra-violet light from a distant terrestrial source. This research, though more than once suggested, does not seem to have been undertaken before.

Various sources of ultra-violet light were tried in the course of the investigation. Burning magnesium proved inconvenient, because of the rapidly falling intensity of the spectrum in the far ultra-violet. A cadmium spark was used for short distances up to 1200 yards, and has the advantage of showing strong lines right through and even much beyond the region of strongest ozone absorption. This would have been the best source for long-distance experiments also, if facilities for producing a powerful spark and skilled assistance in maintaining it for long hours had been at hand, which they were not. I was compelled to use a source which would work without attention, and a quartz mercury vapour lamp answered this purpose well, though its strongest lines are not of quite such short wave-length as might be desired.

The source of light (spark or mercury lamp) was placed behind a quartz lens of 3½ inches diameter, arranged to focus it on the distant station. Since the difference of focus for the visual and extreme ultra-violet regions is very marked, a choice must be made between them, and since the furthest ultra-violet rays are most weakened by atmospheric scattering, the focus was adjusted for them. In the case of the cadmium spark, the line 2313 was chosen.

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[*] 'J. Chem. Soc.,' vol. 39, p. 111 (1881).

[†] 'Roy. Soc. Proc.,' A, vol. 93, p. 577 (1917).

Adjustment was made during day time. The spark-gap was fixed at the calculated distance from the lens. The image of the distant station, as seen by daylight, was viewed by means of an eyepiece, and brought on to the spark-gap. To do this, it was essential temporarily, to stop down the lens to a small aperture, as otherwise the focus adjusted for the far ultra-violet was too indistinct for the visual rays. After adjustment, the diaphragm was removed, and everything left undisturbed till night had fallen.

Nearly the same method was used for the mercury lamp. Here, owing to the large size of the source, chromatic aberration is of less importance, and it suffices to focus for the visual rays. The image of the distant station cannot, however, be well examined with the lamp in place, and it was necessary to substitute cross-wires, which were carried on a support like that of the lamp, and which could be brought into the same position relative to the lens by means of stops.

The mercury lamp was set up at Whitelands, near Hatfield Peveril Station, Essex, where it could be supplied by a private electric installation. The longest range obtainable was across the Chelmer Valley to Little Baddow, a distance of almost exactly four miles (6·45 kilom.). Over most of the range the path of the beam was some distance above the ground, and out of the reach of low-lying mists.

A small portable prismatic camera was used for photographing the spectrum. It was provided with a 60° quartz prism, and a quartz lens of 1 inch aperture and 5 inches focal length. A small telescope with cross-wires was fixed at the proper angle on the top of the instrument, to serve as a finder. When the distant light was on the cross-wires, the spectrum was in focus. The plate holder had of course to be sloped, to allow for the chromatic aberration of the lens, and this made it necessary to recover the exact direction of the source for which the instrument had been adjusted originally.

The camera was designed for broadening the spectrum by slow tilting in the course of the exposure, but this was not found advantageous in practice. Broadening in this way means loss of intensity, and is not required for a bright-line spectrum.* The monochromatic images appear as round dots, not lines, but this is no particular disadvantage.

* It is, of course, much more necessary for dark line stellar spectra.

The camera was taken each night to the place of observation and set up on a portable tripod in the open air Dew on the prism was sometimes a source of trouble Extra rapid plates were used, and in order to determine the limits of transmission, it was necessary to expose till the strong lines in the near ultra-violet were lost in the blur of over-exposure

A spectrum of the cadmium spark taken at 3600 feet (1100 metres) showed no definite indication of ozone, the whole spectrum being transmitted to λ 2313, right through the region, near λ 2536, where ozone absorption is a maximum

The best of the mercury lamp spectra are reproduced in Plate 2 Referring to that plate, No I is the spectrum of the lamp at short range, the wave-lengths of some of the chief lines marked The extreme range ever observed in the solar spectrum is also shown by the arrow

No II is a mercury lamp spectrum, taken with two hours' exposure, on a clear night, at a distance of 4 miles (6 45 kilom.) It will be seen that the spectrum extends as far as wave-length 2536

The length of this spectrum cannot be compared directly with the length of the solar spectrum, taken near the sea-level, for the equivalent thickness of air traversed in that case would be more than 5 miles, and therefore considerably more than the 4 miles range of the experiment * We can, however, compare it with the photographs of the solar spectrum obtained by Simony on the Peak of Teneriffe† at an altitude of 3700 metres (12,200 feet)

Now, at this altitude, the atmospheric pressure, reckoned by the usual logarithmic formula, would be about 0·644 of its value at the earth's surface. Taking the "height of the homogeneous atmosphere" at 0° C as 26,200 feet, the equivalent thickness of air at NTP, measured in a vertical direction above the mountain, would be 16,900 feet The observations are stated to have been made in August, when the sun's declination cannot have been less than 9° north. Assuming the observations to have been made near noon, as no doubt they would be, the sun's zenith distance in latitude 28° north would be not more than 19° For such moderate distances from the zenith, the thickness of air traversed may be taken as inversely proportional to the cosine of the zenith distance This makes the thickness of air traversed not more than 17,900 feet

This is decidedly less than the thickness of air through which the mercury spectrum was photographed. Reduced to 0° C, the latter was about 20,100 feet.

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* The lie of the ground did not allow me to get a longer range.

† See Cornu, 'Comptes Rendus,' vol. 111, p. 941 (1890)

We have, then, the following results to consider —

	Thickness of air	Ultra-violet limit
	feet.	
Solar spectrum from near sea level	29,000*	2948
Solar spectrum from Peak of Teneriffe	17,900	2922
Mercury lamp spectrum ..	20,100	2536

* This figure is a little vague, as the data for calculating zenith distance are not given

It is evident that the lower air is far more transparent than the upper air to ultra-violet rays, if equal masses are considered

This conclusion is opposed to that of Cornu* His method was to observe near sea-level the limit of the solar spectrum for various zenith distances, obtaining in this way an empirical formula connecting the spectrum limit with the equivalent thickness of average air traversed (reduced to N.T.P) He then went up to a high altitude above sea-level, and found that the increased length of the spectrum for a given altitude of the sun was in accordance with what was to be expected from his formula, on the assumption that the lower air had the same absorbing power as the average He concluded, therefore, that this latter assumption was justified

The results now brought forward, however, clearly prove that it was not justified Cornu's method is not an advantageous one, as great strain is thrown on the accuracy with which small variations in the position of the spectrum limit are determined This limit is only a moderately definite quantity, and however carefully the photographic and instrumental conditions may be defined and reproduced, the limit will depend to some extent on the amount of atmospheric haze. These considerations may perhaps account for what I think must, in any case, be considered an erroneous conclusion, for an indirect inference that the lower atmosphere will not transmit rays of short wave-length cannot stand against direct experimental proof that it will do so

I have spoken so far of the long-distance mercury spectrum as not extending beyond wave-length 2536. But there is no evidence, nor is it probable, that this is a limit in the same sense as the observed end of the solar spectrum It is known that the solar spectrum cannot be appreciably lengthened by multiplying the exposure many times. But the line 2536 was barely visible in a photograph which was given one hour's exposure, whereas it comes out clearly with two hours'. Circumstances prevented me from trying longer exposures, but there would be no inherent difficulty in exposing for 10 hours on a winter night. Moreover, a more powerful lamp might be used. With these and other improvements, the spectrum of the lamp might probably be prolonged to the point where oxygen absorption begins to be important.

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* 'Comptes Rendus,' vol 90, p. 940 (1880)

According to the observations of Fabry and Buisson,[*] the line 2536 is more strongly absorbed by ozone than any other in the mercury spectrum. Since this radiation gets through, it is evident that the apparent limit of the photographed spectrum is not conditioned in any way by ozone.

Although the ultra-violet spectrum is transmitted as far as λ 2536, yet if we compare the long distance spectrum No. II with No. III taken at short distance, it will be noticed how rapidly the latter falls off in intensity as the wave-length diminishes. The exposures were adjusted to give about the same intensity in the yellow and green lines in each spectrum.

There is a cause quite distinct from selective absorption by ozone or any other special constituent of the air, which will account, at any rate in part, for an effect of this kind. I mean the scattering of light by "small particles." These particles are of two kinds—the molecules themselves and the larger particles which give rise to atmospheric haze. The action of the former alone would not be enough to explain the observed diminution of intensity. It can be shown from established theory that the coefficient of absorption K, for light of wave-length λ, has the value

$$K = \frac{32\pi^3(\mu-1)^2}{3N\lambda^4},\text{[†]}$$

where μ is the refractive index and N the number of molecules per cubic centimetre under standard conditions.

From this we find, for λ 2536,

$$K = 2{\cdot}95 \times 10^{-6},$$

which gives for the transmission at 6·44 kilom. (4 miles) the fraction 0·15.

A similar calculation carried out for the less refrangible part of the spectrum shows that here the loss by scattering is almost negligible. Judging by the great increase of exposure required to extend the spectrum to λ 2536, it appears that the transmission for this line cannot be nearly so much as 0·15, and therefore that scattering by pure air will not account for the observed absorption.

In the lower air, however, suspended particles, small compared with the wave-length, but large compared with the molecules, produce a more important effect than the latter. The amount of this effect is so variable from day to day that only a general notion of its magnitude can be obtained from known data.

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[*] 'Journal de Physique,' March, 1913.

[†] Lord Rayleigh, 'Phil. Mag.,' vol. 47, p. 375 (1899); see also Schuster's 'Optics,' 2nd ed., p. 326 (1909).

The measurements of Abbot and Fowle* on solar radiation for different altitudes of the sun at Washington and at Mount Wilson, show that 1 mile of air measured vertically from sea-level on average clear days transmits 0·745 of the radiation of wave-length 3987. This makes the coefficient of absorption of such radiation $1·83 \times 10^{-6}$ cm $^{-1}$.

Ignoring changes of refractivity, and assuming as a rough approximation that the absorption coefficient varies as λ^{-4}, we get for λ 2536 the coefficient of absorption $1·125 \times 10^{-5}$ cm $^{-1}$, and the transmission of this ray by 6·44 kilom (4 miles) of air would be only 0·0007.

This value, which can only be considered as giving the roughest idea of the true transmission under the actual weather conditions of my experiment, would perhaps more than account for the observed enfeeblement of the line 2536, even if no ozone were present at all.

Making use of Fabry and Buisson's values (*loc. cit.*) for the absorption of λ 2536 by ozone, we find that this computed reduction of intensity (0·0007), actually due to atmospheric scattering, could be produced by a layer of ozone 0·26 mm. thick. Although it appears from what has been said that the experiments afford no positive evidence of any ozone in the lower atmosphere, yet they do enable us to give a maximum estimate of the ozone content, though one that may be much in excess of the actual value.

It is impossible to estimate from the present observations how much of the enfeeblement of the ultra-violet spectrum is due to ozone, and how much to scattering. To do this would require very elaborate photometric studies of the transmission of the various wave-lengths, and even so the results would probably not be very secure.†

Let us suppose, however, for a moment that scattering were put out of account. If λ 2536 were enfeebled by ozone only how much ozone could be present? To determine this, spectrum photographs were taken through known thicknesses of the gas. An ozone tube was arranged with quartz windows cemented on with sodium silicate, and a sulphuric acid manometer, as shown (see figure). The quartz windows were 18 mm. apart. The ozoniser was filled with oxygen, allowing the gas to wash through, and to bubble out through the acid below. The pressure was then reduced by slight suction so that the acid stood at a convenient level in the tube. The discharge was passed for a short time so as to produce ozone, and when the temperature had settled down, the level was read again in order to estimate the contraction.

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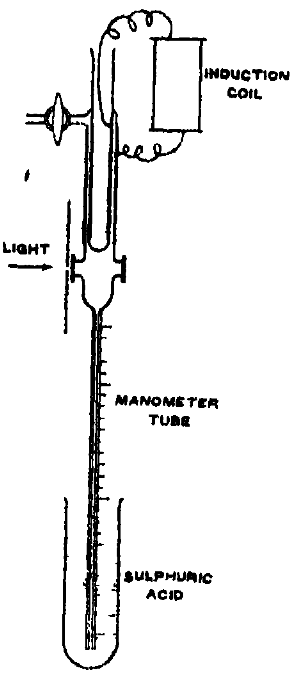
* 'Ann. Astr. Obs. of Smithsonian Institution,' vol. 2, p. 113 (1908).

† In any case, nothing could be attempted in this way without establishing a station from which the distant light could be conveniently observed whenever the weather was suitable, over a long period. I had to be content with a much less ambitious programme, working without assistance during a short summer holiday in the country.



The volume of the whole apparatus was measured as 12 c.c. Calibration of the narrow tube gave its volume as 32·5 c.mm. per centimetre of length. From these data it may be calculated, taking change of both pressure and volume into account, that 1 cm. rise of sulphuric acid level means 0·890 per cent. by volume of ozone present in the oxygen. Hence, in the distance of 18 mm. between the quartz windows each centimetre rise of the acid level means a thickness of *pure ozone* (at N.T.P.) equal to 0·152 mm.

A small metal diaphragm was illuminated by the mercury lamp, and the ozone apparatus which has been described was placed in front of it. The spectrum was photographed by the prismatic camera, placed about 15 metres off. The exposure was adjusted so that the yellow and green mercury lines, which are not much affected by ozone absorption, had the same intensity as in the long distance experiments.

Without any ozone the spectrum No. III was obtained. It will be observed that the line 2536 is lost in the blur of over-exposure.

Spectrum No. IV was obtained with a rise of the sulphuric acid equal to 0·7 cm., hence with 0·11 mm. of pure ozone in the absorbing layer. The whole spectrum is still transmitted, but with considerable enfeeblement in the region about 2900.

No. V was obtained with a rise of the sulphuric acid equal to 1·8 cm., hence 0·27 mm. of ozone in the absorbing layer. The line λ 2536 is just visible on the negative, but will probably not come out in the reproduction. It is decidedly less intense than in the long distance spectrum No. II. A slight increase in the ozone beyond this was found to obliterate λ 2536 altogether.

We may conclude then that even if the low intensity of λ 2536 in the long distance spectrum were wholly due to ozone absorption, it would be accounted for by less than 0·27 mm. of ozone in 4 miles of air.

We have already seen that it is quite probable that an effect equivalent to

0·26 mm ozone is really due to atmospheric scattering. The close agreement of the two figures is, no doubt, largely accidental, but still, allowing for the somewhat uncertain deduction to be made for scattering, it cannot be said that any undoubted effect remains to be attributed to ozone absorption. In any case it is certain that the ozone cannot exceed 0 27 mm

Now Fabry and Buisson (*loc cit*), observing the solar intensity at λ 3000 for various zenith distances, concluded that the observed effects would be explained if the light of the vertical sun passed through the equivalent of 5 mm. of pure ozone, measured at N T P At this rate, for 4 miles of average air at N T P., the ozone would be equivalent to 4 mm of the pure gas, at least 15 times the amount that can be present in 4 miles of air near sea-level It appears clearly, therefore, from comparison of optical data only, that the air near the ground contains very much less ozone than the average for the whole atmosphere So far as the optical evidence now available goes, this might be attributed to local impoverishment of the air near ground-level, due for instance to the oxidising action on organic matter, or alternatively it might be due to a stratum rich in ozone, on the outer confines of the atmosphere A repetition of the mercury lamp experiments at a high altitude on the Alps would help to decide between these alternatives

The same general conclusion as to the relative poverty in ozone of low-lying air has been drawn previously, on the strength of chemical determinations of ozone, both by Fabry and Buisson (*loc cit*), and by Hayhurst and Pring[*] and also by Pring[†] The chemical results of the various experimenters are, however, so hopelessly discrepant with one another that the purely optical method seems to afford a much-needed confirmation It is wholly free from the main difficulty besetting chemical determinations—I mean the difficulty of distinguishing between ozone and other chemically active substances which may be present in the air, such as hydrogen peroxide and oxides of nitrogen

Summary

1 The lower atmosphere is found to be comparatively transparent to ultra-violet light. The line λ 2536 can be detected in the spectrum of a mercury lamp 4 miles distant.

2. The solar spectrum, even when observed from high altitudes when the equivalent thickness of air overhead (reduced to N.T.P.) is less than 4 miles, is limited by atmospheric absorption to λ 2922 Air near the ground-level is, therefore, much more transparent to ultra-violet light than the upper air

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[*] 'Chem. Soc Trans,' vol 97, p. 868 (1910)

[†] 'Proc Roy Soc,' A, vol 90, p 204 (1914).

3 Since the limitation of the solar spectrum is almost certainly due to ozone, it follows that there must be much more ozone in the upper air than in the lower

4 Scattering by small particles acts in the same way as ozone to absorb ultra-violet radiation from a distant source, and this action makes quantitative estimation difficult Even if the observed enfeeblement of λ 2536 were entirely due to ozone, 0 27 mm. of pure ozone in 4 miles of air would suffice to produce it Taking scattering into account, the quantity is probably much less, and there is no evidence from this investigation that any ozone is present in the lower air

DESCRIPTION OF PLATE

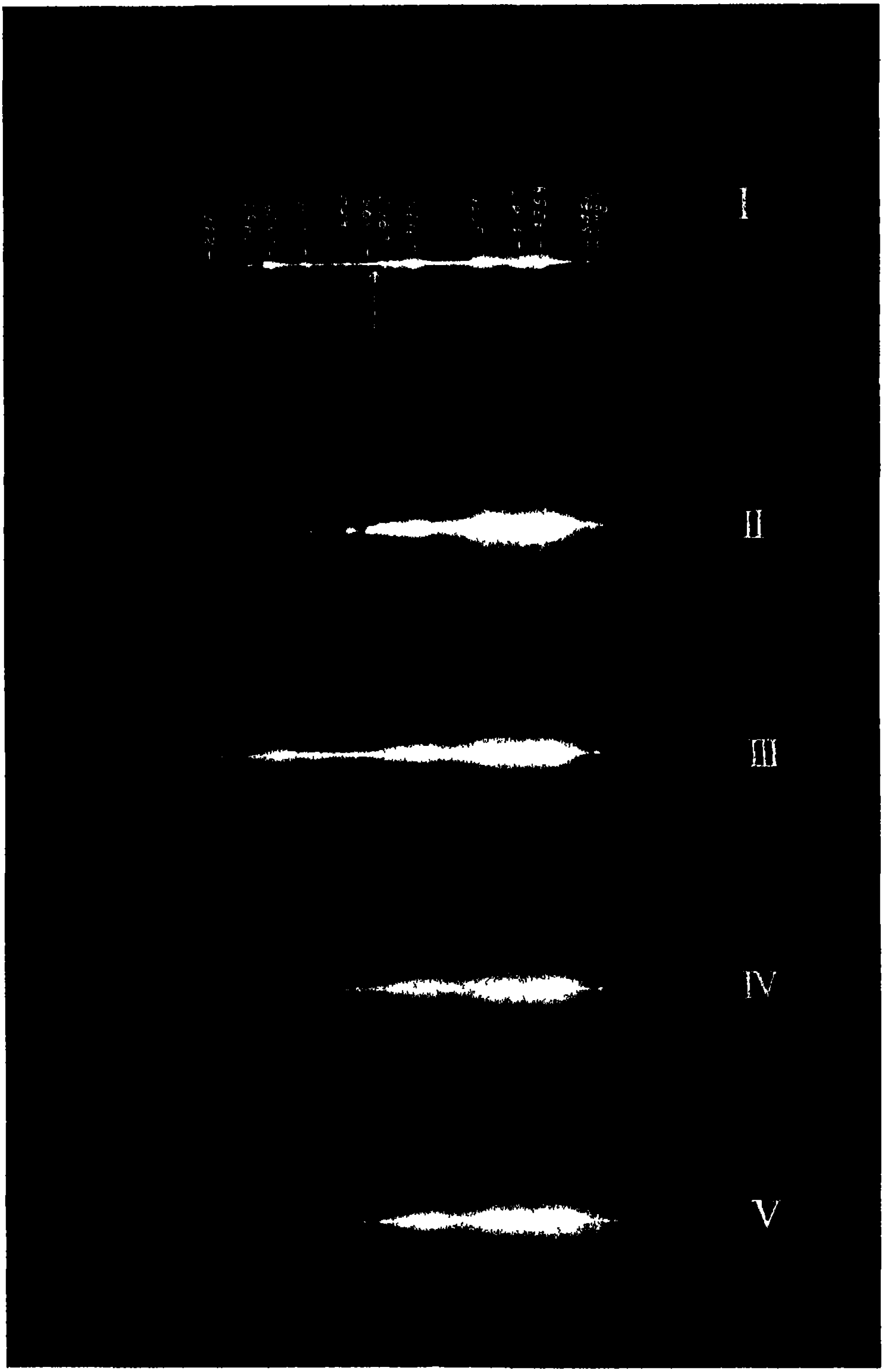
I Reference spectrum of mercury The arrow shows the extreme limit observed for the solar spectrum under the most favourable conditions

II Spectrum of mercury lamp 4 miles distant The greater part of the spectrum is unavoidably lost by over exposure, in order to bring out the ultra violet end, which extends to λ 2536

III Spectrum of the same lamp at short distance No ozone interposed This and the remaining spectra are given an exposure bringing the yellow and green lines on the extreme right to the same intensity as the long distance spectrum II

IV The same, through 0 11 mm thickness of ozone The ultra-violet much enfeebled, though less than in II

V The same, through 0 27 mm thickness of ozone The ultra-violet feebler than in II showing that the ozone in 4 miles of air amounts to less than 0·27 mm


I
II
III
IV
V

The Photo-electric Action of X-Rays.

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(Received December 28, 1917.)

The fundamental law of photo-electric activity states that

$$\tfrac{1}{2}mv^2 = h\nu - w, \tag{1}$$

where the left-hand side represents the maximum kinetic energy of the liberated electrons, h is Planck's constant, ν is the frequency of the exciting light, and w is a constant which measures the work necessary for an electron to escape from the substance, and whose value is characteristic of the material under consideration. The equation (1) was first given by Einstein[*] as a deduction from the view that the energy of radiation was distributed in discrete quanta. However, I succeeded[†] in showing that it followed from Planck's radiation formula, so that it evidently has a wider basis than the restricted and doubtful hypothesis used by Einstein.

On the experimental side the evidence for several years was somewhat conflicting, but in 1912 I showed,[‡] as a result of experiments made in collaboration with Dr. K. T. Compton, that the equation represented the photo-electric behaviour in the visible and in the ultra-violet as far as $\lambda = 2 \times 10^{-5}$ cm., for the metals sodium, aluminium, magnesium, zinc, tin, and platinum as accurately as it could be determined. The experiments also showed that the differences in the values of the constants w for different metals were connected with the corresponding contact potentials V by the relation

$$w_n - w_m = e(\mathrm{V}_n - \mathrm{V}_m). \tag{2}$$

This was confirmed both by direct measurement of the energy of the emitted electrons, and by determining the threshold frequencies ν_m given by the obvious relation

$$w_m = h\nu_m. \tag{3}$$

When these experiments were started I thought it improbable that equation (1) would turn out to be correct, on account of the very grave objections to the form of quantum theory on which it had up to that time been based by Einstein. After making the experiments I felt that there was no reasonable doubt of its entire validity. I confess that this view has not been shared by all the more recent writers on the subject. However, as it has been confirmed by the independent experiments of Hughes,* and by the very careful critical work of Millikan,† who has shown that the equation can be used to determine the value of h with very great accuracy, there can at the present time be no doubt of its validity.

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[*] 'Ann. d. Physik,' vol. 17, p. 145 (1905).

[†] 'Phys. Rev.,' vol. 34, p. 146 (1912); 'Phil. Mag.,' vol. 24, p. 570 (1912).

[‡] 'Science,' vol. 35, p. 783 (1912); 'Phil. Mag.,' vol. 24, p. 575 (1912).

In the X-ray region the researches of Sadler,‡ Beatty,§ Whiddington,‖ and Moseley,¶ taken together,** have shown that the *maximum* energy of electrons liberated by X-rays of frequency ν is very closely represented by

$$\tfrac{1}{2}mv^2 = h\nu \qquad (4)$$

This relation is evidently consistent with (1), since the electrons with the maximum velocity will come from the more superficial parts of the atom, and for such electrons w is negligible in comparison with $h\nu$ in the case of X-rays. The fact that (4) represents the experimental results with sufficient accuracy cannot, therefore, be considered an argument against equation (1) being the complete expression of the facts in this case.

The study of the generation of X-rays by the impact of electrons of various velocities has shown that equation (4), and in all probability equation (1), operates both ways, that is to say, there is an upper limit to the frequency of the radiation generated by the impact of electrons with energy $\tfrac{1}{2}mv^2$ which is given by equation (1). This can be deduced from the earlier experiments of Kaye†† and Whiddington,‡‡ and has been confirmed by the recent very accurate work of Duane and Hunt.§§. There is evidence that the same condition governs the excitation of ordinary spectral lines by means of electron impacts of definite velocity, but in this case secondary complications arise which make the phenomena less easy to disentangle.

The foregoing statements and facts show that equation (1) represents a condition possessing a very high degree of generality. It evidently embraces the whole gamut of the radiation scale in its interaction with every type of matter. So far, it has only been demonstrated as a limiting condition. That is to say, when matter is subjected to the action of radiation of frequency ν, no electron ever acquires a total energy, kinetic ($\frac{1}{2}mv^2$) and potential (w), in excess of $h\nu$ On the other hand, there is nothing in the evidence referred to above which can be said to contravene the view that equation (1) completely represents the transference of energy between radiation and the electrons contained in matter, and that it is not merely confined to a statement of the limiting velocities or frequencies attainable in such interchanges The experimental evidence necessarily largely relates to solid substances, and, before an electron can escape from such a substance, it must, as a rule, lose a great deal of its energy in atomic encounters So far as the results available have been analysed, or are capable of being analysed, they have been found to be consistent with the view that individual energies of electrons below the limiting value all arise from such losses The reversibility of the equation likewise suggests that it is a true equation, and not an inequality Moreover, the experiments of C T R. Wilson,* on the photography of the tracks of the emitted electrons in gases, indicate that for monochromatic radiations they would consist of a limited number of definite lengths rather than a series extending from zero to an upper limit Certain theoretical arguments might lead us to contemplate the occurrence of energies corresponding to integral multiples of $h\nu$, but the group of facts under consideration has afforded no evidence of their existence up to the present The evidence so far considered, and it is drawn from a very wide field, points to the absorption and emission of radiation, in so far as it is concomitant with emission or absorption of electrons, that is to say, with the transference of energy from the radiant to the material form, as taking place primarily in whole quanta, and nothing but whole quanta

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* 'Phil. Trans.,' A, vol. 212, p. 205 (1912).

† 'Phys. Rev.,' vol. 7, pp. 18, 355 (1916).

‡ 'Phil. Mag.,' vol. 19, p. 337 (1910).

§ 'Phil. Mag.,' vol. 20, p. 320 (1910).

‖ 'Roy. Soc. Proc.,' A, vol. 86, pp. 360, 370 (1912).

¶ 'Phil. Mag.,' vol. 27, p. 703 (1914).

** Richardson, 'Electron Theory of Matter,' Chap. XIX.

†† 'X-Rays,' 1914 edition, p. 124.

‡‡ 'Roy. Soc. Proc.,' A, vol. 85, p. 323 (1911).

§§ 'Phys. Rev.,' vol. 6, p. 166 (1915).

In the Bakerian Lecture for 1916 Barkla has adduced data on the electron emission especially associated with the emission of the K series of fluorescent X-radiations, from which he concludes that the absorption of radiation associated with such emissions is not confined to whole quanta, but may take place in amounts which vary continuously between one quantum and two quanta In fact, on p 359, he states "There is no evidence of absorption of X-radiation in whole quanta, though the conditions are frequently such as to give an approximation to this"

There is no doubt, however, that the absorption and emission of X-rays, more especially by heavy atoms, is a complex phenomenon, and it appears that the new facts, when considered critically, so far from lending support to the view that absorption of X-radiation takes place continuously, and in amounts which may vary by any fraction of a whole quantum, seem rather to be in favour of the contrary hypothesis.

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* 'Roy. Soc. Proc.,' A, vol. 87

Broadly speaking, the method of investigation employed by **Barkla** is to determine the ratio between the energy of the secondary K fluorescent radiation and that of the accompanying electronic radiation for a given absorption of primary X-rays of varying frequency ν. The method adopted involves the inclusion of the energies of the *soft* L, M, etc., radiations in the measured electronic radiant energy. If ν_K, ν_L, ν_M, etc., are the frequencies of the fluorescent K, L, M, etc., rays he finds approximately that the energy of the K fluorescent radiation is to that of the accompanying electronic and L, M, etc., fluorescent radiations as $h\nu_K : h(\nu+\nu_L+\nu_M+\dots)$. Assuming that these radiations represent the whole of the absorbed energy, this is equivalent to saying that the energy of the K fluorescent radiation is to the whole energy absorbed as $h\nu_K : h(\nu+\nu_K+\nu_L+\nu_M+\dots)$. When $\nu = \nu_K$ there is one quantum of K fluorescent radiation emitted and also one electron emitted for every two quanta of primary radiation absorbed. When $\nu_K/\nu =$ zero there is, in the limit, one quantum of K fluorescent radiation and also one electron emitted for every quantum of primary absorbed radiation. For intermediate values of ν one quantum of fluorescent K radiation and one electron are emitted as a consequence of the absorption of amounts of primary radiation which increase continuously from one to two quanta according to the frequency of the primary radiation. It is also assumed, on grounds which appear to me insufficient, that the energy of the electronic radiation is invariably equal to one quantum $h\nu$ of the primary radiation. The demonstration that there is approximately *one* electron emitted for each quantum of K fluorescent radiation emitted rests fundamentally on this assumption. It has been proved beyond reasonable doubt that the maximum energy of the electrons in question is approximately equal to $h\nu$, and it is not unreasonable to suppose that this magnitude will correspond closely with the actual value of the energy of one group of electrons on emission, in accordance with the argument on p. 271. On the other hand, it is to be anticipated, as will be shown below, on account of the complexity of the phenomena of X-ray emission, that in general there will be other groups of electrons for which the maximum energy may in general have any value between $h\nu$ and zero. The experiments hitherto made which bear on this point have been directed to the determination of the maximum energy of the whole group of electrons, and would not be expected to detect the simultaneous presence of groups having a less degree of maximum energy than that value. If these criticisms are correct, Barkla's interpretation of his results is open to question, and they cannot be held to prove that absorption of X-radiation takes place continuously. In any event they could hardly be held to prove the thesis, unless it were established that the alternative hypothesis were incapable of accounting for them

Having due regard to our knowledge of X-ray phenomena, let us consider what we should expect on the assumption that absorption takes place in whole quanta, and that the fundamental law of the matter is expressed by equation (1). We know from Barkla's researches that the emission of fluorescent X-rays is closely connected with increased absorption of the primary rays, and it follows from W H Bragg's[*] experiments that this increased absorption, connected with the K emission, does not set in until a frequency equal to that of the shortest K line has been obtained Consequently, as has been pointed out by J J. Thomson,[†] Bohr,[‡] and Kossel,[§] the first process involved in the emission of the K radiation is the removal of an electron from the innermost ring to a point near the surface of or outside the atom, and the various K lines result from the subsequent filling up of the gap in the innermost ring by electrons from the various outer rings

[*Note added February* 22, 1918—Considerations brought forward by Bohr strongly support the view that the innermost ring is the one which is concerned in the emission of the K radiations If this position should prove to be well founded it would seem necessary to look for the origin of the more penetrating J radiations, which Barkla has recently found to be given off by atoms of relatively low atomic weight, in the displacement of electrons from the nucleus of the atom]

If these were the only processes occurring we should have, on the quantum theory immediately under consideration, in the first place, the absorption of a quantum $h\nu$ of radiation by a K electron from the primary rays. By equation (1) this would give rise to an emitted electron having kinetic energy $\frac{1}{2}mv^2 = h\nu - w$. The energy w is only lost for the time being, and during the process of restoring the normal atom which fills up the innermost ring again, it is re-emitted in the form of K, L, M, etc, radiations whose frequencies satisfy the equation

$$w = h\nu_K + h\nu_L + h\nu_M + \quad + \epsilon \tag{5}$$

It is possible that some small part of the energy w may be converted into radiations of long wave-length or into other forms of energy which are unable to give rise to ionisation In that case such energy would be lost so far as ionisation experiments are concerned. To take account of this possibility the quantity ϵ has been included in (5) It is to be expected that the lost energy ϵ will be small compared with $h\nu$, but that it will be relatively most important in the neighbourhood of $\nu = \nu_K$ The values of ν_K, ν_L, etc, in equation (5) are to be understood as average values, being averaged according to the frequency of occurrence of the processes corresponding to the emission of the various K, etc, lines According to the view taken, which Kossel has shown to be in substantial accord with the known facts, in considering the restoration to the normal condition of any particular atom which has lost a K electron, one or more of the various frequencies L, M, etc, may be absent Thus, if an electron jumps to the K ring from the next ring but one we shall get a K_β emission and no L emission but only emissions of lower frequency (M, etc) than the L emissions, and so on It is to be remembered that in Barkla's experiments on bromine the L and M radiations on account of their high absorbability cannot be distinguished from electronic radiations, so that their energy is added to the measured electronic energy

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[*] 'Phil. Mag,' vol 29, p. 407 (1915), see also Webster, 'Phys Rev,' vol 7, p. 599 (1916).

[†] 'Phil Mag.,' vol 23, p. 456 (1912).

[‡] 'Phil. Mag.,' vol 26, p 1 (1913), vol 34, p. 394 (1915)

[§] 'Verh d Deutsch. Physik Ges.,' vol 16, p. 953 (1914)

If the foregoing were the only phenomena to be taken account of, we should have, on the view developed above, for every quantum $h\nu$ of primary radiation absorbed the following amounts of energy in other forms

A

(1) Energy of K fluorescent radiation, $h\nu_K$
(2) Energy of true electronic radiation, $h(\nu-\nu_K-\nu_L-\nu_M-\quad)-\epsilon$
(3) Energy of apparent electronic radiation $h(\nu_L+\nu_M+\quad)$
(4) Lost energy, ϵ

But Barkla's experiments show that in the neighbourhood of $\nu = \nu_K$ the proportion of the absorbed energy which reappears as energy of electronic radiation is much greater than that which is indicated by the scheme A Without understanding precisely how this emission of electronic radiation comes about, we can see that it must occur if the emission of the K fluorescent radiation is to set in as soon as the frequency of the exciting primary radiation reaches a value identical with or very close to that of the K_γ ray as Bragg's results indicate * For a quantum $h\nu_\gamma$, of radiation of this frequency, represents

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* Webster, 'Phys Rev,' vol 7, p. 607 (1916), from careful observations of the radiation from a rhodium anticathode, concludes that the special absorption commences exactly at the frequency of the K_γ ray, and quotes Hull as having made similar observations with tungsten and molybdenum. The energy of the electrons necessary to excite the various K rays corresponds on the quantum relation [illegible] frequency some 1·0–1·5 per cent higher than this value

the work done by an electron in moving from a position in an outer ring (probably the outermost ring) to an unoccupied position in the innermost (K) ring. But the normal atom contains one electron more than that just considered, and the work to remove an electron from the K position in a normal atom to the γ position will, therefore, be greater than $h\nu_{K_\gamma}$. This difference will be further accentuated by the electrostatic repulsion of the individual electrons in the outer rings, which will tend to prevent the K electrons from undergoing a displacement to the outer positions such as is a necessary antecedent to the emission of the K fluorescent radiation These difficulties disappear if an outer electron is first removed, so that, on these grounds, it is to be anticipated that at the limiting frequency each quantum of K fluorescent radiation liberated will be accompanied by the emission of one electron from the outer layers According to the view proposed the energy for this liberated electron must come from the absorption of one quantum of the primary radiation

When $h\nu$ is large compared with $h\nu_K$ the energy given by the primary radiation to a K electron is large compared with the energy necessary to escape from the atom, so that it is not to be expected that the emission of a K electron in this case will be affected appreciably by the absence of a single electron from the surface layers Thus this special emission of electronic radiation should not occur when ν becomes very great For intermediate values of ν it is to be anticipated that it will fall away continuously from the value of one electron for each quantum of fluorescent emission at the critical frequency to zero when ν becomes infinite

An alternative statement of this part of the position, which is possibly only a recapitulation of the experimental facts, is to say that, in connection with the intense absorption of the primary rays by electrons in the interior of the atom, which sets in as soon as ν exceeds the critical frequency, there is a special absorption of energy in the exterior layers The increased tendency to absorb energy in the exterior layers may be caused by instability, arising from the presence of the large amounts of energy absorbed or possessing a configuration suitable for absorption in the interior layers In any event, the emission of such electrons from the outer layers is a condition which alone admits of the liberation of K electrons at the limiting frequency. With increasing frequency, its influence becomes less and less important.

On the view just developed, there will be associated with each quantum of K fluorescent radiation emitted an emission on the average of p electrons from the outer layers of the atom. The value of p will increase from zero for $\nu = \infty$ to the value unity when ν falls to the critical frequency The energy absorbed from the primary radiation by these p electrons will be $ph\nu$ This will give rise to the emission of p electrons, each with a kinetic energy $\frac{1}{2}mv^2 = h\nu - w'$, where w' is the work for these electrons to escape from the atom On the subsequent return of the atom to the normal condition, the energy w' will reappear as energy of soft fluorescent X-radiations such as L, M, etc, radiations, depending on the depth within the atom from which the electron has been ejected, together with, possibly, some radiation of wave-length too long to cause ionisation, which will be lost so far as *the ionisation measurements are concerned*, and may be represented by ϵ' In the experiments under consideration, the energies of the soft fluorescent radiations are indistinguishable from that of the purely electronic radiation, so that, as far as the measurements are concerned, the radiation thus absorbed is represented in effect by—

B

(1) Energy of K fluorescent radiation $= 0$
(2) and (3) Energy of true and apparent electronic radiation $= p(h\nu - \epsilon')$
(4) Lost energy $= p\epsilon'$

On this view, the complete disposition of the energy will be given by adding the corresponding quantities in schemes A and B together, and the various measured energies, if we do not distinguish between the true electronic energy and that arising from soft fluorescent X-radiations, as these are not separated by the measurements, will be in the proportion of the numbers given in the following scheme (C):—

C

E_A = Energy absorbed from primary radiation $= (1+p)h\nu$
E_K = Energy of K electronic radiation $= h\nu_K$
E_e = True + apparent energy of electronic radiation
$$= h[(1+p)\nu - \nu_K) - \epsilon - p\epsilon'.$$
E_l = Lost energy $= \epsilon + p\epsilon'$

It will be observed that the relation between E_e and E_A is more complex than that between E_K and E_A on account of the possible effect of the lost energy having to be taken into consideration From the Table given by Barkla on p 345 of the Bakerian Lecture for 1916 it appears that the lost energy may possibly exceed 12 per cent. of the total absorbed energy in the neighbourhood of the limiting frequency, but the data are too scanty and irregular to form any reliable conclusions as to this matter. For this reason, in testing the application of the theory to the experimental data I shall keep to the simpler relation between E_K and E_A

As regards the function p all that we are entitled to expect, so far, is that it will be a function of ν, involving parameters depending on the structure of the particular atom concerned, which approaches the value zero as ν approaches infinity and which approaches the value unity as ν approaches the value of the critical frequency ν_{K_γ} A very general function satisfying these conditions is

$$p = e^{-\sum_q \alpha_q \{(\nu-\nu_{K\gamma})/\nu_{K\gamma}\}^{\sigma q}} \tag{6}$$

where $q = 0, 1, 2, 3$, etc, the α_q's are constant coefficients, and σ may be a constant proper fraction or unity From the relations (C) we have

$$E_K/E_A = \frac{\nu_K}{(1+p)\nu} = \frac{\lambda}{(1+p)\lambda_K} = f \tag{7}$$

and

$$\frac{\partial f}{\partial\left(\frac{\lambda}{\lambda_K}\right)} = \frac{1}{1+p}\left[1 - \frac{p}{1+p}\sum_q \alpha_q \frac{\lambda_{K\gamma}}{\lambda}\left(\frac{\lambda_{K\gamma}-\lambda}{\lambda}\right)^{\sigma q-1}\right], \tag{8}$$

and at the critical value $\lambda_{K\gamma}(\equiv c \div \nu_{K\gamma})$,

$$\left[\frac{\partial f}{\partial\left(\frac{\lambda}{\lambda_{K\gamma}}\right)}\right]^{\lambda_{K\gamma}} = \frac{1}{2}\left(1 - \frac{\alpha_{1/\sigma}}{2}\right) \tag{9}$$

The values of p can be found empirically by means of the relation

$$p = \frac{E_A}{E_K} \cdot \frac{\lambda}{\lambda_K} - 1$$

Values of $\log_{10} p$ obtained in this way are given in the following Table —

$\lambda \times 10^8$	0·87	0 71	0·56	0·49	0 35
$\log_{10} p$	0	−0 042	−0·126	−0·195	−0·320
$\frac{\lambda_{K\gamma}-\lambda}{\lambda}$	0·056	0 296	0·643	0·878	1 630

I can find no record of the wave-lengths of the K_γ line for bromine having been measured, but Friman* records a line L_{β_1} for this element for which $\lambda = 8·141 \times 10^{-8}$ cm. Using Kossel's relation we can assume as a sufficient approximation

$$\nu_{K\gamma} = \nu_K + \nu_{L_{\beta_1}} \tag{10}$$

and, putting $\lambda_{K_A} = 1·038 \times 10^{-8}$, this gives

$$\nu_{K_\gamma} = 3·27 \times 10^{18}\ \text{sec}^{-1} \text{ and } \lambda_{K_\gamma} = 0·918 \times 10^{-8}\ \text{cm} \tag{11}$$

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* 'Phil Mag.,' vol 32, p 497 (1916).

This value gives the data in the third row of the preceding Table

On plotting the values of $\log p$ against $\frac{\lambda_{K\gamma}}{\lambda} - 1$, it appears that they can be represented, if not with absolute accuracy, at any rate to within the degree of accuracy of the measurements, by a single term of (6), viz., the term for which the index $\sigma q = 1$ After determining the coefficient it appears that

$$p = e^{-0\cdot460[(\lambda_{K\gamma}-\lambda)/\lambda]} = 10^{-0\cdot200[(\lambda_{K\gamma}-\lambda)/\lambda]} \qquad (12)$$

Substituting in (9) this gives

$$-\left[\frac{\partial f}{\partial\left(\frac{\lambda}{\lambda_K}\right)}\right]_{\lambda_{K\gamma}} = 0\cdot385 \qquad (13)$$

The value of this differential coefficient measured from the curve on p 386 of Barkla's paper is 0·376, which agrees satisfactorily with (13)

The complete expression for the fraction of the absorbed radiation which is transformed into fluorescent radiation of the K series is thus —

$$\frac{E_K}{E_A} = \frac{\lambda}{(1+p)\lambda_K} = \frac{\lambda}{(1+e^{-0\cdot46[(\lambda_{K\gamma}-\lambda)/\lambda]})\lambda_K} \qquad (14)$$

The values calculated from this formula are given in the fourth column of the following table —

λ	p	λ/λ_K	$\frac{E_K}{E_A} = \frac{\lambda}{(1+p)\lambda_K}$	$\frac{E_K}{E_A}$ observed	Percentage error	VII
$1\cdot035 \times 10^{-8}$		1·00				
$0\cdot87 \times 10^{-8}$	0·973	0·84	0·426	0·42	+1·4	+2·4
$0\cdot71 \times 10^{-8}$	0·871	0·686	0·366	0·36	+2·2	+5·6
$0\cdot56 \times 10^{-8}$	0·748	0·542	0·312	0·31	+0·65	+6·5
$0\cdot49 \times 10^{-8}$	0·667	0·473	0·285	0·29	−1·7	+6·9
$0\cdot35 \times 10^{-8}$	0·472	0·338	0·230	0·23	0·0	+8·7

For comparison the observed values of E_K/E_A are shown in the fifth column The next column shows the difference of these numbers expressed as a percentage of the observed value. In every case the agreement is within the probable limits of the errors of measurement and is best for the shorter waves, where the measurements are admittedly most accurate. The last column gives the percentage error given by the corresponding numbers which Barkla calculates from his formula in column III of Table VIII on p. 345 of his paper It will be seen that this formula gives a worse agreement in every single instance, and I suspect that the disagreement exceeds the probable errors of measurement in the case of the shorter waves. Moreover, all the deviations are in the same direction, and are greatest for the shorter waves, for which the experiments are most accurate

It will be observed that the formula (14) for E_K/E_A contains only one arbitrary constant, the constants λ_K and λ_{K_γ} being predetermined by the X-ray spectrum of the atom. If, as the experiments seem to prove, the emission of fluorescent K-radiations is bound up with the simultaneous emission of electrons from positions near the surface of the atom, it is difficult to see how the complete expression for E_K/E_A can avoid containing at least one constant depending on the configuration of the electrons in the atom, in addition to the constants ν_K and ν_{K_γ}, which, as is shown by the success of Bohr's theory, merely involve energy relations.

Another view of the phenomena, which could be made to give expressions having some resemblance to those obtained, would be to suppose that radiation generated by the displacement of an electron to the central regions of an atom is liable to special absorption by the external electrons of the atom in which it is generated. One would naturally expect such absorption to be greatest for the softer rays in the neighbourhood of $\nu = \nu_{K_\gamma}$, and to diminish to zero as ν approached infinity. However, I feel that this view contains an element of artificiality which it is undesirable to introduce into the interpretation of the phenomena, unless their more exact investigation makes it necessary.

As regards the precise form found for p, it does not seem wise to attempt to push the interpretation of this too far at present. The expression given certainly covers the experimental numbers to within the limits of experimental error, but it does not cover them exactly, and it is possible that a more precise experimental investigation of the formula would lead to a function which, whilst it would have to agree very closely with the expression found over the range of data now available, might have a different mathematical form, and consequently be capable of a different physical interpretation.

The main conclusions to be drawn from this paper may be summarised as follows:—

(1) The ratio E_K/E_A of the energy E_K emitted in the form of K secondary X-radiation to the energy E_A of the primary radiation (wave-length λ) absorbed is found, in the case of bromine, to be expressed to within the degree of accuracy of the observations by the formula

$$E_K/E_A = \frac{\lambda}{(1+p)\lambda_K},$$

where $p = e^{-0\cdot 46(\lambda_{K_\gamma}-\lambda)/\lambda}$, λ_K is the average wave-length of the K radiations, and λ_{K_γ} is the wave-length of the shortest K radiation.

(2) The formula implies, or, at any rate, can be so interpreted as to

imply, the emission of one electron for every quantum ($h\nu$) of primary radiation absorbed It likewise involves, for the emission of each quantum of K radiation, the absorption of one quantum of primary radiation when $\nu = \infty$, of two quanta of primary radiation when $\nu = \nu_{K_\gamma}$, the limiting frequency (threshold frequency) for excitation, and of amounts of primary energy varying between one and two quanta for exciting radiations of intermediate frequency

(3) Inasmuch as the formulæ have been deduced, and have been given a plausible explanation, on the hypothesis that the primary radiation is absorbed in whole quanta, it does not appear that the facts under consideration can be held to support the contrary hypothesis.

[*Note added February* 22, 1918.—To these considerations may be added the general argument which is indicated, although perhaps insufficiently emphasised, in the preliminary part of this paper, and to which I have for some time felt that very great weight should be attached It will be observed that, in the interpretation of the phenomena under discussion, both as given in this paper and as given by Prof Barkla, and, in fact, in any reasonable interpretation of such phenomena, the quantity $h\nu$, where ν is the frequency of the exciting radiation, plays a conspicuous part Now, the quantity ν has nothing whatever to do with the particular kind of matter with which the radiation reacts, it is a quantity whose value is determined by the radiation alone. It follows that the quantum relation is dominated by something in the structure or properties of radiation, although it may be that the limitations imposed by such structure or properties are inoperative except in regard to certain reactions with matter, such as absorption or emission, whereby the frequency of the radiation may be altered.]

Some Problems in the Theory of Radiation.

By ARTHUR SCHUSTER, Sec. R.S.

(Received January 26, 1918.)

1 The following paper deals with the oscillatory energy taken up by a simple resonator under the action of white light, and the translatory energy imparted to a molecule by radiation

The first problem has already been treated by Planck: it is solved here in a very simple manner, and the method used, when applied to the second problem, leads to the important result, that a molecule at rest within an enclosure of uniform temperature will, under the action of the radiations, be set in motion with an acceleration that will increase its speed until the average energy reaches a definite value. If the Rayleigh-Jeans laws of radiation be assumed to hold, the ultimate average energy due to radiation alone is two-thirds of that derived from the kinetic theory of gases.

2 The connexion between radiation and absorption is sometimes illustrated by means of a pendulum acted on by a succession of blows. If these be delivered always in the same direction, and at intervals equal to the period of free oscillation, the amplitude rapidly increases; and the pendulum may then be said to absorb the energy of the blows When the intervals are irregular, the impulses varying in phase and direction, it is further argued that their effects will be neutralized, so that the pendulum remains at comparative rest. I used at one time to try and illustrate the distinction experimentally in the lecture-room, but met with no success. However much I delivered the blows indiscriminately and irregularly, the pendulum always showed a continuously increasing amplitude of oscillation. I ultimately abandoned the experiment, thinking that there was some unconscious psychological influence which forced the experimenter to time his blows—to some extent—in unison with the time of swing. I ought to have known better, for anyone who has given attention to the nature of white light should have seen at once that the continuously increasing oscillation of the pendulum set in motion by irregularly delivered blows, each imparting the same velocity irrespective of the phase, is the true analogy of what occurs in nature when white light is absorbed in a responsive medium If there were no continuously increasing energy of motion there could be no absorption of white light.*

The complete theory shows that with regular intervals equal to that of the free period, the *amplitude* increases continuously with the time, while, when the blows succeed each other irregularly, it is the average *energy* that is proportional to the time. In the absence of friction the energy increases indefinitely in both cases.

3. Let a particle perform simply periodic oscillations under the action of a force directed to a fixed centre, the velocity being represented by $p \cos n(t-\tau)$. If at the time t_1, an additional velocity q be imparted to the particle in its line of motion, the subsequent velocity will be represented by

$$p \cos n(t-\tau)+q \cos n(t-t_1),$$

for it is easily seen that the last term introduces no change of position at time t_1, but increases the velocity by the quantity q. The result of the sudden change of velocity is therefore the same as if an additional oscillation $q \cos n(t-t_1)$ had been superposed on the original one. It follows that the effect of successive blows q_1, q_2, etc., delivered at times t_1, t_2, etc., may be obtained by taking the resultant of a number of oscillations determined by their maximum velocities q_1, q_2, and phases nt_1, nt_2, etc.

If the intensities of the blows be equal, and the phases distributed at random, it follows, from a well-known theorem of Lord Rayleigh, that the expectancy of the resultant intensity is Nq^2t, where N is the number of blows delivered in unit time. The measure of the intensity here is the square of the maximum velocity of the particle. Writing m for its mass, the expectancy of maximum kinetic energy is therefore $\frac{1}{2}mNq^2t$, and the same expression may stand for the sum of the potential and kinetic energies in the resulting oscillations. When the motion of the particle is damped so that its average energy Q diminishes according to the exponential law, the decrease of energy per unit time, due to damping, will be proportional to Q say, λQ. A state will ultimately be reached where the diminution due to damping will be equal to the energy supplied by the impulses, and the expectancy of the energy will then be

$$Q = Nmq^2/2\lambda. \tag{1}$$

The condition that the oscillator has only one degree of freedom and that the impulses are delivered in its line of motion may now be dropped. If we resolve the impulses and the velocity of the oscillator along three co-ordinate axes, the result applies to each co-ordinate separately, the average square of the component of the blows along one co-ordinate being equal to a third of the square of the resultant, and, adding up the three resulting intensities, the final result remains the same. It is understood, of course, that if the oscillator has only one degree of freedom, and the blows are delivered indiscriminately in all directions, we should only get one-third of the resultant intensity.

4. Before applying our results, a few words may be said on the method of treating white light as a series of impulses. This was first suggested by Gouy, and favourably commented on by Rayleigh, and I have used it with advantage, but its value has never been sufficiently recognized. A non-homogeneous radiation is completely defined by the distribution of energy, which—when resolved spectroscopically—is contained within a definite range of frequencies. Two sets of disturbances, however they may differ in detail, give, so far as our observational powers go, identical results so long as their energy distribution is the same. In the first consideration of the subject, light is naturally introduced as a periodic phenomenon, homogeneous radiations alone being dealt with. In some of the more complicated effects we are then led to consider white light as a superposition of homogeneous radiations, though we can never obtain a truly continuous spectrum in this manner. If we have recourse to Fourier's theorem, certain difficulties arise which may easily be overcome, but they are often best avoided by considering a non-homogeneous radiation to consist of a superposition of similar disturbances having a finite and preferably short duration. If it be required to represent completely a given radiation, each impulse must—when resolved spectroscopically—contain the required distribution of energy, but in the important series of problems in which the range of wave-length affecting the solution of the problem is small, this condition is not essential. Within the effective range, the energy may be looked upon as constant, and the remainder of the spectrum does not matter. For my purpose it is sufficient to consider the impulse to be confined to a narrow layer of thickness $c\tau$ along a plane progressive wave-front (c = velocity of light). Within the layer I take the electromotive force E to be constant. There is an equal magnetic force at right angles to E, and the energy transmitted in unit time per unit surface of the wave-front is $E^2c\tau/4\pi$. If there are N equal impulses following each other in unit time the energy per unit volume is $NE^2\tau/4\pi$. It will be shown in § 10 that when the disturbance is analysed spectroscopically, the energy per unit volume and unit range of frequency is connected with the impulse $E\tau$ by the relation

$$U = NE^2\tau^2/2\pi. \qquad (2$$

5. We are now prepared to write down the result for the ultimate average energy of an electric oscillator under the action of light impulses. The velocity imparted to the electron at each impulse is $Ee\tau/m$, and equation (1) therefore becomes

$$Q = NE^2e^2\tau^2/2\lambda m,$$

$$= 2\pi e^2U/\lambda m$$

λ represents the fractional loss of energy of the oscillator due to its radiation per unit time, and this is shown in the electromagnetic theory of light to be equal to $2e^2\omega^2/3mc^3$, $\omega/2\pi$ being the frequency. Hence

$$Q = 3\pi c^3 U/2\omega^2$$

This agrees with Planck's solution of the problem, except in so far as his *oscillator is assumed to have only one degree of freedom, which reduces the* average energy to one-third of the above value. The Rayleigh-Jeans law of radiation may be put into the form

$$3\pi c^3 U = 4\omega^2 W,$$

where W represents the average kinetic energy of the molecule of a gas, and is proportional to the absolute temperature. This gives finally

$$Q = 2W \tag{3}$$

As half the energy of the oscillator is potential, we conclude that the ultimate kinetic energy of an oscillator having three degrees of freedom is equal to the translatory energy of a molecule in accordance with the law of equi-partition. This has already been pointed out by Planck. The proposition is subject to the same limitations as the Rayleigh-Jeans law, and holds therefore only for low frequencies, or high temperatures, the condition being that the product of temperature and wave-length shall be large.

It is instructive to compare the above method of treatment with that employed by Planck. Both assume the intensity distribution to be uniform within the effective range of wave-length, and the result is therefore not affected by considering it to remain uniform throughout the spectrum. Planck, starting from homogeneous oscillations, has to calculate the result of each within the narrow limits of frequencies closely adjoining that of the free period of the oscillator. He thus obtains the resulting energy by integration. I avoid the integration by starting from the integrated light disturbance which, in the case of a disturbance having uniform intensity along a frequency spectrum, consists of a series of impulses within each of which the electric force is constant.

6. We may now deal with the mechanical effect on the oscillator as a whole which is due to the magnetic force carried by the impulse. If M be the mass of the molecule, and v the velocity of the oscillating electron at any time, its direction forming an angle θ with the magnetic force, the velocity generated by the impulse, considering that the magnetic and electric forces are numerically equal, is $Ee\tau v \sin\theta/cM$. We have to find the average square of the velocities for all possible combinations of θ and v. Collecting together the cases in which v has the same numerical value, we note that

the average square of $\sin^2\theta$ is $\frac{2}{3}$. The rate of increase of energy of the molecule, to which we attach a mass M, is therefore,

$$\tfrac{1}{4}c^{-2}M^{-1}Ne^2E^2\tau^2v^2 = \tfrac{1}{3}2\pi c^{-2}e^2Uv^2/M, \tag{4}$$

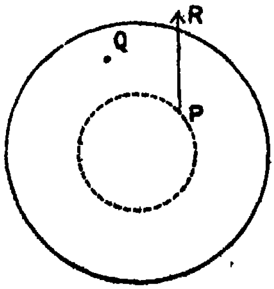
where v^2 now stands for the average square of the oscillatory velocity of the electron

Using a result obtained by M Abraham,[*] which will be further discussed in the next paragraph, it appears that a moving oscillator is acted on by a retarding force which is equal to $\frac{2}{3}\omega^2v^2e^2u/c^5$, if u be its translatory velocity. The diminution of energy per unit time is, therefore, equal to this expression multiplied by u. Equating the rate of increase and diminution of energy, we obtain for the ultimate state·

$$\begin{aligned} \omega^2Mu^2 &= \pi c^3U, \\ \tfrac{1}{2}Mu^2 &= \tfrac{1}{2}\pi c^3U\omega^{-2}, \\ &= \tfrac{2}{3}W = \tfrac{1}{3}Q \end{aligned} \tag{5}$$

The kinetic energy of the molecule tends, therefore, to become equal to two-thirds of that obtained by the kinetic theory of gases, and equal to one-third of the energy of internal motion

7 The retarding force acting on the moving resonator is most easily determined by calculating the momentum transmitted to the enclosure; but the law of action and reaction is not applicable, because a momentum that leaves the oscillator in the direction of motion in time dt will reach the enclosure in time $dt(1-u/c)$ Generally, if the direction of transmission be defined by θ, the angle it forms with the direction of motion, the ratio of the rate at which the momentum is received to that at which it leaves the oscillator is $1-\beta\cos\theta$ The force on the enclosure is therefore $1-\beta\cos\theta$ times that acting on the oscillator. The difference is made up by an accumulation of momentum in the medium. That such an accumulation actually takes place may be shown directly Consider a point P within a spherical enclosure radiating equally in all directions. At any point Q we may associate (according to Poynting) a momentum density with the energy density. If its value at unit distance from P be M, its value at Q will be M/r^2. If the angle between PQ and PR, the direction of motion, be θ, and we only consider the momentum transmitted along PR, we may confine ourselves to the effective part of the momentum density, which is $M\cos\theta/r^2$ To obtain its total value we must sum up over the whole sphere. It is clear



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[*] 'Elektromagnetische Theorie der Strahlung,' p. 116.

that, as the point P moves, the momentum towards R will diminish, and that in the opposite direction increase, hence there is an accumulation of momentum in the direction opposed to that of the motion The total momentum transmitted in the direction PR is easily obtained if we note that the integral of $M \cos \theta / r^2$ is the same as that of the component, in the direction PR, of an attracting mass spread with uniform density M over the sphere, the law of action being that of the inverse square This resultant force is then equal to the component resolved along PR of the attraction of a sphere, concentric with the enclosure and passing through P. If we take as axis of y a line passing through the centre parallel to PR, we find for the resultant momentum resolved along the axis of y the expression $-4\pi My/3$ The momentum M may now be expressed in terms of the energy of radiation The energy density being Mc/r^2, the amount traversing a closed surface enclosing P in unit time is $4\pi Mc^2$, and if this be denoted by I, we find for the rate of increase of the momentum resolved in the direction of motion within the enclosure $\frac{1}{3}c^{-1}I\beta$ This must, therefore, be the excess of force acting on the enclosure over that acting on the radiant source

8. We have now to determine the value of the force retarding the translatory velocity of a radiating body in virtue of its own radiation Abraham has given the complete solution of this problem for an oscillator moving with a velocity which is great compared with that of the periodic motion (*loc cit*). This is not the case in the problem here considered, but it will be shown that his solution holds generally to the first order of magnitude in u/c The subject being important, it is permissible to enter into more detail than would be necessary for our main object.

Abraham's equations give the electric and magnetic forces due to an electron possessing any given velocities and accelerations. It will be found convenient to separate at once the oscillatory velocity v from the translatory velocity u, and to write for the total velocity $u+v$, both quantities being looked upon as vectors. Abraham's vector-equations, by simple transformations, then become

$$\mathrm{E} = \mathrm{A}\left\{r_1(v_1, r_1) - v_1 - \beta u_1(v_1, r_1) + \beta v_1(r_1, u_1)\right\},$$

$$\mathrm{H} = \mathrm{A}\left\{[v_1, r_1] + \beta(\dot{v}_1, r_1)[u_1, r_1] - \beta(r_1, u_1)[v_1, r_1]\right\}.$$

Round brackets here denote scalar products and square brackets vector products. The suffix "1" denotes that the vector has unit length, so that v_1 would simply indicate the direction of the acceleration, and (v_1, r_1) the cosine of the angle between the direction of the acceleration of the electron and the radius vector, drawn from the electron to the point at which the electric and magnetic forces are required. In the second equation it has been assumed that the velocity and acceleration are in the same direction, so that it can only be applied to linear oscillations. The quantity β denotes u/c. All terms of the equations (omitting the factor A) have zero dimensions, and may be expressed in terms of angular co-ordination A has the value $ec^{-2}r^{-1}v[1-c^{-1}((u+v)r_1)]^{-3}$ If we wish to express our results in rectangular co-ordinates, we must distinguish the case of an oscillation at right angles to the direction of motion u from that in which the oscillation is parallel to u In the former case we obtain, if u be along the axis of y, and the oscillations along z,

$$\left.\begin{aligned} E_1 &= A\frac{xy}{r^2}, & H_1 &= -A\left(\frac{y}{r}-\beta\frac{y^2+z^2}{r^2}\right),\\ E_2 &= A\left(\frac{y^2}{r^2}-\beta\frac{z}{r}\right), & H_2 &= A\left(\frac{x}{r}-\beta\frac{xy}{r^2}\right),\\ E_3 &= -A\left(\frac{x^2+y^2}{r^2}-\beta\frac{y}{r}\right), & H_3 &= -A\beta\frac{xz}{r^2} \end{aligned}\right\} \qquad (6)$$

In the case of longitudinal oscillations, the terms in β disappear, and we have

$$\left.\begin{aligned} E_1 &= A\frac{xy}{r^2}, & H_1 &= A\frac{z}{r},\\ E_2 &= -A\frac{x^2z^2}{r^2}, & H_2 &= 0,\\ E_3 &= A\frac{yz}{r^2}, & H_3 &= -A\frac{x}{r} \end{aligned}\right\} \qquad (7)$$

The energy density is $A^2\{(x^2+y^2)+\beta^2(y^2+z^2)-2\beta yr\}/4\pi r^2$ for transverse vibrations, and $A^2(x^2+z^2)/4\pi r^2$ for longitudinal vibrations

The above equations hold without limit as regards the velocity, but, if we wish to proceed further, we have to examine the factor A, which contains a term $c^{-1}(v+u)r_1$, or $\beta\cos(r,u)+\beta'\cos(r,v)$ if $\beta'=v/c$. When $\beta=0$, we return to the stationary oscillator, and β' introduces a small correction to the Hertzian equations. Its effect would—*inter alia*—be to introduce higher harmonics.[*] But all effects of the first power of β' are periodic, averaging out during a complete oscillation, and for our purpose we may therefore neglect them altogether. Confining henceforward our investigation to terms not higher than the first power of β, we may put $c^2rA=(1+3\beta y/r)ev$

To obtain the rate at which the momentum leaves the oscillator, we have according to § 7, to multiply our expression by $(1-\beta\cos\theta)$. Taking all these matters into consideration, and introducing polar co-ordinates,

$x = r \sin\theta \sin\phi$, $y = r \cos\theta$, $z = r \sin\theta \cos\phi$, the total rate of momentum leaving the oscillator is

$$(4\pi c^4)^{-1} e^2 v^2 \int_0^{\pi}\int_0^{2\pi} \{(\cos^2\theta + \sin^2\theta \sin^2\phi)(1 + 5\beta\cos\theta) - 2\beta\cos\theta\} \cos\theta \sin\theta \, d\theta \, d\phi$$

The term not affected by β disappears, when integrated over the sphere, and we ultimately find $\frac{2}{3}e^2\dot{v}^2c^{-4}\beta$ for the retarding force acting on the oscillator As we need only consider average values, we may substitute $\omega^2 v^2$ for $\dot{v}^2$, and we thus obtain the result used in the previous article For longitudinal vibrations, we obtain similarly for the rate of loss of momentum

$$(4\pi c^4)^{-1} e^2 v^2 \int_0^{\pi}\int_0^{2\pi} \sin^3\theta \cos\theta (1 + 5\beta\cos\theta) \sin\theta \, d\theta \, d\phi$$

This integral is found to have the same value as the previous one

The rate of emission of energy by a stationary simple oscillator is $2e^2\dot{v}^2/3c^3$, and the retarding force is therefore $\mathrm{I}u/c^2$, independently of the direction of vibration Hence the same expression holds for any particle, or, indeed, for a solid body emitting white light By an altogether different method Lorentz arrived at the same result. Poynting, on the other hand, gives $\frac{2}{3}\mathrm{I}u/c^2$ for the retarding force in the body of his paper, and reduces that still further in a note at the end The discrepancy lies in the value given to the intensity sent out by a moving radiating surface in a direction forming an angle θ with the direction of motion. According to Poynting, this is proportional to $1 + 2\beta\cos\theta$, while our equations, based mainly on Abraham's calculations, lead to $1 + 4\beta\cos\theta$

9 We may now briefly cover the ground again, with a view to dismissing all unnecessary assumptions The most important of these is, that the molecule acts as a simple oscillator, *i.e.*, emits homogeneous radiations. The calculation remains substantially the same, for an electron performing orbits round a nucleus. If the electric and magnetic forces of the incident radiation be concentrated in impulses, each impact adding a velocity V, the square of the velocity, which originally was v^2, will have increased to $v^2 + \mathrm{V}^2 + 2v\mathrm{V}\cos\gamma$, where γ is the angle between V and v. The angle γ being arbitrary, the average increase of the square of the velocity will be V^2. The rate of increase of internal energy is therefore the same as that previously found $\mathrm{NE}e^2\tau^2/2m$. But we cannot, without knowing the laws of intra-molecular action, decide what fraction of that increased energy is kinetic, because the value of an instantaneous increment of velocity does not give any information on the change of the average kinetic energy in the new orbit Such an increment, carrying the particle further away from

the centre, is indeed, as a rule, accompanied by a *diminution* of the average energy of motion

There is a small increase of internal motion due to the magnetic force, which is $ec^{-1}\mathrm{H}v\sin\theta$, and as H is numerically equal to E, the impulse due to the force is $vc^{-1}\sin\theta$ times that due to the electric impulses. We may therefore disregard it, as we retain in each case only the first term of the forces The action of the magnetic force becomes important in connexion with the translatory motion, because it is there the dominating factor Our expression for the rate of increase of this kinetic energy remains as before $\mathrm{NE}^2e^2\tau^2(v\sin\theta)^2/2\mathrm{M}c^2$ Although the above equations still rest on the assumption that the effect of the radiation is concentrated in a series of impulses, I do not think that this assumption need be retained Whatever may be the distribution of the intensities when analysed by Fourier's theorem, it is difficult to resist the conclusion that the electric and magnetic forces affecting the internal and translatory motion respectively are as E to $\mathrm{H}c^{-1}v\sin\theta$, E and H being numerically equal The corresponding increase of energy must be, in the long run, proportional to the time, and, the angle between the magnetic forces and the direction of velocity of the electron being arbitrary, we may substitute $\frac{2}{3}v^2$ for the average square of $v\sin\theta$ We may, therefore, write for the rate of increase of the internal energy $\mu c^2/m$, and for that of the translatory energy $2\mu c^2v^2/3\mathrm{M}c^2$ Without entering into the question of the meaning of μ, it seems to me that these relations are independent of any particular manner of resolving the radiation into its components We may now turn to the retarding force If K be that force, the rate of diminution of energy is Kv But, in the case of electrons, the force is not necessarily directly opposed to the velocity, and we must, therefore, use the scalar product (Kv), or the velocity multiplied by the component of force parallel to it, the other component, being at right angles to the velocity, does no work Taking the most important term we find in this way for the rate of diminution of internal energy $2e^2(vv)/3c^3$ When there is a balance between the rates of gain and loss of energy

$$m(vv) = \tfrac{3}{2}\mu c^5. \tag{8}$$

To obtain the force retarding the translatory velocity, I must refer to the equations of Abraham * We replace the velocity of the electron by its two parts u and v, the former relating to the translatory motion. We may then reject the periodic terms. The forces in that case, omitting terms of the second order, reduce to $2e^2u(vv)/3c^5$. Again balancing the gain and loss of energy, we find

$$\mathrm{M}(vv)u^2 = \mu c^3v^2 \tag{9}$$

* *Loc. cit*, p. 123.

Equations (8) and (9) lead to

$$Mu^2 = \tfrac{2}{3}mv^2 \tag{10}$$

This is in agreement with (5), where Q represents the sum of the average kinetic and potential energies when the two are equal But in the general case the relation between the total internal energy and the kinetic energy depends on the law of force The relation has been worked out by Clausius,* who shows, *e g*, that if the law be that of the inverse square, impulses like those here considered would, while increasing the total energy of the internal motion, actually diminish its kinetic energy by an equal amount The absorption of light would result mainly in an increase of potential energy equal to twice the work done, while the internal kinetic energy and the translatory energy would diminish in the same proportion as the absorbed energy It appears, therefore, that the distribution of energy between the internal and external energy depends on the forces regulating the internal motion, but no reasonable law for these forces can be given that would lead to an exact equi-partition

10. It remains to prove equation (2) referred to in § 4, for a fraction of the total energy of the light impulses, which we may consider to be contained within a definite range of frequency If a vector $f(t)$ defining the radiation be introduced such that the energy per unit volume is proportional to $[(f(ct)]^2$, we may express $f(t)$ in terms of Fourier's series. The integral $\int_{-\infty}^{+\infty} f(\lambda)\cos\omega\lambda\,d\lambda$ and the corresponding sine integral will, in the case of a beam of light of infinite extent, become infinite, but we avoid the consequent difficulty by taking the limits of the integrals to be finite, though indefinitely large Between the limits ±T, we now have

$$\pi f(t) = \int_0^\infty d\omega \int_{-T}^{+T} f(\lambda)\cos\omega(\lambda - t)\,d\lambda,$$

we derive a second integral† giving the total energy of radiation

$$\pi \int_{-T}^{+T} [f(t)]^2\,dt = \int_0^\infty (A^2 + B^2)\,d\omega,$$

where

$$A = \int_{-T}^{+T} f(\lambda)\cos\omega\lambda\,d\lambda, \qquad B = \int_{-T}^{+T} f(\lambda)\sin\omega\lambda\,d\lambda.$$

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* 'Phil. Mag.,' vol 42, p 327 (1847).

† Rayleigh, 'Phil Mag.,' vol. 27, p. 460 (1889); Schuster, 'Phil. Mag.,' vol. 37, p. 533 (1894)

The average value of $[f(t)]^2$ during the time 2T is therefore

$$(2\pi T)^{-1}\int_0^\infty (A^2+B^2)\,d\omega$$

Though A and B are infinite, A^2 and B^2 increase on the average in proportion to the time, and the energy transmitted per unit range of frequency $\omega/2\pi$ soon converges with increasing T to a definite limit. Applying the result to the case where $f(t)$ has a constant value E during an interval τ at time t, and zero values at all other times, we find

$$A = 2E\omega^{-1}\cos\omega t\sin\tfrac{1}{2}\omega\tau, \qquad B = -2E\omega^{-1}\sin\omega t\sin\tfrac{1}{2}\omega\tau$$

$$A^2+B^2 = 4E^2\omega^{-2}\sin^2\tfrac{1}{2}\omega\tau$$

If a number of equal impulses follow each other at irregular intervals, A and B will depend on T, but A^2+B^2 has the same value for each impulse. We may therefore obtain the average energy by multiplying the expression by the total number of impulses, of which we shall suppose there are N per unit time. We then obtain for the average value of $[f(t)]^2$

$$\pi^{-1}NE^2\int_0^\infty 4\omega^{-2}\sin^2\tfrac{1}{2}\omega\tau\,d\omega$$

To calculate the energy per unit volume we must apply the factor $(4\pi)^{-1}$, and for small values of τ we then find the energy-density per unit range of frequency to be $NE^2\tau^2/2\pi$.

On the Series of Legendre.

By W H YOUNG, M.A., Sc D., F R S

(Received October 17, 1917)

In recent communications to the Society, I have confined myself largely to the Theory of Fourier Series, partly because much seemed to me still to require doing in this subject, partly because I believed its thorough investigation to be the natural preparation for the study of other series of normal functions It has, indeed, been known for some time that the behaviour of, for instance, series of Sturm-Liouville functions exactly corresponds to that of Fourier series. The introduction that I have recently made into Analysis of what I have called *restricted Fourier series* enables us to notably extend the range of such analogies I propose in the present communication to illustrate this remark with reference to series of Legendre coefficients

Whereas Fourier series may be said to be "naturally unrestricted," in virtue of the fact that the convergence of the integrated series to an integral necessarily involves the tendency towards zero of its own general term, so that the consideration of the more general type of series does not at once suggest itself, Legendre series may be said to come into being "restricted," even when the coefficients are expressible in what may be called the Fourier form by means of integrals involving Legendre's coefficients. In other words, such series correspond precisely to restricted Fourier series, instead of to ordinary Fourier series like the analogous series of Sturm-Liouville functions

Among these restricted Fourier series, those which I distinguished as *of the first class*, namely, those whose first integrated series converge to integrals in some interval or intervals, play the most important part. With regard to such series I have been able to demonstrate, among others, two fundamental theorems :—

I *If the typical term of the series tends to zero, in which case I distinguish the series as ordinary restricted Fourier series (R.F. series); these series behave precisely in the same way as ordinary Fourier series in the interval or intervals to which they are restricted.*

II *If this condition is not fulfilled, then the properties of the series which coincide with those of ordinary Fourier series are those in which the summation is performed in the Cesàro manner, index unity; properties involving ordinary* *convergence, or ordinary upper and lower functions, have in this case to be excluded.*

These theorems have an immediate application to Legendre series, and the mode of application is more general than might be at first supposed. It is, in fact, quite unnecessary to postulate in our theorems that the coefficients of the Legendre series should be expressible in the Fourier form. In other words, if we denote the Legendre series by

$$\sum_{n=0}^{\infty} a_n \mathrm{P}_n(\cos\theta), \tag{1}$$

then it is *not* necessary for the validity of our theorems that the "Fourier" relation

$$a_n = \frac{2n+1}{2}\int_0^{\pi} f(\cos\theta)\,\mathrm{P}_n(\cos\theta)\sin n\theta\,d\theta \tag{2}$$

should hold.

When (2) does hold, it is known that the series (1), when integrated term-by-term, converges to an integral throughout its interval of periodicity. The converse also is true, but this is not the case when all that we require is that the integrated series of (1) should converge to an integral *in one or more sub-intervals*.

It seems worth while to particularly call the attention of the Society to the fact, firstly, that the larger class of Legendre series, so obtained, possess, as regards the internal points of all such intervals, all the properties of the smaller class for which (2) holds: this follows immediately from our theory. As already pointed out, however, even if (2) holds, the series is, so to speak, naturally restricted owing to the failure of $a_n\mathrm{P}_n(\cos\theta)$ to tend to zero. This fact has certainly been responsible for the isolated character of the results hitherto obtained with respect to these series. For their satisfactory discussion, the previous consideration of restricted Fourier series is absolutely essential.

In a recent communication to the Society, I gave actual examples of restricted Fourier series of the ordinary type (R.F. series) which converge in the ordinary way, thus justifying from the practical standpoint their introduction. It will be gathered from what has been said, however, that their importance in theory is even greater than that in practice.

We are, indeed, able to completely dispose of the theory of series of Legendre coefficients, in virtue of the properties of the trigonometrical series above referred to.

In fact, by means of the asymptotic expansion of $\mathrm{P}_n(x)$ we are able to express a Legendre series, when multiplied by a suitable function, as the sum of a restricted Fourier series and a series of terms which converges uniformly in any closed interval which excludes the end-points of the interval of periodicity.

We thus at once obtain the theorem that, at an internal point of an interval to which such a series is restricted, equally whether this be the whole interval of periodicity of $P_n(\cos\theta)$ or not, the conditions of convergence of Legendre series are *the same* as those of an ordinary Fourier series, provided only, either the summation be effected in the Cesàro manner, index unity, in both series, or

$$a_n P_n(\cos\theta) \to 0, \qquad (n \to \infty),$$

which is the same thing as

$$n^{-\frac{1}{2}} a_n \to 0, \qquad (n \to \infty),$$

in which latter case, equally whether the summation employed be ordinary or Cesàro, the conditions are the same Moreover, this is true with regard to uniform convergence, as well as with regard to convergence *per se*. We have, be it understood, to suppose that, if the coefficients a_n do not satisfy condition (2), the integrated series converges to an integral in the interval, or intervals, of restriction. Corresponding theorems, however, hold good when, without this being the case, we can still assert that, for some integer p, the p-th integrated series converges to a p-ple integral in each such interval. This is in accordance with what might be expected by reference to the higher class of restricted Fourier series.

It will be remarked that this method not only reduces the discussion of Legendre series to that of Fourier series, of a more or less restricted type, but that it does so without the intervention of any properties of Legendre coefficients, except those involved in the asymptotic expansion above referred to

Further, the considerable analytical difficulties which arise when, for example, *fractional* Cesàro summation is employed, are entirely avoided. The work has been done once for all in discussing Fourier series, nothing remains but to interpret these results in the light of our fundamental theorems

It must not be forgotten, however, that these remarks only relate to *internal* points of the whole interval of periodicity When the extremities are considered, real singular points as they are for a Legendre series, even when it satisfies condition (2), the character of the discussion at once changes. I may, however, remark, *en passant*, for the information of Fellows of the Society, that in this case also the problem may be reduced to one involving the Fourier series of an auxiliary function, so that the results of the theory of these latter series can be again utilised

It is interesting to remark, in view of the practical applications, that this theory has an important bearing on the question of the term-by-term integration of Legendre series when multiplied by another function provided we remain in the interior of the interval of periodicity, or of the interval of restriction, of the series The theorems which in such cases hold for Fourier series hold for Legendre series

With regard to this latter class of theorems, it is evident that the restriction which excludes the end-points of the interval is essential, owing to the fact that the theorems which hold good for the convergence at the extremities are not the same as at an internal point, even in the case when (2) is satisfied Speaking generally, we may say that, with certain exceptions of a notable character, the theorems relating to the end-points of the interval are obtainable from those relating to the internal points of the interval, by replacing ordinary convergence by Cesàro convergence, or by increasing the index of the Cesàro convergence This is true accordingly in general for integration theorems involving the whole closed interval, as compared with those involving only a portion of the interval Yet this complication is unnecessary, in that one of these theorems is deducible from the Theorem of Parsival for a series of normal functions

In view of the great importance of these integration theorems in practice, I propose to make a further communication with regard to them on a subsequent occasion

It should be scarcely necessary to add that the considerations here exposed apply, *mutatis mutandis*, to other series of normal functions, provided these normal functions are expressible, when multiplied by suitable functions, in a similar manner to the coefficients of Legendre.

On the Scattering of Light by Spherical Shells, and by Complete Spheres of Periodic Structure, when the Refractivity is Small.

By Lord Rayleigh, O.M., F.R.S.

(Received January 28, 1918.)

The problem of a small sphere of uniform optical quality has been treated in several papers.* In general, the calculations can be carried to an arithmetical conclusion only when the circumference of the sphere does not exceed a few wave-lengths. But when the relative refractivity is small enough, this restriction can be dispensed with, and a general result formulated.

In the present paper some former results are quoted, but the investigation is now by an improved method. It commences with the case of an infinitely thin spherical *shell*, from which the result for the complete uniform sphere is derived by integration. Afterwards application is made to a complete sphere, of which the structure is symmetrical but periodically variable along the radius, a problem of interest in connection with the colours, changing with the angle, often met with in the organic world.

The specific inductive capacity of the general medium being unity, that of the sphere of radius R is supposed to be K, where K−1 is very small. Electric displacements being denoted by f, g, h, the primary wave is taken to be

$$h_0 = e^{int}e^{ikx}, \tag{1}$$

so that the direction of propagation is along x (negatively), and that of vibration parallel to z. The electric displacements in the scattered wave, so far as they depend upon the *first power* of (K−1), have at a great distance the values

$$f_1, g_1, h_1 = \frac{k^2\mathrm{P}}{4\pi r}\left(\frac{\alpha\gamma}{r^2}, \frac{\beta\gamma}{r^2}, -\frac{\alpha^2+\beta^2}{r^2}\right), \tag{2}$$

in which
$$\mathrm{P} = -(\mathrm{K}-1)\ e^{int}\iiint e^{ik(x-r)}\,dx\,dy\,dz \tag{3}$$

In these equations r denotes the distance between the point (α, β, γ), where the disturbance is to be estimated, and the element of volume $(dx\,dy\,dz)$ of the obstacle. The centre of the sphere R will be taken as the origin of co-ordinates. It is evident that, so far as the secondary ray is concerned, P depends only on the angle (χ) which this ray makes with the primary ray. We will suppose that $\chi = 0$ in the direction backwards along the primary ray, and that $\chi = \pi$ along the primary ray continued. The integral in (3) may then be found in the form

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* 'Phil. Mag.,' vol. 41, pp. 107, 274, 447 (1871); vol. 12, p. 81 (1881); vol. 47, p. 375 (1899); 'Roy. Soc. Proc.,' A, vol. 84, p. 25 (1910); vol. 90, p. 219 (1914); 'Scientific Papers,' vol. 1, pp. 87, 104, 518; vol. 4, p. 397; vol. 5, p. 547; vol. 6, p. 220 (not yet published).

$$\frac{2\pi R^2 e^{-ik\rho}}{k\cos\frac{1}{2}\chi}\int_0^{\frac{1}{2}\pi} J_1(2kR\cos\tfrac{1}{2}\chi\,.\,\cos\phi)\cos^2\phi\,d\phi, \tag{4}^*$$

ρ denoting the distance of the point of observation from the *centre* of the sphere. In the paper of 1914 I showed that the integral in (4) can be simply expressed by circular functions in virtue of a theorem given by Hobson, so that

$$P = -(K-1)\,.\,4\pi R^3\,.\,e^{ik(at-\rho)}\left(\frac{\sin m}{m^3}-\frac{\cos m}{m^2}\right), \tag{5}$$

where

$$m = 2kR\cos\tfrac{1}{2}\chi. \tag{6}$$

In (5) the optical quality of the sphere, expressed by $(K-1)$, is supposed to be uniform throughout. In view of an application presently to be considered, it was desired to obtain the expression for a spherical *shell* of infinitesimal thickness dR, from which could be derived the value of P for a complete symmetrical sphere whose optical quality varies along the radius. The required result is obtained at once from (5) and (6) by differentiation. We find

$$dP = -(K-1)\,.\,4\pi R^2\,dR\,.\,e^{ik(at-\rho)}\,.\,\sin m/m, \tag{7}$$

expressing the value of P for a spherical shell of volume $4\pi R^2\,dR$. The simplicity of (7) suggested that the reasoning by which it had been arrived at is needlessly indirect, and that a better procedure would be an inverse one, in which (7) was established first, and the result for the complete sphere derived from it by integration. And this anticipation was easily confirmed.

Commencing then with a spherical shell of centre O and radius OA equal to R, let xO be the direction of the primary and Oρ that of the secondary ray (fig. 1). Draw Oζ in the plane of Ox, Oρ, and bisecting the angle between these lines and let ζ be a co-ordinate measured from O in the direction Oζ, so that the plane AOA, perpendicular to Oζ, is represented by $\zeta = 0$. The angle xOζ is $\frac{1}{2}\chi$, as in our former notation. We have now to consider the phases represented by the factor $e^{ik(x-r)}$ in P. For the point O, $x = 0$, $r = \rho$, and the exponential factor is $e^{-ik\rho}$. As in the ordinary theory of specular reflection, the same is true for every point in the plane AOA and therefore for the element of surface at AA whose volume is $2\pi R\,dR\,d\zeta$. For points in a plane

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* Given in the 1881 paper.

BB parallel to AA at a distance ζ the linear retardation is $-2\zeta\cos\frac{1}{2}\chi$, as in the theory of thin plates, and the exponential factor is $e^{-ik\rho}\,e^{2ik\zeta\cos\frac{1}{2}\chi}$. The

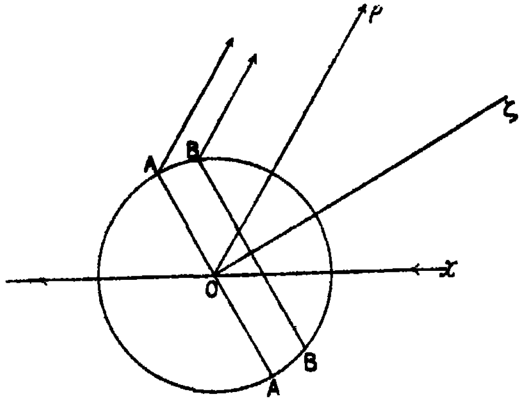

Fig 1

elementary volume at BB is still expressed by $2\pi R\,dR\,d\zeta$, and accordingly by (3)

$$dP = -(K-1)\;2\pi R\,dR\;e^{i(nt-k\rho)}\int_{-R}^{+R} d\zeta\, e^{2ik\zeta\cos\frac{1}{2}\chi} \tag{8}$$

The integral in (8) is $2R\sin m/m$, m being given by (6), and we recover (7) as expressing the value of dP for a spherical shell of volume $4\pi R^2\,dR$.

The value of dP for a spherical shell having been now obtained independently, we can pass at once by integration to the corresponding expression for a complete sphere of uniform optical quality, thus recovering (5) by a simpler method not involving Bessel's functions at all. And a comparison of the two processes affords a demonstration of Hobson's theorem formerly employed as a stepping stone.

When P is known, the secondary vibration is given by (2), in which we may replace r by ρ. So far as it depends upon P, the angular distribution, being a function of χ, is symmetrical round Ox, the direction of primary propagation. So far as it depends on the other factors $\alpha\gamma/\rho^2$, etc., it is the same as for an infinitely small sphere, in particular no ray is emitted in the direction defined by $\alpha = \beta = 0$, that is in the direction of primary vibration. There is no limitation upon the value of R if (K−1) be small *enough*, but the reservation is important, since it is necessary that at every point of the obstacle the retardation of the primary waves *due to the obstacle* be negligible.

When R is great compared with $\lambda (= 2\pi/k)$, m usually varies rapidly with R or k, and so does P, as given for the complete uniform sphere in (5). An exception occurs when χ is nearly equal to π, that is when the secondary ray is nearly in the direction of the primary ray continued ($\beta = \gamma = 0$). In this case m is very small,

$$\frac{\sin m}{m^3} - \frac{\cos m}{m^2} = \frac{1}{3},$$

and $|P|$ is independent of k, and is proportional to R^3. The *intensity* is then

$$|P^2| = \frac{16(K-1)^2\pi^2R^6}{9} \tag{9}$$

The haze immediately surrounding a small source of light seen through a foggy medium is of relatively great intensity And the cause is simply that the contributions from the various parts of a small obstacle agree in phase

But in general when R is great, so also is m, and $|P|$ varies rapidly and periodically with k along the spectrum We might then be concerned mainly with the mean value of $|P^2|$ Now

$$|P^2| = (K-1)^2 \quad 4^2\pi^2R^6(\sin m - m\cos m)^2 m^{-6},$$

of which the mean value is

$$(K-1)^2 \quad 8\pi^2R^6(1+m^2)m^{-6},$$

or approximately, since m is great,

$$(K-1)^2 \quad 8\pi^2R^6m^{-4}$$

When we introduce the value of m from (6), this becomes

$$\text{Mean } |P^2| = \frac{(K-1)^2\pi^2R^2}{2k^4\cos^4\frac{1}{2}\chi} = \frac{(K-1)^2R^2\lambda^4}{32\pi^2\cos^4\frac{1}{2}\chi} \tag{10}$$

The occurrence of λ^4 shows that this is in general very small in comparison with (9)

If, instead of a sphere of uniform quality, we have to deal with one where $(K-1)$ is variable, we must employ (7). The case of greatest interest is when $(K-1)$, besides a constant, includes also a periodic part For the constant part the integration proceeds as before, and for the periodic part, where $(K-1)$ varies as a circular function of R, it presents no difficulty It may suffice to consider the particular case where $(K-1)$ is proportional to $\sin m$, m as before being given by (6), for this supposition evidently leads to a large augmentation of P, analogous to what occurs in crystals of chlorate of potash, to which a plane periodic structure is attributed* It will be observed that the wave-length of the structure now supposed varies with χ, as well as with k or λ. Thus, if $K-1 = \beta m$,

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* 'Phil. Mag.,' vol 26, p. 256 (1888); 'Scientific Papers,' vol. 3, p. 204

$$P = -\pi\beta e^{i(nt-kr)} R^3 \frac{m^2 - m \sin 2m + \frac{1}{2}(1 - \cos 2m)}{m^4}, \tag{11}$$

when the integration is taken for a complete sphere of radius R. If m is moderately great, that is, if R be a large multiple of λ, the first term on the right of (11) preponderates, and we may use approximately

$$P = -\frac{\pi\beta R^2 e^{i(nt-kr)}}{2k \cos \frac{1}{2}\chi} \tag{12}$$

Thus, if $(K-1)$ has no constant part

$$|P| = \frac{\pi\beta R^2}{2k \cos \frac{1}{2}\chi} = \frac{\beta R^2 \lambda}{4 \cos \frac{1}{2}\chi} \tag{13}$$

The relation between the wave-length of the structure (Λ) and that of the light is expressed by

$$\Lambda = \tfrac{1}{2}\lambda / \cos \tfrac{1}{2}\chi \tag{14}$$

It seems probable that a structure of this sort is the cause of the remarkable colours, variable with the angle of observation, which are so frequent in beetles, butterflies, and feathers.

Graphical Solution for High-Angle Fire.

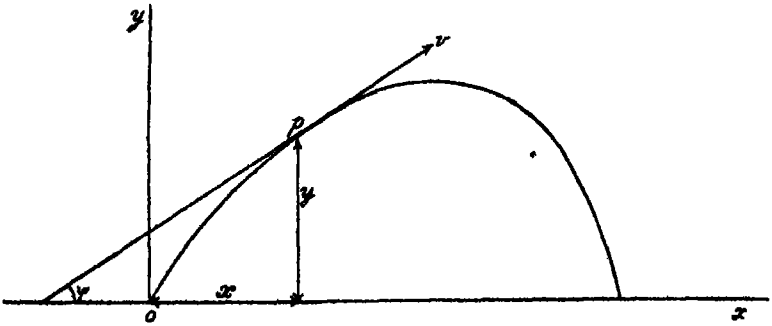
By Prof A. N WHITEHEAD, F R.S, Imperial College of Science and Technology

(Received November 20, 1917)

The point of the following methods is the reduction of the calculation of the path of a shell in high-angle fire to processes which can be either graphically constructed or calculated, once and for all, the results applying to all trajectories and shells of any ballistic constant

It is assumed that the law of resistance R(v) is largely empirical, and is not to be reduced to any neat mathematical formula Furthermore the variation of the density of the atmosphere is taken account of, and its dependence on the height is assumed to be given by some empirical formula

Let W be the weight of the projectile, v its velocity, ψ the angle between its direction of motion and the horizontal, viz, $0 \leq \psi \leq 90°$ for the shell ascending, and $-90° \leq \psi \leq 0$ for the shell descending The resistance of the air is $(W/Cg) \times R(v)$, C being the ballistic constant of the shell, corrected for the state of the atmosphere. Thus C will vary slowly as the height (y) changes



Resolving tangentially and normally, the equations become

$$\left.\begin{aligned} C\frac{dv}{dt} &= -R(v) - Cg\sin\psi \\ \frac{d\psi}{dt} &= -\frac{g\cos\psi}{v} \end{aligned}\right\}, \tag{1}$$

Put

$$\tanh z = \sin\psi \tag{2}$$

Then ψ will be called the "angular elevation" at the time t, and z the "hyperbolic elevation."

Eliminate t and ψ between equations (1) and (2), obtaining

$$\frac{d\log_e v}{dz} = (Cg)^{-1} \times R(v) + \tanh z \tag{3}$$

Let ρ be the density of the atmosphere at the position of the shell. Then we can put

$$C = B/\rho, \tag{4}$$

where B is a constant depending on the special projectile and on the standard projectile and standard state of the atmosphere.

We assume that ρ is some function of y. Put

$$\rho = f(y) \tag{5}$$

Thus (3) can be written

$$\frac{d\log_e v}{dz} = B^{-1} \times f(y) \times g^{-1} R(v) + \tanh z \tag{6}$$

Also it is easy to prove that

$$\frac{dy}{dz} = -g^{-1} v^2 \tanh z \tag{7}$$

These equations determine v and y as functions of z, assuming given initial conditions.

We proceed to explain a graphical method by which, given any simultaneous set (v, y, z) and also given the increment dz to z, the corresponding set $(v+dv, y+dy, z+dz)$ can be determined. The point of the method is that it proceeds by the use of five curves, the same for all projectiles and all trajectories. These curves can, therefore, be reproduced once for all on suitably arranged sheets of the proper dimensions. The actual construction required to find a series of values of (v, y, z) proceeding by small given differences dz, is such as to make a very moderate demand on a skilled draughtsman.

The five curves are

$$\eta = \tanh z, \qquad \eta = g^{-1} R(v),$$
$$\eta = \log_e v, \qquad \eta = g^{-1} v^2, \qquad \eta = f(y).$$

When z (or ψ) is positive, the shell is ascending; when z is negative, the shell is descending. It may, perhaps, be convenient for the draughtsman to treat the latter case separately.

Then putting $z' = -z$, equations (6) and (7) become

$$\frac{d\log_e (v)}{dz'} = \tanh z' - B^{-1} \times f(y) \times g^{-1} R(v), \tag{6'}$$

$$\frac{dy}{dz'} = -g^{-1} v^2 \tanh z'. \tag{7'}$$

Thus the graphical modification for the descending shell, if treated separately, is of the slightest description, and need not be further alluded to.

In Diagram I we use equation (6) and proceed from v, z, y and $z+dz$ to $v+dv$ With O as centre the plane is divided into four quadrants, and the curves in each quadrant are referred to the bounding lines of that quadrant considered as positive axes for that quadrant

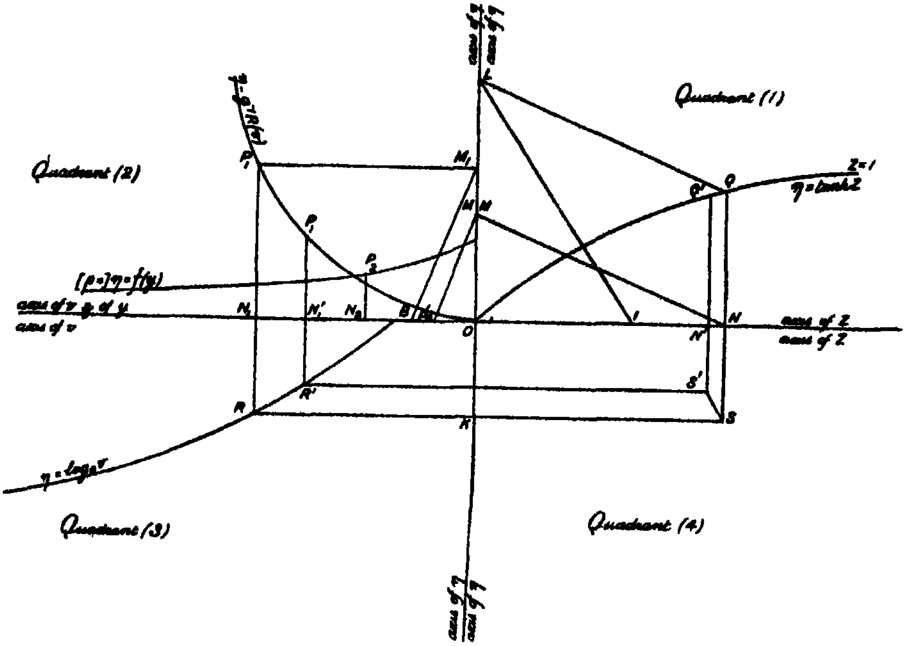


DIAGRAM I

In Quadrant (1) the curve $\eta = \tanh z$ is drawn, also $\mathrm{ON} = z$, $\mathrm{QN} = \tanh z$, $\mathrm{OI} = 1$, $\mathrm{ON}' = z+dz$ (dz being negative)

The point M on OY is determined from the second quadrant (as explained below), so that

$$\mathrm{OM} = \mathrm{B}^{-1} \times f(y) \times g^{-1}\mathrm{R}(v).$$

Then QL, in Quadrant (1), is parallel to NM Thus

$$\mathrm{OL} = \mathrm{B}^{-1} \times f(y) \times g^{-1}\mathrm{R}(v) + \tanh z$$

In Quadrant (2) the curves

$$\eta = g^{-1}\mathrm{R}(v) \quad \text{and} \quad \eta = f(y)$$

are so drawn that the horizontal axis in this quadrant acts both as axis of v and axis of y Then, on the horizontal axis

$$\mathrm{ON}_1 = v, \qquad \mathrm{ON}_2 = y, \qquad \mathrm{OP}_2' = \mathrm{P}_2\mathrm{N}_2 = f(y), \qquad \mathrm{OB} = \mathrm{B}$$

On the vertical axis, $OM_1 = P_1N_1$, and $P_2'M$ is parallel to BM_1.

In Quadrant (3) the curve $\eta = \log_e v$ is drawn, so that the horizontal axis is the axis of v, as in Quadrant (2) for one of the curves. Thus $RN_1 = \log_e v = OK$

In Quadrant (4), S is the point (z, η), where $\eta = \log_e v$

Then SS′ is drawn parallel to LI, to meet the vertical through N′, where $ON' = z+dz$, and dz is assumed to be negative, as will be the case Thus S′, of equation (6), corresponds to S, when z becomes $z+dz$ Then, working backwards, R′ is determined in Quadrant (3), and thence N_1', so that

$$ON_1' = v+dv$$

Hence $v+dv$, corresponding to $z+dz$, is determined

Diagram II gives the graphical construction to determine $y+dy$ in terms of $z+dz$, starting from the given simultaneous set (v, z, y) The quadrants are used separately as in Diagram I, but only two quadrants are now required, which (on the same plan as in Diagram I) are numbered Quadrant (1) and Quadrant (4) respectively

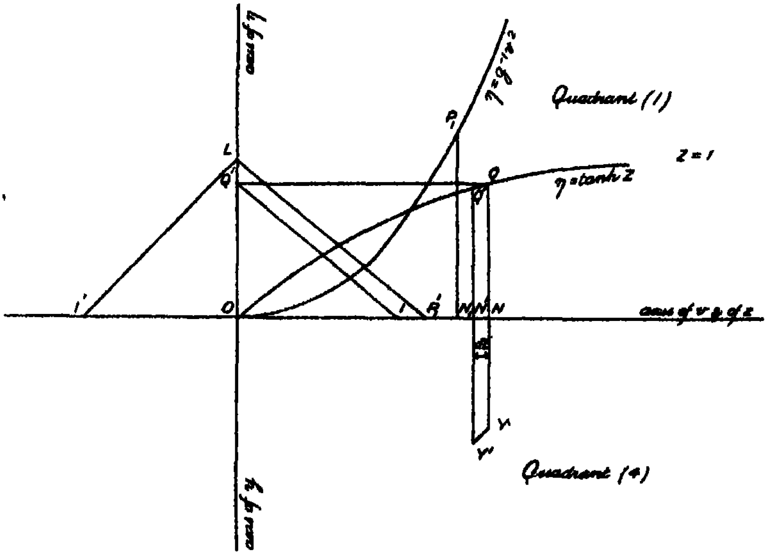


DIAGRAM II.

In Quadrant (1) the curves

$$\eta = \tanh z \quad \text{and} \quad \eta = g^{-1}v^2$$

are so drawn that the horizontal axis in this quadrant acts as the axis of z for the former curve, and as the axis of v for the latter curve.

Then on the horizontal axis

$$ON_1 = v, \quad OP_1' = P_1N_1, \quad OI = 1, \quad \text{and} \quad ON = z, \quad ON' = z+dz,$$

(dz negative in figure)

On the vertical axis

$$OQ' = QN = \tanh z,$$

and L is determined by drawing $P_1'L$ parallel to IQ'

On the horizontal axis produced backwards

$$OI' = 1$$

In Quadrant (4), Y is the point (z, y), and YY' is drawn parallel to I'L, to meet the vertical through N' in the point Y', which is $(z+dz, y+dy)$

Thus $v+dv$ and $y+dy$ have now been determined in terms of v, z, y and dz. The procedure can be repeated, starting from $(v+dv, z+dz, y+dy)$, and so on indefinitely

It is evident that it will not be necessary for a skilled draughtsman to draw many (or, indeed, any) of the straight lines which are here drawn for the purposes of explanation

It will be noted that the effect of height on air-resistance is taken into account in each step of the construction, and there is no smoothing out into zones The effect of different states of the atmosphere on different days might be allowed for by choosing a sufficient number of such typical states and issuing different sets of skeleton sheets, each set corresponding to the form of the curve which belongs to that state of the atmosphere for which that set of sheets is to apply.

Finally we note the form taken by the equation giving the dissipation of energy into the atmosphere, namely,

$$\left.\begin{aligned}\frac{d}{dz}(\tfrac{1}{2}v^2+gy) &= B^{-1}f(y)\times g^{-1}v^2R(v)\\ &= (Cg)^{-1}v^2R(v)\end{aligned}\right\}. \tag{8}$$

By this graphical method v and y have been tabulated in terms of z The remaining elements of the path are given by

$$t = -\frac{1}{g}\int_{z_0}^{z} v\,dz, \qquad x = -\frac{1}{g}\int_{z_0}^{z} v^2 \operatorname{sech} z\,dz. \tag{9}$$

These quadratures can be effected either graphically as above, or by any other suitable method of approximation Also y is given by equation (7) above.

If a closer approximation is desired by an analytical method, we can proceed by using the graphical method as a rapid first approximation

It must be remembered that the whole point of any procedure, analytical or graphical, is to reduce the solution as far as possible to processes which can be tabulated (or drawn) once and for all, suitable for all projectiles and all paths

Write u for the horizontal velocity, so that $u = v \sec\psi$. Resolving horizontally and normally

$$C\frac{du}{dt} = -R(u\sec\psi)\cos\psi,$$

$$u\frac{d\psi}{dt} = -g\cos^2\psi$$

Then [*cf* equations (4) and (5)]

$$Bgu^{-1}\frac{du}{d\psi} = f(y)\times R(u\sec\psi)\times\sec\psi \tag{10}$$

For a certain range of relatively low values of v

$$R(v) \propto v^2, \text{ approximately},$$

for a certain range of high values of v

$$R(v) \propto v^3, \text{ approximately},$$

for the intermediate range, no one value of n will give approximately $R(v) \propto v^n$ But, by dividing the whole range of important velocities for gun fire into a sufficient number of ranges, we can for each range find a value of n, say, n_{pq}, which to sufficient accuracy allows us to write

$$(V_p \text{ to } V_q), \qquad R(v) = \mu v^{n_{pq}}.$$

These values of μ and n_{pq} can be found from the empirically constructed tables for $R(v)$ by the method of least squares, Thus, if $\nu' = \log_e\mu$, and N_{pq} be the number of tabulations within the range V_p to V_q

$$N_{pq}\nu + n_{pq}\Sigma\log_e v = \Sigma\log_e R(v),$$

$$\nu\Sigma\log_e v + n_{pq}\Sigma\{\log_e v\}^2 = \Sigma\{\log_e v \quad \log_e R(v)\}$$

The determination of n_{pq} for each range can be effected once and for all

With this value for n_{pq},

$$R(u\sec\psi) = R(u)\{\sec\psi^{n_{pq}}\}, \qquad (V_p \text{ to } V_q) \tag{11}$$

Then equation (10) becomes

$$Bg\int_{u_0}^{u}\frac{du}{uR(u)} = \sum_{(pq)}\int f(y)(\sec\psi)^{n_{pq}+1}\,d\psi \tag{12}$$

The integration on the left-hand side of (12) can be extracted from tables for the function $g\int^{u}\frac{du}{uR(u)}$, which is already tabulated in the Gunnery text-books

Thus the only operation special to the path is the calculation of the right-hand side of (12)

Now y is determined as a function of ψ by the preceding graphical method Accordingly within each range (V_p to V_q) or portion of such range,

$$\int f(y)(\sec\psi)^{n_m+1}d\psi$$

can be calculated by any convenient method of quadrature Thus u (and v) are determined as functions of ψ, and all the elements of the path (viz, (x, y and t) are determined in terms of ψ

The Lommel-Weber Ω Function and its Application to the Problem of Electric Waves on a Thin Anchor Ring

By J R AIREY, M A, D Sc

(Communicated by Prof J W Nicholson, F R S Received December 7, 1917)

The function $\Omega_n(x)$ defined by the integral $\frac{1}{\pi}\int_0^\pi \sin(x\sin\phi - n\phi)\,d\phi$, or, when n is an even integer, by $\frac{2}{\pi}\int_0^{\pi/2}\sin(x\sin\phi)\cos n\phi\,d\phi$, is closely related to $J_n(x)$, which was first given by Bessel under the form $\frac{1}{\pi}\int_0^\pi \cos(x\sin\phi - n\phi)\,d\phi$

The Ω function, although of less importance in its application than the J function, occurs in a number of problems in mathematical physics, more especially those relating to the interference and diffraction of light.

The function $\Omega_0(x) = \frac{2}{\pi}\int_0^{\pi/2}\sin(x\sin\phi)\,d\phi$ was employed by Lord Rayleigh[*] in the problem of the reaction of the air on a vibrating circular plate, and by H Struve[†] in the theory of diffraction in telescopes H F Weber[‡] applied the function $\Omega_{\frac{1}{2}}(x)$ and, in addition, found a number of interesting properties of $\Omega_\nu(x)$, including the recurrence formula and the development in series of ascending and descending powers of the argument.

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[*] Lord Rayleigh, 'Theory of Sound,' vol 2, p. 164, equation (5)

[†] H. Struve, "Beitrag zur Theorie der Diffraction an Fernröhren," 'Annalen der Physik und Chemie,' vol. 17, p. 1013

[‡] H. F Weber, "Die wahre Theorie der Fresnel'schen Interferenz-Erscheinungen," 'Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich,' vol. 24, p 48

Lommel* proved that the integral $\frac{1}{\pi}\int_0^\pi \sin(\nu\phi - x\sin\phi)\,d\phi$ is represented by the following series when ν is an even number, $2n$,

$$\Omega_{2n}(x) = -\frac{2}{\pi}s_{0,2n}(x) \qquad \text{or} \qquad -\frac{2}{\pi}\left[S_{0,2n}(x) + Y_{2n}(x)\right],$$

where

$$s_{0,\nu}(x) = \frac{x}{(1^2-\nu^2)} - \frac{x^3}{(1^2-\nu^2)(3^2-\nu^2)} + \frac{x^5}{(1^2-\nu^2)(3^2-\nu^2)(5^2-\nu^2)} - \ldots,$$

$$S_{0,\nu}(x) = \frac{1}{x} - \frac{(1^2-\nu^2)}{x^3} + \frac{(1^2-\nu^2)(3^2-\nu^2)}{x^5} - \ldots,$$

and $Y_{2n}(x)$ is the Bessel function of the second kind according to Lommel's definition. A similar formula gives the value of the function when ν is odd

Functions of higher order occur in the problem of electric waves on a thin anchor ring If a is the radius of the circular axis of the ring, ϵ the radius of the circular section, m any integer and $\Pi = e^{im\phi}e^{ipt}/\rho$, Lord Rayleigh† found that approximately

$$\int_0^\pi \frac{(a^2\alpha^2\cos\phi - m^2)\cos m\phi\,d\phi}{\sqrt{(\epsilon^2 + 4a^2\sin^2\frac{1}{2}\phi)}} + \int_0^\pi \frac{(e^{2ia\alpha\sin\frac{1}{2}\phi} - 1)(a^2\alpha^2\cos\phi - m^2)\cos m\phi\,d\phi}{2a\sin\frac{1}{2}\phi} = 0 \tag{1}$$

The second integral determines the imaginary part of $(a^2\alpha^2 - m^2)L$, where $L = \log\frac{8a}{\epsilon}$, and it has been shown‡ that this can be expressed in terms of Bessel functions of nearly equal order and argument, viz

$$\frac{im^2\pi}{2}\{J_{2m-1}(2m) - J_{2m+1}(2m)\} \qquad \text{or} \qquad im^2\pi\ \frac{d}{dx}\ J_{2m}(x), \tag{2}$$

and $x = 2m$

On the other hand, the real part of $(a^2\alpha^2 - m^2)L$ is made up of contributions from both integrals The second integral takes the form

$$\frac{m^2}{2a}\int_0^{\pi/2} d\psi\,\frac{e^{ix\sin\psi} - 1}{\sin\psi}\,\{\cos(2m+2)\psi - 2\cos 2m\psi + \cos(2m-2)\psi\}, \tag{3}$$

where $x^2 = 4a^2\alpha^2 = 4m^2$ nearly To evaluate this, Lord Rayleigh introduces the integral

$$\frac{2}{\pi}\int_0^{\pi/2} d\psi\cos 2m\psi\quad e^{ix\sin\psi} = J_{2m}(x) + iK_{2m}(x) \tag{4}$$

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* Lommel, "Zur Theorie der Bessel'schen Functionen," 'Math Ann,' vol. 16, p. 188.

† Lord Rayleigh, "Electrical Vibrations on a Thin Anchor Ring," 'Roy. Soc Proc.,' vol. 87, p 194

‡ *Loc cit.*, p 196.

The K function is no other than the Ω function, but the term -1 in the numerator $e^{ix\sin\psi}-1$ in (3), although not affecting the imaginary part involving the J functions, reduces the result for the real part to

$$m^2\pi\left\{\frac{d}{dx}\quad\Omega_{2m}(x)-(-1)^m\frac{2}{\pi(2m-1)(2m+1)}\right\}$$

But the contribution to the real part of $(a^2\alpha^2-m^2)\,\mathrm{L}$ from the first integral is

$$(-1)^m\quad\frac{2m^2}{(2m-1)(2m+1)}=(-1)^m m^2\pi\quad\frac{d}{dx}\,\frac{x/2}{\Gamma(m+\frac{3}{2})\,\Gamma(-m+\frac{3}{2})}$$

and $(-1)^m\dfrac{x/2}{\Gamma(m+\frac{3}{2})\,\Gamma(-m+\frac{3}{2})}$ is the first term of $\Omega_{2m}(x)$

Consequently, equation (21) in Lord Rayleigh's paper takes the simple form

$$(a^2\alpha^2-m^2)\,\mathrm{L}=m^2\pi\,\{i\mathrm{J}_{2m}'(x)+\Omega_{2m}'(x)\}\quad\text{and}\quad x=2m,\tag{5}$$

$$=\frac{m^2\pi}{2}\,\{i\,[\mathrm{J}_{2m-1}(2m)-\mathrm{J}_{2m+1}(2m)]+[\Omega_{2m-1}(2m)-\Omega_{2m+1}(2m)]\},\tag{6}$$

since*

$$\Omega_{2m-1}(x)-\Omega_{2m+1}(x)=2\quad\frac{d}{dx}\quad\Omega_{2m}(x)\tag{7}$$

Hence

$$-\alpha=\frac{m}{a}\left\{1+\frac{\pi}{4\mathrm{L}}\{i\,[\mathrm{J}_{2m-1}(2m)-\mathrm{J}_{2m+1}(2m)]+[\Omega_{2m-1}(2m)-\Omega_{2m+1}(2m)]\}\right\}\tag{8}$$

and the equivalent wave lengths λ_m are given by

$$\lambda_m=\frac{2\pi a}{m}\left\{1-\frac{\pi}{4\mathrm{L}}\,[\Omega_{2m-1}(2m)-\Omega_{2m+1}(2m)]\right\}\tag{9}$$

Where m is small, $\Omega_{2m-1}(2m)$ and $\Omega_{2m+1}(2m)$ are readily obtained from the series in ascending powers of m,

$$\Omega_{2m+1}(x)=(-1)^{m+1}\sum_{s=0}^{s=\infty}\frac{(-1)^s(x/2)^{2s}}{\Gamma(s+m+\frac{3}{2})\,\Gamma(s-m+\frac{1}{2})},\tag{10}$$

or, in terms of the Bessel functions† $\mathrm{J}_{2s}(x)$,

$$\Omega_{2m+1}(x)=-\frac{2}{\pi}\left[\frac{\mathrm{J}_0(x)}{2m+1}+(4m+2)\sum_{s=1}^{s=\infty}\frac{\mathrm{J}_{2s}(x)}{(2m+1)^2-4s^2}\right]\tag{11}$$

The asymptotic expansion given by Lommel and Weber‡ cannot be usefully employed where the order and argument are large Expressions giving the values of $\Omega_n(n)$, $\Omega_{n-1}(n)$, etc, can be derived from a result due to Sonine and Schlafli.

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* Nielsen, 'Cylinderfunktionen,' p 49.

† Nielsen, 'Cylinderfunktionen,' p 67

‡ *Loc cit.*, p 48

Sonine's result is[*]

$$Y_n(x) = \int_0^\pi \sin(x\sin\phi - n\phi)\,d\phi - \int_0^\infty e^{-x\sinh\theta}\left[e^{n\theta} + (-1)^n\ e^{-n\theta}\right]d\theta,$$

the function $Y_n(x)$ being the same as Hankel's and equal to $-2G_n(x)$

Schlafli found that[†]

$$Y_n(x) - (\log 2 - \gamma)\,J_n(x) = \tfrac{1}{2}\int_0^\pi \sin(x\sin\phi - n\phi)\,d\phi$$

$$-\tfrac{1}{2}\int_0^\infty e^{-x\sinh\theta}\left[e^{n\theta} + (-1)^n e^{-n\theta}\right]d\theta$$

Here $\qquad Y_n(x) - (\log 2 - \gamma)\,J_n(x)$ is equal to $-G_n(x)$

Expressed in the form

$$\Omega_n(x) = Y_n(x) + \frac{1}{\pi}\int_0^\infty e^{-x\sinh\theta}\left(e^{n\theta} + \cos n\pi\ e^{-n\theta}\right)d\theta, \tag{12}$$

the Neumann function $Y_n(x)$ is equal to $-2/\pi\ G_n(x)$ Very complete Tables[‡] of $G_n(n)$ and $G_{n-1}(n)$ with other related functions have been computed for a large number of values of n The integral in (12) can be evaluated by the "Saddle Point" method of contour integration,[§] or by the less elaborate method given below Taking the case where the order and argument are equal and n is an even integer,

$$\Omega_n(n) = -\frac{2}{\pi}G_n(n) + \frac{1}{\pi}\int_0^\infty e^{-n\sinh\theta}\left(e^{n\theta} + e^{-n\theta}\right)d\theta \tag{13}$$

In the integral $\dfrac{1}{\pi}\displaystyle\int_0^\infty e^{-n(\sinh\theta-\theta)}\,d\theta$, change the variable by substituting t for $\sinh\theta - \theta$ then if λ is written for $(6t)^{1/3}$,

$$\lambda = \theta + \frac{\theta^3}{60} + \frac{\theta^5}{8400} - \ldots$$

Reversing this series,

$$\theta = \lambda - \frac{\lambda^3}{60} + \frac{\lambda^5}{1400} - \frac{\lambda^7}{25200} + \ldots$$

and

$$d\theta = \left(1 - \frac{\lambda^2}{20} + \frac{\lambda^4}{280} - \frac{\lambda^6}{3600} + \ldots\right)d\lambda.$$

Writing a for $\left(\dfrac{6}{n}\right)^{1/3}$ and applying $\displaystyle\int_0^\infty e^{-nt}\,t^{p-1}\,dt = \frac{\Gamma(p)}{n^p}$ to each of the terms in the integrand, we find

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[*] Sonine, "Fonctions Cylindriques," 'Math. Ann.,' vol 3, p 27

[†] Schlafli, "Bessel'schen Functionen," 'Math Ann.,' vol 3, p 143

[‡] 'Report of the Mathematical Tables Committee of the British Association,' 1916, pp 92–96

[§] Debye, 'Math Ann.,' vol 67, pp 535–558, Brillouin, 'Annales de l'École Normale,' vol 33, pp. 17–69

$$\frac{1}{\pi}\int_0^\infty e^{-n(\sinh\theta-\theta)}\,d\theta$$

$$=\frac{1}{3\pi}\int_0^\infty e^{-nt}\left(6^{1/3}t^{-2/3}-\frac{6}{20}+\frac{1}{280}6^{5/3}t^{2/3}-\frac{1}{3600}6^{7/3}t^{4/3}+\quad\right)dt$$

$$=\frac{1}{3\pi}\left[\alpha\quad\Gamma(\tfrac{1}{3})-\frac{\alpha^3}{20}+\frac{\alpha^5}{420}\Gamma(\tfrac{2}{3})-\frac{\alpha^7}{8100}\Gamma(\tfrac{1}{3})+\quad\right] \tag{14}$$

In the integral $\frac{1}{\pi}\int_0^\infty e^{-n(\sinh\theta+\theta)}\,d\theta$, on substituting t for $\sinh\theta+\theta$, reversing the series and proceeding as before, we get

$$\frac{1}{\pi}\int_0^\infty e^{-n(\sinh\theta+\theta)}\,d\theta=\frac{1}{\pi}\left\{\frac{1}{(2n)}-\frac{1}{2(2n)^3}+\frac{2}{(2n)^5}-\quad\right\} \tag{15}$$

It has been shown* that

$$G_n(n)\doteq\tfrac{1}{4}\left\{\alpha\quad\Gamma(\tfrac{1}{3})+\frac{\alpha^5}{420}\Gamma(\tfrac{2}{3})-\frac{\alpha^7}{8100}\Gamma(\tfrac{1}{3})-\;.\right\}$$

Therefore

$$\Omega_n(n)=-\frac{2}{3\pi}G_n(n)+\frac{1}{\pi}\left\{\frac{1}{(2n)}-\frac{1}{2(2n)^3}+\frac{2}{(2n)^5}-\quad\right\}$$

$$-\frac{1}{3\pi}\left\{\frac{3}{5(2n)}-\frac{10449}{134750(2n)^3}+\quad\right\}$$

$$=-\frac{1}{6\pi}\left\{\alpha\quad\Gamma(\tfrac{1}{3})-\frac{2\alpha^3}{5}+\frac{\alpha^5}{420}\Gamma(\tfrac{2}{3})-\frac{\alpha^7}{8100}\Gamma(\tfrac{1}{3})+\frac{15973\alpha^9}{9702000}-\quad\right\} \tag{16}$$

For $\Omega_{n-1}(n)$, the integrals in (10) are

$$\frac{1}{\pi}\int_0^\infty e^{-n(\sinh\theta-\theta)}e^{-\theta}\quad d\theta\text{ and }\frac{1}{\pi}\int_0^\infty e^{-n(\sinh\theta+\theta)}e^{\theta}\,d\theta$$

The factor $e^{\pm\theta}$ is expanded in a series of ascending powers of λ, e g in the first integral $e^{-\theta}=1-\lambda+\frac{9\lambda^2}{20}-\frac{\lambda^3}{10}+\frac{\lambda^4}{280}-\frac{9\lambda^5}{2800}+\quad$ and we obtain the result

$$\Omega_{n-1}(n)-\Omega_n(n)=\frac{1}{6\pi}\left\{\alpha^2\Gamma(\tfrac{2}{3})+\frac{\alpha^4}{30}\Gamma(\tfrac{1}{3})-\frac{37\alpha^6}{1050}+\frac{23\alpha^8}{113400}\Gamma(\tfrac{2}{3})\right.$$

$$\left.-\frac{947\alpha^{10}}{74844000}\Gamma(\tfrac{1}{3})+\frac{4\alpha^{12}}{15875}-\;..\right\}$$

$$=-\frac{2}{3\pi}\left\{G_{n-1}(n)-G_n(n)+\frac{111}{350n^2}-\frac{209}{2560n^4}+\quad\right\}, \tag{17}$$

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* Debye, 'Math Ann,' vol 67, p 557

$$G_n(n)=-\frac{\pi}{2}Y_n(n)=-\frac{i\pi}{4}\left\{H_n^{(\;)}(n)-H_n^{(1)}(n)\right\}$$

'Phil. Mag.,' vol. 31, p. 526 (June, 1916).

the last term in each expansion being given approximately When x is equal to n and n is an even number, the recurrence formula

$$\Omega_{n-1}(x)+\Omega_{n+1}(x) = \frac{2n}{x}\Omega_n(x)+\frac{4\sin^2 n\pi/2}{\pi x}$$

reduces to

$$\Omega_{n-1}(n)-\Omega_{n+1}(n) = 2\{\Omega_{n-1}(n)-\Omega_n(n)\} \tag{18}$$

Schlafli's polynomials $S_n(x) = \int_0^\infty e^{-x\sinh\theta}(e^{n\theta}-(-1)^n e^{-n\theta})\,d\theta$ may be employed to simplify the calculation, since the integral in (13) is equal to $S_n(n)+2\int_0^\infty e^{-n(\sinh\theta+\theta)}\,d\theta$.

Then

$$\Omega_n(n) = -\frac{2}{\pi}\left\{G_n(n)-\tfrac{1}{2}S_n(n)-\left(\frac{1}{2n}-\frac{1}{2(2n)^3}+\frac{2}{(2n)^5}- .\right)\right\}, \tag{19}$$

$$\Omega_{n-1}(n) = -\frac{2}{\pi}\left\{G_{n-1}(n)-\tfrac{1}{2}S_{n-1}(n)+\left(\frac{1}{2n}+\frac{1}{(2n)^2}+\frac{1}{2(2n)^3}- \quad\right)\right\}. \tag{20}$$

and

$$\Omega_{n-1}(n)-\Omega_n(n) = -\frac{2}{\pi}\left\{[G_{n-1}(n)-G_n(n)]-\tfrac{1}{2}[S_{n-1}(n)-S_n(n)]\right.$$
$$\left.+\left(\frac{1}{n}+\frac{1}{(2n)^2}-\frac{1}{(2n)^4}+\frac{11}{2(2n)^6}- .\right)\right\} \tag{21}$$

If n is not very large, the functions $S_{n-1}(n)$ and $S_n(n)$ can be found without difficulty from

$$S_n(x) = \sum_{s=0}^{s \leqq [(n-1)/2]} \frac{(n-s-1)!}{s!}\left(\frac{2}{x}\right)^{n-2s} \tag{22}$$

Table of $\Omega_{2m-1}(2m)-\Omega_{2m+1}(2m) = \Delta$

m	Δ	m	Δ	m	Δ
1	0 308478	6	0·093342	11	0·061822
2	0 196336	7	0·084049	12	0 058277
3	0 149466	8	0 076754	13	0·055198
4	0 122976	9	0·070849	14	0 052493
5	0 105667	10	0 065955	15	0·050095

The corresponding values of R for $m = 1$ to $m = 6$ are

m	R	m	R
1	0 4846	4	3 0907
2	1 2336	5	4 1495
3	2·1130	6	5 2784

Thus as the value of m increases, *i e* for the higher modes of free vibration, λ_m, the equivalent wave-length, approaches the value $2\pi a/m$.

In the general case, where the order and argument are nearly equal, when $x = n+\kappa$, κ being a small number, the integral

$$\int_0^\infty e^{-x\sinh\theta+n\theta}\,d\theta = \int_0^\infty e^{-x(\sinh\theta-\theta)}\,e^{-\kappa\theta}\,d\theta$$

If $\sinh\theta-\theta = t$ and λ is written for $(6t)^{1/3}$, then $\theta = \lambda-\frac{\lambda^3}{60}+\frac{\lambda^5}{1400}-$

and $d\theta = \left(1-\frac{\lambda^2}{20}+\frac{\lambda^4}{280}-\quad\right)d\lambda$

Expressing $e^{-\kappa\theta}$ in terms of λ we get, after a little reduction,

$$\int_0^\infty e^{-x\sinh\theta+n\theta}\,d\theta = \frac{1}{3}\int_0^\infty e^{-xt}\,6^{1/3}\,t^{-2/3}\{\sigma_0-\sigma_1(6t)^{1/3}+\sigma_2(6t)^{2/3}$$
$$-\sigma_3(6t)+\sigma_4(6t)^{4/3}-\quad\}\,dt$$
$$=\frac{1}{3}\left\{\sigma_0\left(\frac{6}{x}\right)^{1/3}\Gamma\left(\frac{1}{3}\right)-\sigma_1\left(\frac{6}{x}\right)^{2/3}\Gamma\left(\frac{2}{3}\right)+\sigma_2\left(\frac{6}{x}\right)\right.$$
$$\left.-\frac{\sigma_3}{3}\left(\frac{6}{x}\right)^{4/3}\Gamma\left(\frac{1}{3}\right)+\frac{2\sigma_4}{3}\left(\frac{6}{x}\right)^{5/3}\Gamma\left(\frac{2}{3}\right)-2\sigma_5\left(\frac{6}{x}\right)^2+\quad\right\}$$

The first seven values of σ_n are as follows,

$$\sigma_0 = 1$$

$$\sigma_1 = \kappa.$$

$$\sigma_2 = \frac{\kappa^2}{2}-\frac{1}{20}$$

$$\sigma_3 = \frac{\kappa^3}{6}-\frac{\kappa}{15}$$

$$\sigma_4 = \frac{\kappa^4}{24}-\frac{\kappa^2}{24}+\frac{1}{280}$$

$$\sigma_5 = \frac{\kappa^5}{120}-\frac{\kappa^3}{60}+\frac{43\kappa}{8400}$$

$$\sigma_6 = \frac{\kappa^6}{720}-\frac{7\kappa^4}{1440}+\frac{\kappa^2}{288}-\frac{1}{3600}.$$

For the integral $\int_0^\infty e^{-x\sinh\theta-n\theta}\,d\theta = \int_0^\infty e^{-x(\sinh\theta+\theta)}\,e^{\kappa\theta}\,d\theta$, put $t = \sinh\theta+\theta$ and $\lambda = t/2$

Then $\theta = \lambda-\frac{\lambda^3}{12}+\frac{\lambda^5}{60}-\ldots$ and $d\theta = \left(1-\frac{\lambda^2}{4}+\frac{\lambda^4}{15}-\ldots\right)d\lambda.$

Since

$$e^{\kappa\theta} = 1+\kappa\lambda+\frac{\kappa^2}{2}\lambda^2+\left(\frac{\kappa^3}{6}-\frac{\kappa}{12}\right)\lambda^3+\left(\frac{\kappa^4}{24}-\frac{\kappa^2}{12}\right)\lambda^4+\quad ,$$

$$\int_0^\infty e^{-x\sinh\theta-n\theta}\,d\theta = \frac{1}{2}\int_0^\infty e^{-xt}\left\{s_0+s_1\frac{t}{2}+s_2\frac{t^2}{4}+s_3\frac{t^3}{8}+\quad\right\}dt$$

$$= \frac{s_0}{2x}+\frac{s_1}{(2x)^2}+\frac{2s_2}{(2x)^3}+\frac{6s_3}{(2x)^4}+\frac{24s_4}{(2x)^5}+\quad ,$$

where

$$s_0 = 1$$

$$s_1 = \kappa$$

$$s_2 = \frac{\kappa^2}{2}-\frac{1}{4}$$

$$s_3 = \frac{\kappa^3}{6}-\frac{\kappa}{3}.$$

$$s_4 = \frac{\kappa^4}{24}-\frac{5\kappa^2}{24}+\frac{1}{12}$$

$$s_5 = \frac{\kappa^5}{120}-\frac{\kappa^3}{12}+\frac{29\kappa}{240}$$

$$s_6 = \frac{\kappa^6}{720}-\frac{7\kappa^4}{288}+\frac{119\kappa^2}{1440}-\frac{43}{1440}$$

Finally, knowing the development of $Y_n(x)$ or $-\frac{2}{\pi}G_n(x)$ in series of descending powers of x, we obtain the result

$$\Omega_n(x) = -\frac{1}{2\pi}\{\sigma_0\alpha\Gamma(\tfrac{1}{3})-\sigma_1\alpha^2\Gamma(\tfrac{2}{3})-\sigma_3\alpha^4\Gamma(\tfrac{4}{3})+\sigma_4\alpha^5\Gamma(\tfrac{5}{3})-\,.\,.\}$$

$$+\frac{1}{3\pi}\{\sigma_0\alpha\Gamma(\tfrac{1}{3})-\sigma_1\alpha^2\Gamma(\tfrac{2}{3})+\sigma_2\alpha^3-\sigma_3\alpha^4\Gamma(\tfrac{4}{3})+\sigma_4\alpha^5\Gamma(\tfrac{5}{3})-2\sigma_5\alpha^6-\,.\quad\}$$

$$+\frac{\cos n\pi}{\pi}\{s_0\beta+s_1\beta^2+2s_2\beta^3+6s_3\beta^4+24s_4\beta^5+120s_5\beta^6+\quad\},$$

where $\alpha = \left(\frac{6}{x}\right)^{1/3}$ and $\beta = \frac{1}{2x}$.

The coefficients σ_n are the same as those occurring in the asymptotic expansion of $J_n(x)$, but σ_2, σ_5, etc., disappear in the final expression owing to the zero factor $\sin\frac{(n+1)\pi}{3}$. The arguments of the Γ functions associated with these coefficients are integers

Flocculation.

By Spencer Umfreville Pickering, M.A., F.R.S.

(Received December 8, 1917.)

The breaking down of clay soils by frost is largely due to the mechanical disruption of the clods, but a change also occurs in the ultimate particles, which, when agitated with water, are found to sink more rapidly than before. This is due to the shrinking of the individual particles, and then agglomeration into clots. Such a change is the reverse of that occurring in flocculation produced by the addition of acids, etc., where, as will be shown, there is an increase in the size of the individual particles.

All bulky, hydrated precipitates appear to behave as clay does on freezing. The three basic copper sulphates examined were obtained by adding lime-water to copper sulphate;[*] the copper hydroxide, by adding copper sulphate in the calculated proportion of lime-water; the ferric hydroxide, from the chloride and ammonia; all of these were used without separation from the liquids. The alumina was precipitated and well washed, but not dried; the clay was from the Oxford formation, it was puddled with water, and the turbid supernatant liquid decanted after 10 minutes' settlement from the coarser particles; the kaolin was similarly treated.

In each case 100 c.c. of these liquids were kept frozen for two hours at −18°, and, on melting, transferred to graduated cylinders, side by side with unfrozen specimens. Readings of the volumes of the sediment were taken daily till the relative volumes of any pair became practically constant; this occurred after two or three days. The values are entered in Table I. Thus, considerable shrinkage, varying from 40 to 90·3 per cent., has occurred in all these cases.[†] This appears to be due to dehydration. Taking advantage of the fact that the sulphate $10CuO,SO_3$ (which contains also the elements of $CaSO_4$, or of Na_2SO_4 if excess of sodium sulphate is present), emulsifies paraffin oil, its state of hydration when freshly precipitated was determined,[‡] and found to be $42H_2O$. A similar determination with the

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[*] 'Journ. Chem. Soc.,' 1907, p. 1988.

[†] Exceptional behaviour was observed with a third preparation of alumina. It had proved very difficult to wash, and still retained 0·09 per cent. Cl after several months; this jelly, which contained 3·32 per cent. Al_2O_3, on being frozen, became a limpid, almost quite clear, liquid, leaving only 4 per cent. of the total alumina present on the filter paper after filtration; yet the rest of the alumina was but in a state of suspension, being separable on filtration through a very fine porous candle.

[‡] 'Journ. Chem. Soc.,' vol. 95, p. 123 (1909).

Table I.—Volume of Sediment from Frozen Liquids, that from similar unfrozen ones being 100.

Substance	Volume
$4CuO,SO_3$	42
$5CuO,SO_3$	43
$10CuO,SO_3$	16
CuO,xH_2O	9·7
Fe_2O_3,xH_2O	49
Al_2O_3,xH_2O, commercial	19
Al_2O_3,xH_2O, pure	19·4
Kaolin	60
Clay A (1·06 grm. anhyd. solid in 100 c.c.)	18·5
" B (1·81 " " 100 ")	19·1
" " frozen, and repuddled for 10 minutes	33·5
" " " " 20 "	33·3
" " " boiled 30 minutes	18·1
" " " " 2 hours	21·0
" " " 0·001 gr. KOH added to 100 c.c.	19·1
" " " 0·01 " " "	19·1
" " " 0·1 " " "	18·6
" " " 2 " " "	20·5
" " " ditto, and boiled	19·1

frozen precipitate is impossible, as it will not emulsify oil, but the water combined with it is found to be much reduced, for, on being filtered under pressure, avoiding loss by evaporation, the combined and adherent water together amounts to only $18H_2O$. Further evidence as to dehydration was afforded by the fact that the sediment from 100 c.c. of the unfrozen clay suspension B, when dried over sulphuric acid and re-suspended in water, gave a sediment with a relative volume of 21·5, or practically the same as that after freezing. Partial drying by mere exposure to air resulted in no shrinkage of the sediment.

Dehydration by freezing affords an instance of a chemical combination feebler than that uniting similar molecules in a crystal.* It is not a temperature effect, for these sediments when kept at 0° for several days undergo no shrinkage, nor does shrinkage occur when there is insufficient water present to allow of crystallisation, as when air-dried clay is cooled to −18°.

The contraction consequent on dehydration is increased, at any rate in the case of clay, by the particles becoming aggregated into clots. With clay, where, unlike the other substances examined, the particles are of very different magnitudes, freezing reduces the amount of finer particles left in suspension in nearly the same ratio as it reduces the volume of the coarser particles in the sediment.

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* *Cf.* Emerson Reynolds' investigation of the behaviour of platin-cyanides at low temperatures, 'Proc. Roy. Soc.,' A, vol. 82, p. 380.

Re-hydration of frozen clay must occur somehow in the field, but by what agency is not apparent; neither frozen clay nor any of the precipitates examined show any increase in bulk on being left in water for months, nor, in the case of clay, does any of the treatments specified in the last eight entries in Table I produce any appreciable effect, except re-puddling, and there the effect is a limited one, probably resulting merely from the breaking up of the clots, not from their re-hydration. Equally ineffective was repeated air-drying and moistening of the sediment during several weeks, and mixture with some garden soil during such treatment produced no change.

The elements of $CaSO_4$ (or Na_2SO_4) are united with the basic sulphate, $10CuO,SO_3$, in the same loose way as that in which water is, and may be removed by washing, or even by dilution of the liquid; yet their presence modifies considerably the physical and chemical properties of the substance. When heated at 100° in the liquid from which it has been precipitated (which is a saturated solution of calcium sulphate), the basic sulphate gradually becomes grey,* but retains its bulkiness for a long time; if, however, the calcium sulphate is entirely or partially removed, decomposition is very rapid and complete, and nothing but a compact deposit of black copper oxide remains. In numerous series of experiments the effect of various substances on the stability of the basic sulphate has been investigated; amongst inorganic salts, magnesium sulphate, when present in large proportion, is the most effective in imparting stability, whilst some organic substances, such as starch, and especially dextrin, are still more effective. With the latter, the maximum stability is attained when the proportions $C_6H_{10}O_5$ 4Cu is reached, the compound resisting decomposition during about two hours' heating at 100°, though it becomes less bulky than in the absence of dextrin. Smaller proportions of dextrin, though they have a delaying effect on the decomposition, render it complete when it does occur, the product then consisting of the black compact oxide. The point at which this decomposition is at a maximum indicates the existence of another compound with, approximately, $C_6H_{10}O_5$ 10 Cu.

In all cases the combination of the basic sulphates with these substances is reversible; the substances can be removed by washing, and replaced by adding them again to the liquids. These facts have an important bearing on the flocculation of clay, etc., by acids or salts.

As regards flocculation, a significant fact seems to have been overlooked by previous workers, namely, that it is accompanied by a large increase in the volume occupied by the particles, as evidenced by the volume of the deposit eventually obtained on subsidence.

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* 'Journ. Chem. Soc.,' vol. 98, p. 1852 (1910).

Kaolin was taken for the present investigation Samples of this vary greatly in their powers of suspension, that selected was a "purified," but not acid-washed, specimen, it was ground with twice its weight of water, rendered slightly alkaline with ammonia,[*] and boiled vigorously in platinum for 30 minutes, after 24 hours, the upper two-thirds were syphoned off for use This contained 20 per cent of solid, after ignition In the determinations, 10 c c were transferred to a graduated cylinder containing well boiled water, the flocculating solution added, and the whole made up to 100 c c Each series consisted of about ten determinations with the proportion of flocculant increasing (generally) tenfold at each step. The cylinders were left at 22° for 48 hours, when the volumes of the sediment were finally read, and the upper 75 c c of the supernatant liquid syphoned off for the determination of the matter natant in it. Judging by the appearance of the liquids, flocculation is not quite complete till (generally) one stage beyond that at which the suspended matter is too small for determination. The liquids which were quite clear are marked by asterisks in Table II, an increase in the amount of flocculant beyond this point in every case induces a very slight turbidity

The concentrations (equivalents per litre) at which flocculation is complete are approximately·—

0 0003 N *to* 0 001 N	$\frac{1}{2}H_2SO_4$, HCl, HNO_3.
About 0 01 N	H_2CO_3,[†] $\frac{1}{2}K_2SO_4$, KCl, KNO_3[‡].
0 1 N to N ...	NaOH, KOH, $\frac{1}{2}K_2CO_3$[†]

The results are depicted in figs. 1 and 2, plotted against the logarithm of the concentration, with the zero point inserted at log $\bar{6}$. The volumes of the sediments are represented by the continuous lines, and the matter in suspension by the discontinuous ones, the latter on a scale much more open than the former, so as to render the relationship between the two apparent.

The volume occupied by the whole of the solid particles is the property which ought to be represented, but this cannot be determined, and it is not even possible to separate the suspension completely from the sediment, but, even in extreme cases, the suspended matter can amount to only about

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[*] It would have been preferable to flocculate it previously by carbon dioxide, this would have admitted of its being washed.

[†] The equivalent of carbonic acid has been taken as H_2CO_3 because only one of the atoms of hydrogen exists as acid hydroxyl ('Proc Roy Soc.,' A, vol 93, p 548), the other would be inoperative, but in potassium carbonate the second atom of potassium, being in the alkaline condition, would be operative, hence the equivalent there has been taken as $\frac{1}{2}K_2CO_3$

[‡] Or at 0·0001 N

Table II.—Volume of Sediment, and Percentage of Solid Matter natant, on the Addition of Substances to Kaolin

Equivalents in 1 litre	$\frac{1}{2}H_2SO_4$		HCl		HNO_3.		H_2CO_3		$\frac{1}{2}K_2SO_4$		KCl		KNO_3.		KOH		NaOH		$\frac{1}{2}K_2CO_3$.		Sugar	
	Sed.	Nat.	Sed	Nat.	Sed.	Nat	Sed.	Nat	Sed	Nat.	Sed.	Nat	Sed.	Nat	Sed	Nat	Sed	Nat.	Sed.	Nat.	Sed.	Nat.
0	3·7	4·09	3·7	4·09	3·8	3·84	4·2	3·03	3·7	4·05	3·4	3·86	3·4	3·86	3·8	3·84	3·4	4·36	3·6	4·67	2·7	4·72
0·00001	3·5	4·24	4·3	4·26	4·0	3·90	4·0	2·81	3·6	4·08	3·7	3·17	3·7	3·43	3·5	4·34	2·8	4·83	3·1	4·11	2·7	5·18
0·00003	4·2	3·60	6·1	3·33	5·5	2·95	4·3	2·92	5·0	3·94	3·4	3·65	4·4	3·33	3·4	4·77	2·7	5·00	3·1	4·23	3·0	4·68
0·0001	9·9	0·91	8·3	1·15	9·0	0·78	4·8	1·98	5·8	3·75	5·7	2·37	10·8	0·0	3·3	5·12	2·7	5·20	2·8	4·70	3·2	4·87
0·00038	12·0	0·0	12·7	0·0	12·7	0·0	9·1	0·53	7·5	1·44	6·1	0·69	10·8	0·10	3·6	6·20	2·7	5·82	3·0	5·52	2·5	4·67
0·001	11·6	0·0*	13·0	0·0*	12·8	0·0*	11·7	0·18	7·7	0·31	8·9	0·06	9·7	0·07	3·8	5·46	2·6	6·75	3·0	5·58	2·7	5·15
0·01	9·7	0·0*	11·9	0·0*	12·0	0·0*	12·7	0·0	7·7	0·0	8·2	0·0	9·1	0·0	6·0	3·28	4·1	2·95	4·5	2·33	2·4	6·05
0·02	—	—	—	—	—	—	12·5	0·0*	—	—	—	—	—	—	—	—	—	—	—	—	—	—
0·1	9·3	0·0	10·7	0·0	10·8	0·0	—	—	6·7	0·0*	8·0	0·0	8·8	0·0*	7·0	0·0	7·1	0·0	7·2	0·0	2·2	6·70
1·0	7·3	0·0	8·6	0·0	8·2	0·0	—	—	9·6	0·0	10·0	0·0*	9·9	0·0*	9·3	0·0*	7·7	0·0*	10·2	0·0*	1·5	36·5
1·55	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	0·5	61·5
5	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	11·7	0·0	—	—
10	10·5	0·0	11·2	0·0	8·7	0·0	—	—	—	—	—	—	—	—	7·1	0·0	16·9	0·0	—	—	—	—
20	—	—	—	—	11·2	0·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
30	18·3	0·0	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—

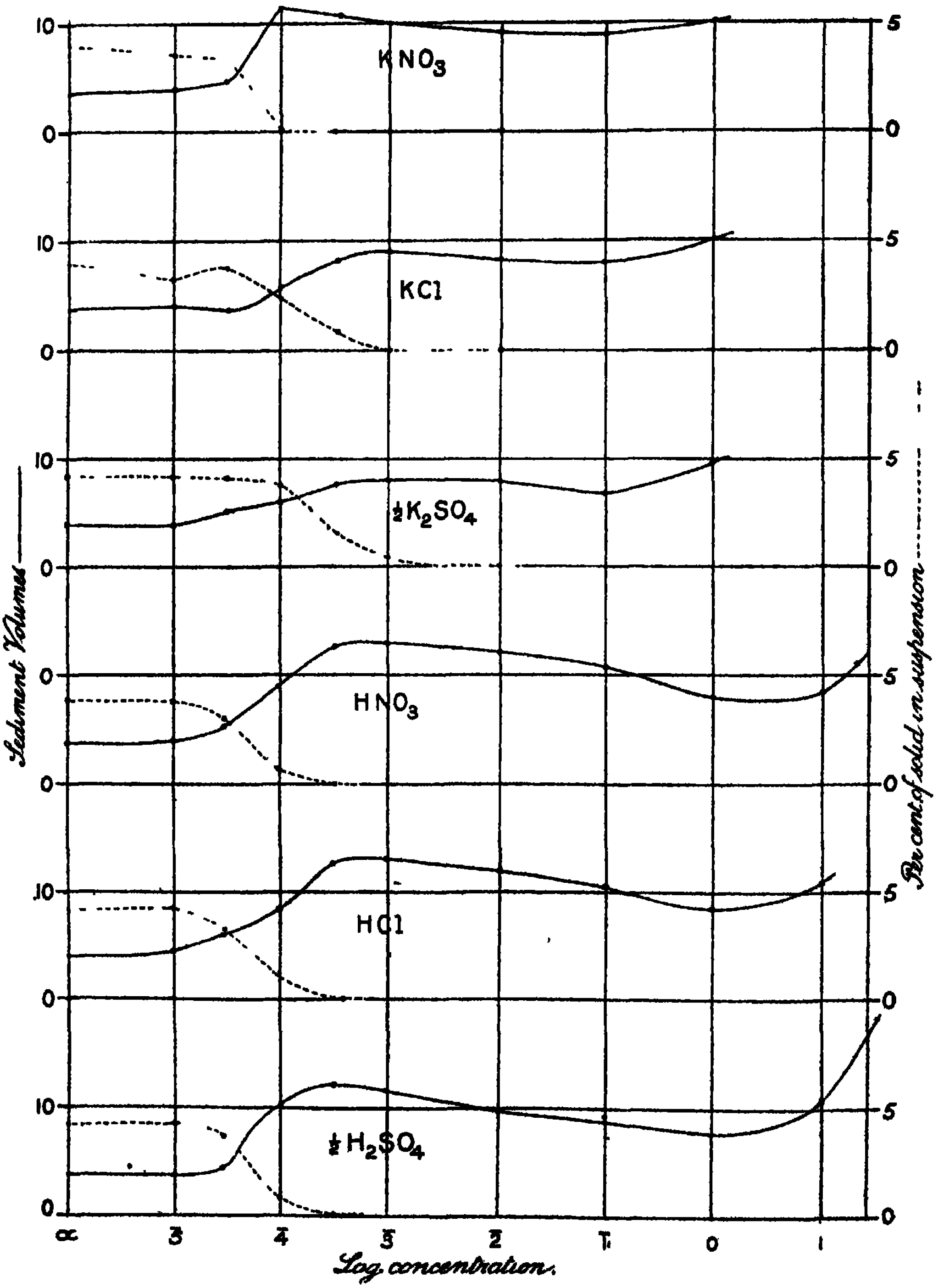
KNO$_3$
KCl
½K$_2$SO$_4$
HNO$_3$
HCl
½H$_2$SO$_4$
10
0
5
Sediment Volumes ——
Per cent of solid in suspension - - - -
∞
3̄
4̄
5̄
2̄
1̄
0
1
Log concentration.

FIG 1

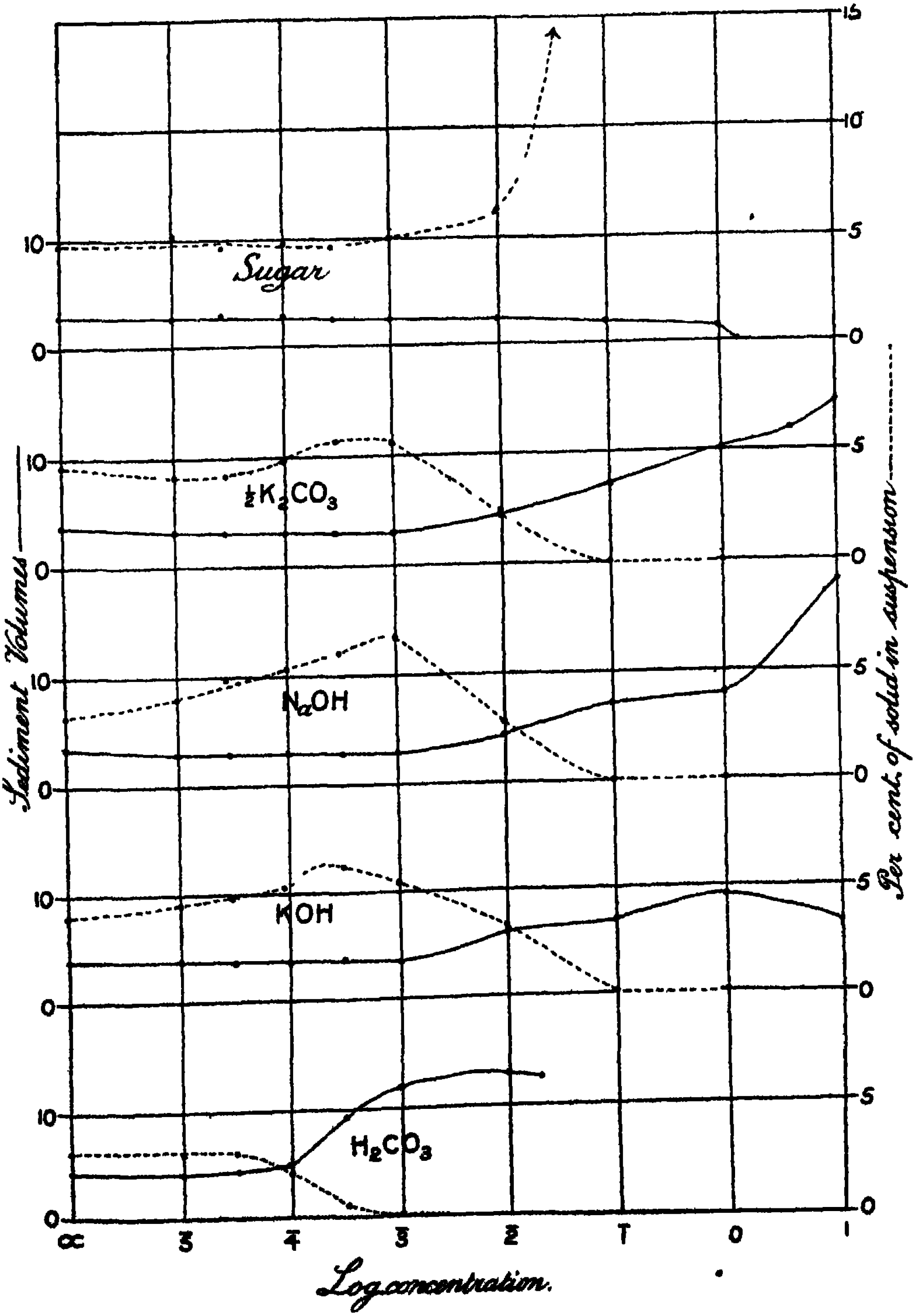
15
10
5
0
5
0
5
0
5
0
5
0
10
0
10
0
10
0
10
0
10
0
Sugar
½K₂CO₃
NaOH
KOH
H₂CO₃
Sediment Volumes
Per cent. of solid in suspension
∞
5̄
4̄
3̄
2̄
1̄
0
1
Log concentration.

FIG. 2.

6 per cent. of the total solid present, and a correction for this, in order to convert the observed sediment volumes into volumes occupied by the total solid, would amount to 0·25 c.c. at most, an almost negligible quantity, especially when the difficulty of reading the volumes of 3 or 4 c.c. at the bottom of a 100-c.c. cylinder is considered.

It is evident that, as regards acids and salts, the increase in the sediment-volumes forms an exact counterpart to the decrease of matter in suspension, the latter becoming nil when the former attains a maximum; where, as in the case of cane sugar (fig. 2), there is no flocculation, there is also no increase in the volume of the sediment. Both phenomena must be attributed to the same cause, and this, apparently, is the combination of the flocculant with the kaolin. An investigation of such combination was possible in the case of acids and alkalis, but only over a limited range; the results were:—

Concentration		Weight of flocculant per 100 gr. kaolin	
		Added.	Absorbed
$\frac{1}{2}H_2SO_4$	0·00033	0·082	0·078 = 95 per cent. of total
	0·001	0·247	0·121 = 49 ,, ,,
	0·01	2·47	0·124 = 5 ,, ,,
HCl	0·00038	0·061	0·060 = 97 ,, ,,
	0·001	0·184	0·075 = 41 ,, ,,
	0·01	1·84	0·074 = 4 ,, ,,
NaOH	0·00033	0·067	0·0
	0·001	0·200	0·009 = 4·5 ,, ,,
	0·01	2·000	0·050 = 2·5 ,, ,,
	0·1	20·000	0·400 = 2·0 ,, ,,

Thus, with the two acids, practically the whole of the flocculant is removed from the liquid up to the point at which flocculation becomes complete, beyond which there is no more removal; a definite quantitative combination must, therefore, occur. The results with soda, which will be discussed later, are confirmatory in proving combination to occur up to the point of flocculation, but, owing to the strength of the solutions concerned, they cannot be extended so as to prove that it becomes complete at this point (0·1 to 1·0 N).

Hall and Morrison* show that the electrical theories of flocculation put forward by Whetham and by Joly cannot be accepted, and, whilst advancing no explanation themselves, reject the idea of combination as a cause, on the ground (pp. 250, 254) that the action is reversible, and that their own conductivity determinations showed that no appreciable fraction of the flocculant was removed from the liquid That reversibility disproves combination cannot be maintained (*cf.* p 317), and, in all the four cases examined by them, a reduction of conductivity was observed, which, moreover, indicated combination of the same order of magnitude as that given by the present direct determinations, namely, 100 molecules of kaolin* to 0 2 $BaCl_2$ (or 0 4 equivalent), and to 0·1 $Al_2(SO_4)_3$ (or 0 6 equivalent). With natrolite the proportions of salt were two or three times as great The present determinations show a combination of 100 $Al_2O_3,2SiO_2$ with 0 28 H_2SO_4 (0 56 equivalent), with 0 46 HCl, and with 2 2 NaOH

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* 'Journ. Agric. Science,' vol. 2, p. 244 (1907).

On flocculation, the Brownian motion of the particles ceases, and, as this motion is but a consequence of the smallness of the particles, flocculation must increase their size No swelling of the particles can be observed under the microscope, hence the increase in size must be due to their combination with invisible liquid matter, a layer of liquid round a solid particle would act as a buffer against molecular bombardment, and hence arrest Brownian motion.

The increased size of the particles is shown by the increased volume occupied by them on subsidence, but this increase is about 200 per cent. in the case of the acids, and 100 per cent in the other cases,† and could not result from the union of such small proportions of the solute with the kaolin, the solute, however, whatever views be taken as to its condition, is certainly united with a large proportion of water, amounting in dilute solutions to several hundred H_2O, as the author's former examination of various properties of solutions shows, and the union of such complex hydrates with the kaolin would quite account for the magnitude of the increase in volume observed

The spheres formed by the solid particles surrounded by liquid molecules would exhibit some surface tension towards the medium, and, as in the case of an imperfect emulsion, these spheres would tend to coalesce into irregular masses, with the solid particles congregated towards the centre Hence the appearance of an attraction between the solid particles

Alterations in the composition of the hydrates as the concentration of the solutions are altered would produce alterations in the sediment volume curves. The results with potassium sulphate show a want of regularity, possibly from this cause, and with potassium nitrate there was a slight recrudescence of turbidity beyond the point at which flocculation had become complete The reaction of the flocculant with traces of impurity in the kaolin would account for the zero point not always being quite in harmony with the volumes for the weakest solutions

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* We have taken Hall and Morrison's weights to refer to that of anhydrous kaolin, the proportion of salt absorbed would be greater if they referred to hydrated kaolin

† A strict comparison between the results with the different substances must not be made, as the various series could not be done at the same time. Carbonic acid, it may be noticed, gives an increase as great as do the other acids, though larger proportions are required to effect it The absence of precautions to eliminate carbon dioxide must throw doubt on some of the previous work on flocculation It is owing to its action that samples of clay, taken from the same spot at different times, are found to vary greatly in suspensibility.

It is clear that the changes suffered by the kaolin do not cease when flocculation is complete, for the volume of the sediment decreases, and subsequently increases, on further addition of the flocculant The decrease is attributable to the dehydrating action of the solute, partly on the hydrates present, partly on the kaolin itself That the latter is modified when the concentration of the solution is great, is evidenced by the fact that in every such case it assumes a grey tinge, and a coarse-grained appearance, this being accompanied by contraction of bulk, as is proved in the case of the otherwise inert cane sugar, where the sediment from a solution of N concentration, when allowed to settle completely, is found to occupy only 80 per cent of that from water This dehydration, like that produced by freezing or other means (p 317) appears to be irreversible, kaolin in a 0 0003N solution of sulphuric acid, on the one hand, and in a 10N solution on the other, were taken, and the acid in both cases was entirely removed by withdrawing three-quarters of the supernatant liquid, and replacing it with water, every 24 hours for 16 days. The sediment volumes had then both attained constancy, but there was a difference of 30 and 40 per cent between them in the two series made *

The final increase in the volume of sediment must be attributed to a cause similar to that producing the first increase, namely, the combination of the solute with the kaolin, but in this case it will be with the partially dehydrated kaolin, and the solute will be present as hydrates of very different constitution from those in dilute solutions The slight turbidity noticeable in these liquids is due to their increased viscidity, as is shown in the case of sugar, where there is a continuous increase in turbidity from 0 001N solutions onwards, till, in a 1·55N solution, as much as 61 per cent of the kaolin remains suspended in the upper 75 c c after 48 hours.

The results with alkalis show that they produce flocculation, just as do acids and salts, but not till the proportions present are a hundred to a thousand times greater. The consequence of this is that the attainment of

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* The volumes represent those of the carbonic acid compounds of the kaolin, for the water used must not be freed from carbon dioxide, otherwise many weeks would have to be allowed for the settlement of the kaolin after each dilution

the maximum sediment-volume at the flocculation point is not always evident, owing to the overlapping of it and the subsequent changes—dehydration and the combination of the dehydrated kaolin This is so with potassium carbonate, partially so with soda, but not with potash, in the last of which cases, it will be noticed, no final rise occurs within the limits of possible experimentation

The alkalis, however, present one feature absent in the case of acids or salts, namely, that the decrease in the amount of matter in suspension is preceded by an increase, forming a hump in the curves, but this cannot be due to a reversal of flocculation, for there is no corresponding depression in the sediment-volumes It is explained by the fact that combination with alkalis requires the presence of these in excess, as the determinations quoted on p 322 show, in a 0 0003N solution, for instance, no measurable amount of combination occurs, yet there must be attraction between the kaolin and the alkali, for they combine when the proportions are altered, and this attraction must result in the kaolin subsiding less rapidly than in pure water, hence the rise in the curve, followed by a fall as the amount of combination becomes appreciable

A true reversal of flocculation would, however, be produced by alkalis in the case of clay flocculated by carbonic acid, as when a clay soil is dressed with lime

On the Nature of Heat, as Directly Deducible from the Postulate of Carnot

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(Received February 1, 1918)

1 The contemplation of the actual working of heat engines, and of their great development in England, with which he was well acquainted, suggested, to the mind of Sadi Carnot, the fundamental principle regulating their operation He postulated that heat can give rise to mechanical work only in the process of carrying through its effort towards an equilibrium [*] This idea involves immediately the whole of isothermal thermodynamics, including the modern thermodynamic potentials of physical chemistry, for it asserts that, in isothermal circumstances, the heat that is present takes no part in the interchanges of mechanically available energy in the material system, and therefore that the available energy is conserved by itself (or in part dissipated if the operation is irreversible) without any reference to the heat-changes which accompany its transformations

An argument—perhaps the most original in physical science, whether as regards simple abstract power or in respect of grasp of essential practical principles—which was based on combining direct and reversed simplified engines operating in parallel, then led Carnot from this general postulate to a quantitative thermodynamic relation, fundamental for all departments of natural knowledge · that all reversible cyclic thermal operations, involving supply and abstraction of heat at the same two temperatures, have equal mechanical efficiency, which is the maximum possible. But he allowed himself, in his demonstration, somewhat reluctantly, and perhaps hastily in order to fix the ideas, to adopt the view then current that heat is substantial, so cannot be annulled or created. This point came right later, without trouble, in the corrected expositions by Clausius and W. Thomson, once a net of misconception, arising partly from confusion between total energy and mechanically available energy, had been cleared away The whole matter ought, however, to be capable of abstract development on broader and more general lines,* and the following statement is now advanced to that end The rough manuscript notes left by Carnot at his death show his own early and very substantial progress towards a more complete doctrine of thermal motive power

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[*] "La production d'une puissance motrice est donc due . . non à une consommation réelle de calorique, *mais à son transport d'un corps chaud à un corps froid*, c'est-à-dire à son rétablissement d'équilibre, équilibre supposé rompu par quelque cause que ce soit, par une action chimique, telle que la combustion, ou par toute autre . . . D'après ce principe, il ne suffit pas, pour donner naissance à la puissance motrice, de produire de la chaleur il faut encore se procurer du froid sans lui la chaleur serait inutile . " ('Réflexions,' ed 1, p 11 (1824), ed. 2, p. 6).

The unlocking of latent heat into sensible form could not be in itself on these principles a lapse towards equilibrium and so a source of motive power, because it is spontaneously reversible, there being no thermal effort in either direction.

2 Suppose that motive power can be gained in a material system, by the carrying out of the effort of some entity, distributed through it, towards an equilibrium. Suppose that the state, as regards fall toward equilibrium, of this entity is determined in each element of mass of the system by only one variable, a potential belonging to it, so that when a path is open it will pass from an element of mass in which this potential is higher to one in which it is lower, also suppose that the effect is in simple proportion to the amount of the entity that enters into this operation, as the idea of a potential implies. Then it can be shown that only two possibilities are logically open The entity may remain unchanged in amount, but may give rise to motive power in subsiding to a lower potential, just as a stream of water does in falling to a lower level, or a gas in expanding towards a uniform pressure The other alternative is that, as regards giving rise to motive power by a reversible cyclic process, and therefore also generally, the scale of measurement of the entity may be so chosen that the entity will be used up to an amount equivalent to the motive power gained, and then, with suitable scale of measurement of the potential, the relation between them is defined by the statement that for all reversible paths between the same terminal configurations the amount of the entity that is added to the system at each potential divided by the potential at which it is added makes the same constant sum Moreover, the first alternative arises in the analysis merely as the limiting form assumed by the second one, when the rate of exchange of the motive power in relation to the entity is indefinitely small.

But an exact general equation can subsist only for an interlacing plexus of possible operations which are not subject to wearing down or exhaustion: the freedom of the system must then be under adequate control, and the test is that its course is capable of reversal in all respects that are involved in doing work The equation represents an optimum; when there is defect in the control, so that the system can slip away, the quantity otherwise always constant must increase

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* In an interesting survey of thermodynamic history, Helmholtz expresses the contrary opinion " . what is still more noteworthy, it is hardly to be supposed that the principle in question could have been deduced from the more correct view, namely, that heat is motion, seeing that we are not yet in a position to establish that view on a completely scientific basis."—'Abhandl,' iii, p. 594, in a review of Lord Kelvin's 'Papers,' from 'Nature,' 1885

Thus, on the basic Carnot idea that motive power arises only from the carrying through of the effort of heat towards equilibrium, and that some of the possible power is dissipated when the operations are not reversible, it follows that there can be *a priori*, only two possible modes of action.

(1) The heat may fall to lower potential unchanged in amount, as in the cases of water-power, gravitational or electric attraction, etc

(2) The heat may be itself consumed in part, in which case the heat can be measured on such a calorimetric scale that there is equivalence between the motive power that is gained and the heat that is lost This alternative proves to be the actual one for heat, which thus ranks, as regards quantity, but not as regards complete freedom of exchange, with the other forms of natural energies, and is not of the nature of a substance

There is no third choice open assuming, of course, that the field of operation partakes of the general characteristic of the order of nature, in that it is rationally explicable, and does not present arbitrary uncorrelated discontinuities.

The fundamental idea of Carnot, in its original form and the most natural one, as above stated in 1824, nearly a century ago, thus involves, in itself alone, either that heat is a substance, or that heat, under the circumstances when it is converted into mechanical energy, always passes in equivalent amount

3. The argument on which these conclusions can be maintained proceeds as follows, any implication as to the nature of heat being at this stage necessarily avoided. If a reversible cyclic engine, exchanging heat with outside bodies at only two temperatures θ_1 and θ_2, were less efficient than some other type of engine, also taking in heat (H_1) at only one temperature θ_1 and giving out heat (H_2) at another θ_2, then the reversible engine, using the work done by this latter engine to operate it in the reverse direction, would restore heat more than H_1 to the source. If, therefore, the reversed engine is operated so as to restore only the same amount of heat H_1 to the source, then in the working of the compound engine constituted of the direct and reversed ones thus coupled in parallel, work will, on the whole, be done, while there would be no abstraction of heat from outside bodies at the higher of the two temperatures, and therefore no fall of heat towards equilibrium of temperature, and this contradicts Carnot's postulate. The work produced would, in fact, have to be done through the mere vanishing of some heat at the lower temperature, if there is any heat-change at all, for no other cause would be assignable, and this process could go on without end. Thus the

negation of the proposition that the efficiency, defined by work done W_{12} divided by heat H_1 received at the higher temperature, is the maximum possible for a reversible engine, as compared with any other engine working between the same two temperatures of supply and rejection of heat, leads to a result deemed to be impossible When this is granted, it follows, after the manner of Carnot, that all such simple reversible engines have the same efficiency, which is a function of the two temperatures θ_1 and θ_2 alone This demonstration seems to be free from any assumption as to the nature of heat

4 Stated in terms of the simplified reversible thermal engine—introduced by Carnot to render the subject amenable to exact reasoning—which takes into its working system heat H_1 at temperature θ_1 alone and rejects H_2 at θ_2 alone, thereby doing work W_{12} available for external use in a cyclic manner, the extended mode of argument, on which the further conclusions stated above are based, may be set out briefly as follows. As just shown, for any such reversible engine,

$$W_{12}/H_1 = F(\theta_1, \theta_2)$$

Also, the same formula must apply to the working of the same engine reversed, for there would otherwise be breach in physical continuity, thus

$$W_{21}/H_2 = F(\theta_2, \theta_1)$$

where

$$W_{12} = -W_{21}$$

Let now, following Carnot, the step of temperature $\theta_1 - \theta_2$ be infinitesimal, say $\delta\theta$ Then we might proceed *tentatively* (in order to exhibit the necessary precautions) to reason as follows

$$\frac{W_{12}}{H_1} = F(\theta_1, \theta_1) + \frac{\partial F(\theta_1, \theta_1)}{\partial \theta_1}(-\delta\theta)$$

$$\frac{W_{21}}{H_2} = F(\theta_2, \theta_2) + \frac{\partial F(\theta_2, \theta_2)}{\partial \theta_2}\delta\theta$$

where in $\frac{\partial F(\theta, \theta)}{\partial \theta}$ the differentiation applies only to the second θ in the functional bracket If we represent the function thus denoted by the reciprocal of $-f(\theta)$, we have, since $F(\theta, \theta)$ must itself vanish,

$$W_{12} = \frac{H_1}{f(\theta_1)}\delta\theta = \frac{H_2}{f(\theta)_2}\delta\theta$$

Thus

$$\frac{H_1}{f(\theta_1)} = \frac{H_2}{f(\theta_2)} = \frac{W_{12}}{\delta\theta}$$

giving for the infinitesimal range with which we are now concerned

$$H_1 - H_2 = W_{12}\frac{f(\theta_1) - f(\theta_2)}{\theta_1 - \theta_2}.$$

If, then, a new scale of temperature Θ is adopted, such that $\partial f(\Theta)/\partial\Theta$ is unity *i e*, if $f(\theta)$ is replaced by Θ, which is W Thomson's absolute scale, we have

$$\frac{H_1}{\Theta_1}=\frac{H_2}{\Theta_2} \quad \text{and} \quad H_1-H_2=W_{12}$$

the only alternative being an exceptional or rather limiting case for which $f(\theta)$ is constant, say, B^{-1}, and then $H_1=H_2$, while $W=BH(\theta_1-\theta_2)$

The argument could now be extended to a finite range of temperature, in Carnot's manner, by coupling engines of infinitesimal range in series, so that the heat from each feeds the next

5 Further scrutiny of this mode of reasoning by way of Carnot's function of a single variable is, however, demanded: for at first sight it would seem to be equally open to us to infer, for the infinitesimal range under consideration,

$$\frac{W_{12}}{H_1}=\frac{\delta\theta}{f(\theta_2)}, \quad \frac{W_{21}}{H_2}=\frac{-\delta\theta}{f(\theta_1)}$$

giving

$$\frac{H_1}{f(\theta_2)}=\frac{H_2}{f(\theta_1)}=\frac{W_{12}}{\delta\theta}$$

and so

$$H_1-H_2=-W_{12}\frac{d}{d\theta}f(\theta)$$

Thus, if we choose the scale of θ so as to ensure equivalence of heat and work $W_{12}=H_1-H_2$, then $-f(\theta)$ becomes Θ and therefore $H_2\Theta_2=H_1\Theta_1$, differing from the previous result

But this procedure is ruled out for the same formula must hold also for infinitesimal ranges shorter than $\theta_1-\theta_2$, starting with θ_1, which shows that the ratio of the work to the heat-supply H_1 must be put equal to a function of the temperature θ_1 of that supply, multiplied by the variable infinitesimal range

6 Indeed, on such an order of ideas as we have just now rejected, this ratio of work to heat ought to be the Carnot function of the mean of the temperatures multiplied by the range, but that would be too narrow a view, as it would necessitate equality of H_1 and H_2, and therefore a substantial nature for heat. In the earlier papers of W. Thomson, he, in fact, does compute the value of Carnot's function at the temperature ½° C in order to apply it to a range of working from 0° C to 1° C, thus here we have probably a main cause of that perplexity which led him even to ignore, provisionally, for a year or two, Joule's principle, in order, as he thought, to save that of Carnot. Clausius was more fortunate his analysis, after Clapeyron, measuring temperature from the first by the expansion of a perfect gas, assumed to

have no internal potential energy, and working concretely with the simple gas cycle, went straight to its goal, and did not encounter at all this arresting type of paradox

7. But even our previous deduction for an infinitesimal range of working requires much further support For the purpose to which it is applied, the expansion of $F(\theta_1, \theta_1 - \delta\theta)$ ought to proceed up to the term involving $\delta\theta^2$, and then, as it stands, the conclusion could not be drawn.

A point of exposition and demonstration, cognate to the present one, arises in a closely related case, that of Hamilton's general dynamical equation of variation of the action, which is sometimes also in this regard faultily developed There also, it is an affair of passing from one state of motion, between assigned terminal configurations, to another that is variationally near it In passing from an actual motion to an adjacent motion, frictionlessly constrained in any manner and so possessing constant energy, and having the same terminal configurations, but without further restriction, the variation of the action, expressed by $\delta\int 2T dt$, always vanishes, so that, provided the path of the dynamical system does not pass beyond the next kinetic focus, the action is minimum But notwithstanding, an analytical equation of variation of the action, in terms of the variations of the terminal configurations alone, and of the energy of the system, subsists only when it is *actual free paths* that are compared Otherwise, *second* differentials of the action, derived from this variational expression regarded as implying that it is a function of initial and final configurations and energy alone, could have no definite values, in fact could not exist The necessity of this distinction, always obvious in special applications of variation of the action, such (*e g.*, Hamilton's own optical rays) as usually form the guide to wider generalisations, is laid down explicity in the general statement in Thomson and Tait's 'Nat Phil,' § 329 Jacobi employs a more algebraic order of ideas in which it is latent

The thermal procedure above, involving adjacent direct and reversed paths differing by reason of an infinitesimal change of temperature, might be reconstituted on a sound basis after this model But it is clearer to construct an argument once for all in terms of finite ranges of temperature.

8 If in the operation of a Carnot reversible engine, absorption from outside of heat H_1 at temperature θ_1 and rejection to outside of H_2 at θ_2 leads to the production of motive power W_{12} which can be transmitted mechanically to external uses, then by the argument developed above

$$W_{12}/H_1 = F(\theta_1, \theta_2). \text{ where } F(\theta_1, \theta_1) = 0$$

If we may postulate as before that this formula must rest on a rational physical basis, then it must apply also to the reversed working of the engine*: thus

$$-W_{12}/H_2 = F(\theta_2, \theta_1).$$

Therefore $$\frac{H_1}{H_2} = -\frac{F(\theta_2, \theta_1)}{F(\theta_1, \theta_2)} = f(\theta_1, \theta_2),$$

here introducing an abbreviated functional symbol $f(\theta_1, \theta_2)$.

But two such engines $\{H_1, H_2\}$ and $\{H_2, H_3\}$ can be coupled to work in series, so that the heat H_2 rejected from the first at θ_2 feeds the second at the same temperature. For this compound reversible engine the formula gives

$$\frac{H_1}{H_3} = -\frac{F(\theta_3, \theta_1)}{F(\theta_1, \theta_3)} = f(\theta_1, \theta_3)$$

Thus we have the functional relation

$$f(\theta_1, \theta_3) = f(\theta_1, \theta_2) f(\theta_2, \theta_3)$$

This expression on the right for $f(\theta_1, \theta_3)$ involves an arbitrary parameter θ_2: moreover, it exhibits that function as the product of a function of θ_1 and a function of θ_3. These features can arise only through a relation of the form

$$f(\theta_1, \theta_2) = g(\theta_1)/g(\theta_2)$$

Thus $$H_1/g(\theta_1) = H_2/g(\theta_2) = H_3/g(\theta_3), = h \text{ say}$$

Moreover $$W_{12} + W_{23} = W_{13}$$

where $$W_{12} = hg(\theta_1) F(\theta_1, \theta_2) = h\phi(\theta_1, \theta_2) \text{ say}$$

so that we have $$\phi(\theta_1, \theta_3) = \phi(\theta_1, \theta_2) + \phi(\theta_2, \theta_3)$$

whatever be the value of θ_2, which requires that

$$\phi(\theta_1, \theta_2) = \psi(\theta_1) - \psi(\theta_2)$$

Thus $$W_{12} = h\phi(\theta_1, \theta_2) = H_1 \frac{\psi(\theta_1) - \psi(\theta_2)}{g(\theta_1)}$$

Therefore finally $$\frac{H_1}{g(\theta_1)} = \frac{H_2}{g(\theta_2)} = \frac{W_{12}}{\psi(\theta_1) - \psi(\theta_2)},$$

so that $$W_{12} = H_1 \frac{\psi(\theta_1)}{g(\theta_1)} - H_2 \frac{\psi(\theta_2)}{g(\theta_2)}$$

9. The scales of measurement of heat and temperature are in these abstract formulæ as yet entirely unspecified. If we now fix the measurement of quantities of heat in terms of a new ideal standard calorimetric substance, whose law of specific heat differs from that of the previous one by the

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* This can, however, be made a matter of deduction: for if a different function of θ_2, θ_1, is assumed to apply to the thermal motor used reversed as a pump for raising heat to higher level of temperature, the same type of argument as in this paragraph will lead to the same final results.

multiplier $\psi(\theta)/g(\theta)$, the ratio of work gained to heat lost in a cycle of finite range will become unity, $\psi(\theta)$ replacing $g(\theta)$ in the final formulæ. This choice of calorimetric substance will therefore establish quantitative equivalence between work gained and heat lost· and there is no reason why this result should remain restricted to reversible processes Finally, we are obviously invited to select the one remaining function $\psi(\theta)$ as the normal measure of the standard temperature Θ, so that now for any finite range of temperature the formulæ for a Carnot engine become

$$\frac{H_1}{\Theta_1}=\frac{H_2}{\Theta_2}=\frac{W_{12}}{\Theta_1-\Theta_2}$$

The one exception to this deduction is its own limiting case, that $g(\theta)$ may be a constant, say B^{-1} Thus $H_1 = H_2$ and $W_{12} = BH\{\psi(\theta_1)-\psi(\theta_2)\}$, so that we would still be invited to select $\psi(\theta)$ as absolute temperature, and the motive power would arise from the fall of the heat, unchanged in amount, through the range of temperature, exactly after the analogy of the fall of a stream of water through a change of level.

10 The whole formal theory of heat engines, and of thermodynamic processes in general, is thus developable in a purely abstract manner from Carnot's own initial idea, that heat can give rise to motive power only in the process of carrying through its effort towards equilibrium Starting from this postulate, it is left to experience merely to decide between two sharply contrasted alternatives, whether (1) heat is virtually a substance doing work merely by falling to a lower level of temperature, or, on the other hand, (2) it is a form of energy, presumably of a fortuitous molecular type, which can be rearranged in part into ordered or mechanical energy by taking advantage of its innate effort to run down towards a dead level of distribution

The kinetic energies of the molecules in a small element of mass will, in fact, mix together far more rapidly towards an equilibrium of distribution in that element than will the energies distributed through a large volume of the substance; thus in the static type of theory to which thermodynamics is restricted, each element of mass is assumed to have already a temperature But the energy existing at this dead level of temperature in the infinitesimal elements can be in part recovered or reconstituted into energy arranged in finite groupings in these elements, thus becoming partly regularised and therefore of mechanical type, when suitable advantage is taken of the further effort towards equalisation of temperature between the elements and throughout the whole mass.

11. It still remains, on this order of ideas, to identify physically the scale

of temperature for which the formulæ become simplified as above, and also to ascertain that the usual calorimetric substances do not differ very substantially, within the usual ranges of temperature, from the ideal one which allows heat to be thus measured as energy

In the ideal perfect gas of Joule and Waterston and Clausius, the molecules exert mutual forces only in the instant of encounter, their range being thus so short that only negligible mechanical work is thereby involved . there is no internal potential energy, and all the energy that such a substance receives passes into the form of kinetic energy of the un-coordinated motions of the molecules, which may be named internal heat The pressure is, as Joule proved, equal to $\frac{2}{3}\tau'$, where τ' is the translatory part of the molecular energy per unit volume this is $\frac{2}{3}k\tau$, where τ is the whole of that energy and k must be constant Therefore, the heat received by a mass of such substance finds a measure in the change of volume produced by it under constant pressure , so that if also temperature is measured by its expansion, this will be the standard calorimetric substance But what is the relation of temperature thus measured to the ideal absolute temperature ? The working cycle for a gas, expounded and scrutinised* in the necessary close detail by Carnot as a typical precise example of his conception of a reversible cyclic process, shows that the two are identical. Thus a gas satisfying approximately the ideal gaseous laws realises, practically, as Clausius was the first formally to disentangle, the normal scales of temperature and of calorimetry, which Carnot's idea by itself had already implicitly contained

The experimental checking of the results, to an extent adequate to ensure confidence, is, of course, a necessary supplement to any abstract argument concerning natural processes , and after that there still remains the concrete dynamical realisation or reconstitution of the subject, so far as possible, in terms of the molecular constitution of matter

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* The development of heat in the gas by compression he has to accept as a fact on the basis of the familiar experiments and of Laplace's correction to the velocity of sound-waves , for by adopting the current view he has debarred himself from the idea of creation of new heat from work. In the algebraic analysis for gas-cycles (ed. 2, footnote, p 40), he is thus led to results confused and in part suspect, and consequent recurring remarks (*cf* p 50) on the doubtfulness of the current theory of heat, which he had adopted' He is thus puzzled by the result of expansion of the gas into a vacuum reservoir, as found by Guy Lussac and Welter, the one vessel gaining as much heat as the other loses , but later, in his posthumous notes (p 91), he attains to nearly complete illumination Thus in the same context ('Notes inédites,' p. 96), we even find a suggestion of the Joule-Thomson experimental method . "Faire sortir de l'air d'un vaste réservoir où il est comprimé, *et rompre la vitesse* dans un large tuyau où se trouvent placés des corps solides, mesurer la température lorsqu'elle est devenue uniforme. Voir si elle est la même que dans le réservoir. Mêmes expériences avec d'autres gaz et avec la vapeur formée sous diverses pressions . . " He points in passing to reduction of temperature with height in the atmosphere as due to the convective equilibrium investigated later by W Thomson (ed. 2, *footnote*, p. 16).

12. If then Carnot's original idea is well founded, that heat can give rise to work only in carrying through its innate effort towards an equilibrium, as water does in its effort towards a lower level, or a gas in its effort to expand towards a uniform pressure, it follows necessarily that heat is measurable as regards quantity in the same terms as motive power The physical alternative, that heat may be virtually a substance, appears in the abstract argument merely as a limiting case, when the ratio of equivalence is indefinitely small and the amount of heat therefore does not sensibly vary

But although the preliminary idea of Carnot, apart from the semi-practical development that accrued to it in his further logical analysis of heat-engines and other thermal processes, was almost co-extensive with the modern doctrine of chemical dynamics, yet in 1824, nearly a century ago, the scientific material did not exist for any systematic search into the ramifications of such a principle of available isothermal energy throughout nature Carnot was duly impressed with the fact that chemical combination in the furnace is an essential part[*] of the phenomenon which he explores and analyses, but he had to take up the position that for the purposes of his discussion it enters only in a preliminary way, as the means of providing the heat with which the engine works, and elsewhere he notes the production of motive power by electric means as beyond the range of his argument.

The principle of available isothermal energy in nature is in fact far wider than its thermal province, though the latter is an inseparable part of it Its development must depend on a reasoned survey of the operations of nature in their wider aspects. Thus it is not so surprising that the principle of Joule should have been recognised in its full scope, as an exact law, only twenty years later. But we find the preliminary notion of conservation and interchange of natural energies already confidently and acutely expanded by Faraday, to whose manifold discoveries in the correlation of the physical agencies it seems, in his own vivid but undefined intuition, to have formed a main guide. No instance of this is historically more instructive than the remonstrance in one of his later researches against the partial interpretation of Volta's original investigations, that the phenomena of the voltaic pile can be founded solely on electric forces of mere contact between different substances. Thus[†]

"2071. The contact theory assumes in fact . . that—without any change in the acting matter or the consumption of any generating force—a current can be produced which shall go on for ever against a constant resistance, or only be stopped as in the voltaic trough by the ruins which its exertion has heaped up in its own course This would indeed be a creation of power and is like no other force in nature We have many processes by which the form of the power may be so changed that an apparent *conversion* of one into another takes place So we can change chemical force into the electric current, or the current into chemical force The beautiful experiments of Seebeck and Peltier show the convertibility of heat and electricity, and others by Oersted and myself show the convertibility of electricity and magnetism But in no case, not even those of the Gymnotus and Torpedo (1790) is there a pure creation of force a production of power without a corresponding exhaustion of something to supply it"

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[*] He recognises that the essential datum is the maximum temperature at which the combustion can be sustained.

[†] 'Exp. Res.,' xvii, January, 1840

"2073 Were it otherwise than it is, and were the contact theory true, then it appears to me, the equality of cause and effect would be denied (2069) Then would the perpetual motion idea be true and it would not be at all difficult, upon the first given case of an electric current by contact alone, to produce an electro-magnetic arrangement which, as to its principle, would go on producing mechanical effects for ever"

In a footnote he quotes from Roget (1827) to the effect that physical effort of any kind appears to be unable to carry itself through without drawing upon and partially exhausting its limited reserves and he might have quoted a footnote of Carnot (*infra*), introducing this very subject of the voltaic pile, had it been known to him

13 In this special controversy about contact forces, which has persisted down to our own time, a sharp discrimination between two distinct ideas, force and energy, is what had been mainly lacking In the eighteenth century a cognate controversy as to whether force should be measured by the *momentum* or by the *vis viva* produced by it, long raged the final stage in appeasing it was the introduction of the necessary new term *energy* by Young* to represent the accumulated *vis viva* of a force, which Leibniz and his school, and many others including the engineer Smeaton, had insisted on taking as its measure, instead of the Newtonian *momentum*. The expansion, rather than correction, of ideas that was thus involved—the distinction between force and energy, between effort and its consummation—may be compared with another classical instance of clarification, W. Thomson's recognition of available energy as distinct from total energy, of which the germ was latent, for development in the fullness of time, in Carnot's term motive power.

Two years after these remarks of Faraday, the idea of energy as conserved and interchangeable in natural processes was grasped firmly and developed in many subtle aspects by the physiologist J R Mayer; while Joule was already engaged in the experiments that confirmed his own precise and practical outlook, and constituted it the fundamental generalisation of physical science Finally, the doctrine was crystallised by Helmholtz in his famous essay of 1847, building on the ideas and experiments of Joule, as a quantitative guide through the correlations of natural agencies,—stimulated thereto in the main, as was also Young, by study of the mathematical physicists of the previous century, and indeed without much immediate local recognition except from the mathematician Jacobi But even this formulation of the conservation and interchange of potential motive power in nature is incomplete until the condition implied in Carnot's fundamental idea is added to it, that the operations must take place at uniform temperature, or else be subject to the other limitations of thermodynamics

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* 'Lectures,' vol. i, p 78 (1807).

But the ideas of Carnot on this subject are as definite as those of Helmholtz or of Thomson On the theoretical abstract side nothing has been added, except by way of further exemplification, to his reasoned footnote on the principle of the *perpetual motion** The principle has, he states, been demonstrated only for mechanical actions "Mais peut-on concevoir les phénomènes de la chaleur et do l'électricité commes dues à autre chose qu'à des mouvements quelconques de corps, et commes tels ne doivent-ils pas être soumis aux lois générales de la mécanique?" And he goes on to assert, as an example, the inevitable exhaustion of the power of the pile of Volta, owing to the work that it performs In the text to which this footnote refers the implication is that heat as well as electricity is a substance, and that the phenomena arise from its disturbance. Thus " . . tout rétablissement d'équilibre [dans le calorique] qui se fera sans production de cette [maximum] puissance devra être considéré comme une véritable perte, . " The principle of the thermal dissipation of available energies, in extension of the simpler principle of the complete availability of isothermal energies, is here almost in sight. His thoughts recur to the latter subject, now however on a definite mechanical theory of the nature of heat, but still in a tentative way, in the posthumous fragments (p. 92) published in 1878 by his brother.

The master thought which presides over all this development of physical science was enunciated by Sadi Carnot in 1824, in a solitary essay nearly contemporary with the chief work of his great countrymen Ampère and Fresnel. As a chapter in scientific method, it seems desirable even now to bring the full individual potentiality of this creative idea into view.

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* 'Réflexions,' ed 1, p. 20; ed. 2, p 12.

[*Added February* 26.—The main thesis developed above is that a close analysis of the postulate of Carnot evolves from it not one but two equations. These may be interpreted, after suitable simplification of scales of measurement of heat and temperature, as asserting that when there is no waste of power, two quantities are conserved, viz., heat *plus* other forms of energy, and entropy when however irreversible features are present, the latter must increase while the constancy of the former need not be disturbed. Yet, as Helmholtz remarked, the distinction between the two constituents of this constant sum, the heat in a body and its other forms of internal energy, has hardly even now become analytically precise

My attention has now been recalled by Sir Alfred Ewing to a very interesting and in certain respects cognate discussion by Prof H L. Callendar,* which turns on the idea that the caloric, whose conservation had been *assumed* by Carnot, is capable of being interpreted as entropy, so far as reversible processes are concerned He there recognises, as above, that the efficiency principle can be established without any assumption as to the nature of heat Then the basic postulate would take the form that motive power can be gained only in carrying through the effort of this caloric towards its equilibrium distribution, and would arise from part of the energy, of its disturbance from equilibrium, being diverted into mechanical work The motive power thus gained would be equal to the caloric that is transferred, multiplied by its fall in potential as measured by the temperature associated with it. but there would be no necessary absolute zero The entity heat would, on this train of ideas, enter as mere residual energy, of amount required to satisfy Joule's experimental law of conservation But wherever the energy of disturbance of the caloric is transformed in a wasteful (because irreversible) manner, there would have to be creation of new caloric, of amount equal to work wasted, divided by temperature, thus implying an absolute zero, it would be imaginable, perhaps, as a sort of degenerate residue from lost motive power The reversible disappearance of free electricity, or even free heat, by becoming latent, is hardly an analogy in point.

This great fluidity of ideas as to specification of heat, or rather of caloric, has its source in the necessity of postulating a provisional calorimetric substance, after the manner of Black, the unit of heat being proportional to its thermal capacity which may be any function of temperature: thus heat will be an equivalent of vanished energy only when it is measured on the proper scale; while even entropy would assume the *rôle* of caloric provided the thermal capacity of the calorimeter were taken proportional to absolute temperature

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* Presidential Address to the Physical Society, February 10, 1911.

Reference may also here be made to Clausius' long-sustained attempt (which received some favour from Willard Gibbs in his Obituary Notice of Clausius, *Collected Papers*, ii, p. 263) to break up the Carnot cycle into two physically distinct parts, the direct transformation of heat Q′ into work at temperature θ', and a compensating transference of heat Q from temperature θ_1 to a lower temperature θ_2.]

The Absorption of the Radiation emitted by a Palladium Anticathode in Rhodium, Palladium and Silver.

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(Communicated by Prof. A. W. Porter, F.R.S. Received December 28, 1917.)

[Plate 3.]

Introduction.

The analysis of a beam of rays from an X-ray bulb by reflection at a crystal face shows that the bulb emits a continuous spectrum of radiation upon which are superposed a few wave-lengths which stand out prominently from the rest, these being the characteristic wave-lengths of the metal of the anticathode. Much note has not been taken hitherto of the general radiation, more attention having been paid to that part of the spectrum in the immediate vicinity of the characteristic rays. In the present work an attempt is made to obtain fuller knowledge of the general radiation by studying the absorption coefficients of the rays in the three elements rhodium, palladium, and silver. In the particular bulb chosen, palladium is used as the metal of the anticathode.

Apparatus.

The apparatus used in the investigation is the Bragg X-ray spectrometer, a full description of which may be found in Bragg's book on "X-ray and Crystal Structure," p. 22 *et seq.* A beam of rays, leaving the anticathode at a grazing angle, passes through two slits before falling on the crystal. The slit nearer to the crystal is movable and can be brought up very close to the crystal. After reflection, the beam passes through a third slit on its way to the ionisation chamber. We shall allude to these three slits as the bulb slit, crystal slit, and chamber slit respectively.

It is essential in an investigation of this nature to use a crystal which gives intense reflections as well as good resolution. It is not always possible to obtain these two requirements in the same crystal. Rock salt, for instance, gives fairly intense reflections, but the definition as a rule is poor; a maximum, instead of being confined to a few minutes of arc, is spread over comparatively big angles. The reason for this is that the crystal is not an accurately built one. It is made up of a number of small crystals that are slightly out of alignment, and when the crystal is rotated the reflection takes place successively at the faces of the smaller crystals. Another crystal may, on the other hand, give good definition, but the atoms of which it is composed are not such as to give intense spectra, because the intensity of the spectra depends, amongst other factors, upon the absorption coefficient of the rays in the crystal, the distribution of weight among the reflecting planes, and the thermal movement of the atoms, each of these factors depending upon the atoms composing the crystal.

The diamond is a crystal which would meet the above two requirements satisfactorily, but as a crystal of this substance was not available a search had to be made for another which would serve the purpose.

Carborundum* proved satisfactory. The reflections from the (111) face of this crystal were good both as regards definition and intensity. The structure of the crystal has not been solved, but a Laue photograph taken by passing the rays through it in a direction perpendicular to the (111) face, *i.e.*, along the hexagonal axis, points to a simple structure. The photograph is given in Plate 3.

The Spectrum of Palladium

In order to map the spectrum of the rays emitted by the bulb, the first step is to determine accurately the position of the crystal and the ionisation chamber when a known wave-length is reflected into the chamber. The intense α-line was chosen for this purpose.

The carborundum crystal is set up on the revolving table of the spectrometer and the crystal slit brought as near as possible to it. The crystal slit is narrowed to 0·4 mm. and the chamber slit left wide open. In a rough preliminary examination the position of the ionisation chamber when the characteristic radiation is reflected into it is approximately found. The chamber is kept stationary in this position whilst the crystal is rotated,

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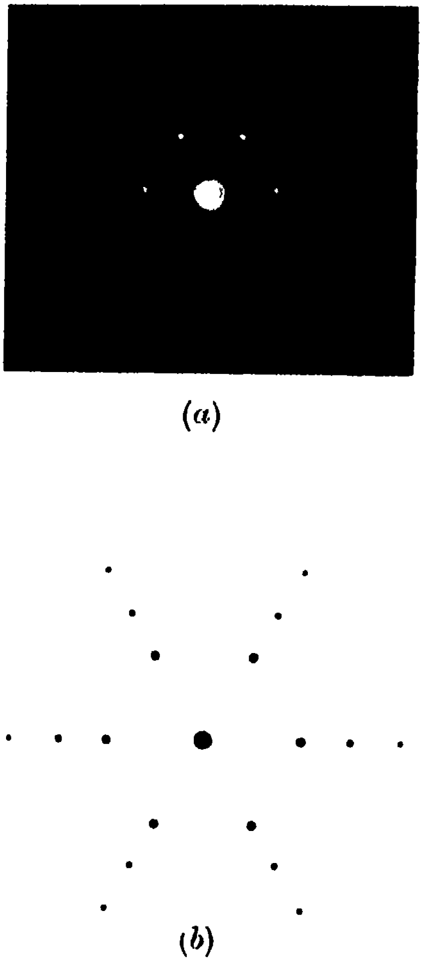
* The writer is greatly indebted to the Carborundum Company, Trafford Park, Manchester, for supplying him with excellent specimens of very nearly pure crystals of this substance. He also desires to express his sincere thanks to Mr. G. G. Blake, Richmond, in whose laboratory the Laue photograph of the crystal was taken, for supplying the necessary apparatus, and for giving liberally of his time to assist with this part of the work.



(*a*) Laue photograph of a crystal of Carborundum Rays passed through the crystal in a direction at right angles to the (111) face

(*b*) Diagram showing in detail the spots observed in the photograph

observations of the ionisations produced being taken every two minutes of arc of rotation of the crystal, and every minute of arc in the immediate neighbourhood of the maximum From these observations the exact position of the crystal when it reflects the characteristic line is determined.

The crystal is fixed in the position just found, and the chamber slit narrowed to 0 2 mm The chamber is now rotated in the same way as the crystal was rotated previously The sharpness of the definition of the reflected beam was very marked, and the position of the chamber when the maximum ionisation was produced could be determined very accurately, so accurately indeed that it could be used to measure the glancing angle of the rays on the crystal after applying the necessary zero correction of the spectrometer scale

Having now fixed the positions of the crystal and the chamber for one known wave-length, the spectrum can be mapped throughout its entire range Starting with the crystal and chamber in the positions just determined, the chamber is rotated at double the rate of the crystal, and readings taken every two minutes of arc of rotation of the crystal The spectrum was examined in this way from a point just beyond the longest characteristic ray to the shortest ray emitted by the bulb The form of the spectrum turned out very approximately the same from each determination, showing that the bulb was working steadily An example of a determination is plotted in fig 1

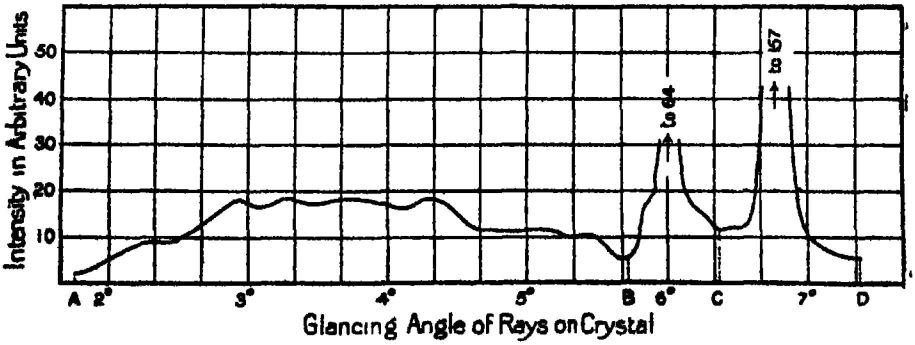


Fig. 1 —Spectrum of the rays from a palladium anticathode

The general scattering of the rays by the crystal was inappreciable. No deflection of the leaf of the electroscope was observed during an exposure of 4 seconds—this was the exposure in each position when the spectrum was mapped—when the chamber was set at an angle different from double the glancing angle at which the rays fell upon the crystal. It was important to investigate this, in order to be certain that the effect observed when the chamber was in its correct position relative to the crystal was due to regular reflection of the rays at the crystal face, and not to a general scattering of the rays by the crystal.

Information Obtained from the Spectrum Curve

The relation between the wave-length and the glancing angle is given by the equation $n\lambda = 2d \sin\theta$, where n is the order of the spectrum and d the distance between the reflecting planes. If λ_1 and λ_2 are two wave-lengths reflected at angles of incidence θ_1 and θ_2 respectively, then, for the same order spectrum,

$$\frac{\lambda_1}{\lambda_2} = \frac{\sin\theta_1}{\sin\theta_2}$$

If the glancing angle corresponding to a given wave-length in the spectrum is known, then the wave-length corresponding to any other glancing angle can be calculated. The mean value of the α-line of palladium obtained by Bragg by reflection in calcite and diamond is 0.586×10^{-8} cm., and, in the case of carborundum, this is reflected at an angle of 6° 45′. The wave-length of the β-line of palladium, calculated from the present results, is 0.520×10^{-8} cm., and that of the γ-line 0.509×10^{-8} cm. The accuracy of the last figure is not as high as that of the β-line. These values are in close agreement with the values previously obtained by Bragg.*

The energy in the spectrum is localised in the characteristic wave-lengths of the metal of the anticathode. Integrating the spectrum curve as plotted in fig. 1, it is found that the ratio of the intensity of the wave-lengths between the points A and B to that of the wave-lengths between the points B and D is 1.12, so that, in this particular case, there is almost as much energy in the characteristic rays as in the whole of the rest of the spectrum investigated. The ratio of the intensity of the radiation between B and C and that between C and D is 1.90. This gives approximately the ratio of the intensity of the α-lines to that of the β- and γ-lines. This integration was also carried out experimentally by rotating the crystal at a definite rate through each of the maxima, and noting the total amount of ionisation produced in each case. The mean value obtained for the ratio by this method was 1.93, which agrees very closely with the value just obtained by the direct integration of the spectrum curve.

The spectrum shows a minimum of intensity at the glancing angle 5° 41′, corresponding to a wave-length 0.493×10^{-8} cm. This suggests selective absorption of this wave-length by the crystal, which would be the case if the characteristic radiation of the J series of one of the atoms composing the crystal were excited. From observations on other crystals, and in view of the results that are to follow, we would assign this wave-length, namely, $0·493 \times 10^{-8}$ cm., to the β-line of the J series of silicon, assuming that the J-radiations are of constitution similar to the K-radiations. The β-lines of the J-radiations of oxygen and carbon, calculated on the basis of the relative values of the wave-lengths of aluminium, oxygen, and carbon, as experimentally determined by Barkla and White,* work out to be about $0·519 \times 10^{-8}$ cm. and $0·550 \times 10^{-8}$ cm. respectively. These figures are in fair agreement with those deduced by Barkla by the ionisation method, but are somewhat higher than the values obtained by him by the absorption method.

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* W. H. Bragg, 'Phil. Mag.,' vol. 29, March, 1915.

Another feature of the spectrum is the rather prominent maximum at about 4° 20′, corresponding to a wave-length of about $0·38 \times 10^{-8}$ cm. It is hoped later to investigate this region of the spectrum more in detail than has been done up to the present.

The shortest wave-length emitted by the bulb under the conditions of working in the experiment was about $0·15 \times 10^{-8}$ cm. The spectrum shows that, from this wave-length to that of the longest characteristic wave-length of the anticathode, there exists a continuous band of wave-lengths, and in this particular case the relative intensities remained approximately constant.

It is, of course, to be expected that the distribution of energy in the spectrum would vary greatly, according to the state of the bulb, and it would be of interest to find what effect the alteration of voltage and current would have on the form of the spectrum curve.

The Intensities of the Different Order Spectra of the Characteristic Lines of Palladium in Carborundum.

It will be necessary to know how the intensities of the spectra decline as we proceed to higher orders, in order to account for some apparent irregularities observed in the absorption of rays from different parts of the spectrum. These were measured in the case of the two peaks, giving the α_1, α_2 and β, γ lines respectively, and in each case the rate of decline was the same. Different widths of slits were used, but it was found that this had but a slight effect upon the intensities. The relative intensities of the orders were as follows:—

100·0, 37·9, 18·0, 10·6, 2·8, 0·6.

These are plotted in fig. 2, and it will be seen that they fall along a smooth curve, except for the fourth order, which is too large. The planes parallel to the (111) face of carborundum are therefore not all similar, such as, for example, the (100) planes of rock salt, and the rate of fall of intensity differs from that of the normal fall of intensity as generally assumed.

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* Barkla and White, 'Phil. Mag.,' vol. 34, October, 1917.

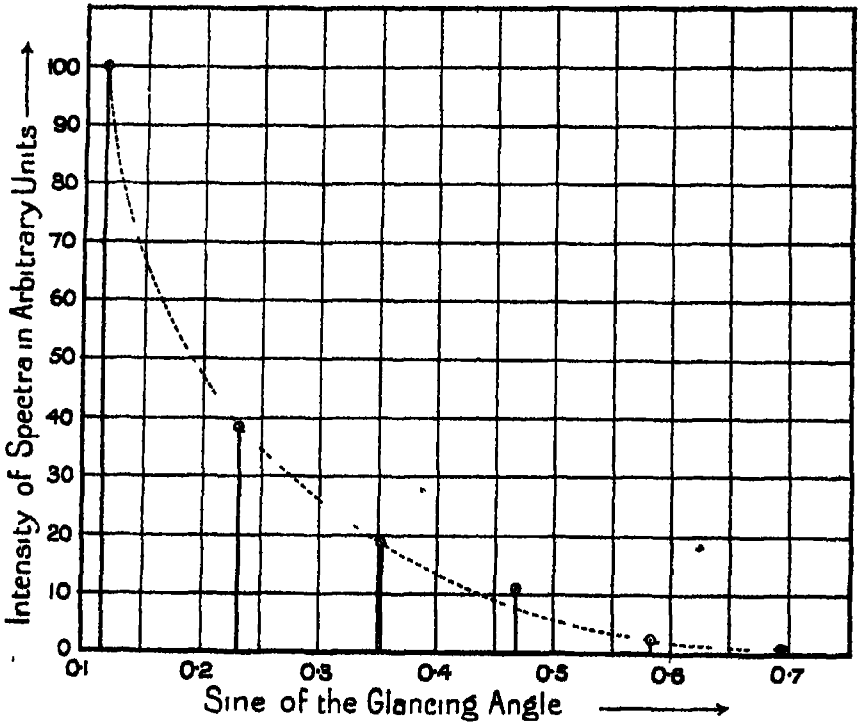

FIG. 2.

We shall assume in the calculation that follows that the different order spectra of all the wave-lengths in the spectrum follow the same rate of decline as do those of the α and β peaks

Measurements of Absorption Coefficients

The absorption of the rays in the spectrum was measured in the three metals rhodium, palladium, and silver For this part of the work the width of the chamber slit was 1 mm, the bulb and crystal slits remaining the same as previously, namely, 2 0 mm. and 0·4 mm. respectively. No appreciable difference was observed in the amount of ionisation produced in the chamber for a given exposure when the absorbing screen was placed near the crystal and when it was placed near the chamber slit. It was finally fixed at about 2 cm in front of the chamber slit The ionisations produced in the chamber, with and without the screen in front of it, were measured, and the value of the absorption coefficient determined by means of the equation $I = I_0e^{-\mu x}$, where I_0 is the initial intensity, and I the final intensity after traversing a screen of thickness x cm The accuracy of the method of measurement employed depends upon the constancy of the bulb, but, by measuring the ionisation without the screen before and after that with the screen, and multiplying observations, fairly accurate mean values were obtained for the absorption coefficients

The present investigation is to be regarded as a preliminary one, and it is hoped with the new apparatus now under construction to employ a method which will be independent of the output of the bulb both as regards wave-length and intensity

The values of the absorption coefficients here determined refer to the total absorption of the radiation in the absorbing screen This absorption is made up of the scattering absorption and the fluorescent absorption, the latter being itself constituted of the absorption of the various fluorescent X-radiations, J, K, L, , excited in the absorbing screen, together with the corpuscular radiations which accompany each of these fluorescent radiations We have then $\mu/\rho = \sigma/\rho + \tau/\rho$, where μ/ρ is the total absorption coefficient, σ/ρ the scattering coefficient, and τ/ρ the fluorescent coefficient Barkla finds that the scattering of X-rays of very short wave-length by equal masses of various substances varies only little with the atomic weight of the scattering element There is a slight increase of the mass scattering coefficient with an increase in the atomic weight of the scatterer, but this increase is more marked with X-radiation of greater wave-length, until for long waves it is very considerable As we are dealing here with short waves we may assume without introducing serious error that the mass scattering coefficient is constant for the absorbing substances we are using, and also that it is independent of the wave-length over the range of wave-lengths that we are investigating. The value found by Barkla for σ/ρ, when the wave-length of the radiation is short, is 0 2 This is small compared with the values of the total absorption coefficient μ/ρ found for the elements rhodium, palladium, and silver, with the wave-lengths here employed It does not exceed 4 per cent of the smallest value of μ/ρ determined, so that its effect is almost within the limits of experimental error. We shall, however, apply this correction for the scattering to bring the results into line with those already published by Barkla and White* for the light elements, in whose case the scattering plays a prominent part.

In fig. 3, the logarithm of $(\mu/\rho - 0{\cdot}2)$, *i.e.*, the logarithm of τ/ρ, is plotted against the logarithm of the corresponding wave-length in the case when the absorbing screen is rhodium. The curves obtained with palladium and silver are similar to that of rhodium. The three curves are very near

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* Barkla and White, *loc. cit.*

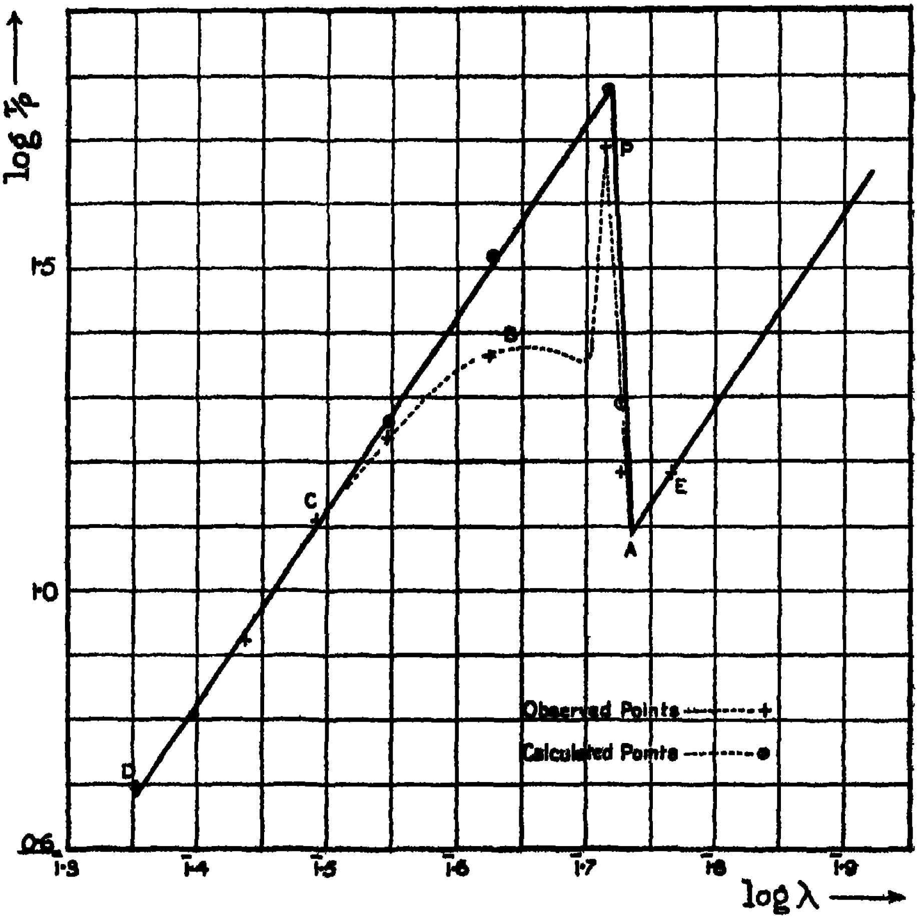

Fig. 3.

together, and in order to avoid confusion only the rhodium curve is included in the diagram.

The graph shows that the absorption of the rays is normal until we reach the point A, where, as we proceed to shorter wave-lengths, the absorption coefficient increases rapidly and follows approximately the curve ABC, except for a singular point P, which is very far removed from this curve. The reason for this apparent irregularity will shortly be explained. Beyond the point C the absorption curve follows approximately a straight line course. The radiations corresponding to this latter part of the curve constitute the "end radiations" of the bulb.

To explain the existence of the curved portion ABC of the absorption curve, we have to refer to the spectrum of the rays from the bulb mapped out in fig 1 It will be seen that the radiations reflected at angles greater than 4° have superposed upon them radiations of shorter wave-length which constitute the second order spectra of the shortest waves emitted by the bulb It is therefore to be expected that the mean wave-lengths of the rays corresponding to the curved portion of the absorption curve will be shorter than those calculated from the angle of incidence on the assumption that the rays are all of the first order spectrum Consequently the absorption coefficient of the rays in the absorbing screen will be too small The effect of the superposed second order spectra will be considerable for radiations reflected from the crystal face at angles in the neighbourhood of 5°, because the intensity of the second order spectra forms a fairly high percentage of the total intensity of the radiation in this portion of the spectrum On the other hand, the effect of the superposition of the second order spectra on the wave-lengths of the characteristic lines will be small, because the intensity of these lines is so great compared with that of the radiations superposed upon them. This is the reason why the point P in the absorption curve, which represents the absorption of the intense β-line, is so far removed from the curve ABC It will be seen from the curve that the correction applied to the observed value of the absorption coefficient of the β-waves to obtain the correct value is quite appreciable In the case of the α-wave, however, which is represented by the point E, the correction is negligible

The exact shape of the observed absorption curve depends upon the form of the spectrum emitted by the bulb, and it should be pointed out that the dotted portion ABC represents only very roughly the conditions obtaining in the particular case in question

The above considerations show that the homogeneity, or, in other words, the purity, of the characteristic lines emitted by a bulb and isolated by reflection at a crystal face will depend, to a certain extent, upon the state of working of the bulb.

Calculation of the True Absorption Coefficient of each Wave-length in the Spectrum.

Assuming that the longer wave-lengths in the spectrum are modified by the superposed wave-lengths of the second order spectra of the shorter wave-lengths, we can easily calculate the true absorption coefficient of each wave-length in the spectrum.

Let ${}_2I_0$ be the intensity of the shorter wave superposed upon the longer wave whose intensity is ${}_1I_0$, so that the total intensity is $({}_1I_0 + {}_2I_0)$. If I_1 be the intensity of the longer wave after transmission through an absorbing screen of thickness x cm, we have

$$I_1 = {}_1I_0 e^{-\mu_1 x},$$

where μ_1 is the true total absorption coefficient of the longer wave in the screen.

Similarly for the shorter wave-length

$$I_2 = {}_2I_0 e^{-\mu_2 x},$$

where μ_2 is the true total absorption coefficient of the shorter wave in the screen.

The value of the total absorption coefficient μ, as measured in the present work, is given by the relation

$$I_1 + I_2 = ({}_1I_0 + {}_2I_0) e^{-\mu x},$$

i.e.

$${}_1I_0 e^{-\mu_1 x} + {}_2I_0 e^{-\mu_2 x} = ({}_1I_0 + {}_2I_0) e^{-\mu x},$$

so that

$$e^{-\mu_1 x} = \frac{({}_1I_0 + {}_2I_0) e^{-\mu x} - {}_2I_0 e^{-\mu_2 x}}{{}_1I_0} \qquad (1)$$

which gives the value of μ_1, the true total absorption coefficient of the longer wave without the superposition of the shorter component.

In the present case it suffices to consider only the superposition of the second order spectra, since we are only investigating that portion of the spectrum from the shortest wave-lengths to those of the characteristic lines. This forms but a small part of the whole spectrum, and for longer waves we would have to consider the superposition of spectra of higher order than the second. The expression for the true total absorption coefficient in the general case when several spectra are superposed upon the wave-length considered would be

$$e^{-\mu_1 x} = \frac{e^{-\mu x} \Sigma_1 I_0 - ({}_2I_0 e^{-\mu_2 x} + {}_3I_0 e^{-\mu_3 x} \ldots)}{{}_1I_0} \qquad (2)$$

This expression holds for any wave-length in the spectrum. To evaluate the expression on the right-hand side of equation (1), we require to know the intensity of the shorter wave-length, that of the longer wave-length, and the true total absorption coefficient μ_2 of the shorter wave-length; μ is known, being the quantity measured in the experiment.

An examination of the end radiations of the bulb showed that the rays reflected at definite angles in this part of the spectrum were homogeneous as far as could be determined by absorption in sheets of silver. The short waves superposed upon the longer waves in the region ABC of the absorption curve will be the second order spectra of these homogeneous radiations, and since the rate of decline of intensities of the various order spectra is known from fig 2, which is assumed to be the same for each wave-length in the spectrum, the initial intensity ${}_2I_0$ of the shorter wave can be calculated. Also the true total absorption coefficient μ_2 of this radiation is directly determined in the experiment

The form of the spectrum mapped out on different occasions showed the same characteristics, and the relative intensities of the different wave-lengths remained approximately constant The intensity ${}_1I_0$ of the longer wave-length may therefore be obtained by subtracting the calculated intensity of the shorter wave-length from the observed total intensity

We have therefore the data required to calculate the value of the true total absorption of the pure radiation corresponding to any part of the spectrum Carrying out this calculation for the points observed in the region ABC, the calculated points fall very approximately along the continuation of the straight line DC up to the wave-length corresponding to the point P Afterwards there is a rapid fall, until that point is reached at which the characteristic radiation is excited, when the absorption coefficient begins to increase again with increase of wave-length of the exciting radiation

Table I gives the true fluorescent coefficient τ/ρ in rhodium, palladium, and silver for a number of wave-lengths The figures give the observed values corrected for scattering and for the general radiation emitted by the bulb

Table I

Absorber					
Rhodium		Palladium		Silver	
$\lambda \times 10^8$	τ/ρ	$\lambda \times 10^8$	τ/ρ	$\lambda \times 10^8$	τ/ρ
0 226	4 92	0 226	5 18	0·226	5·58
0 249	6 44	0 268	8 61	0·268	9 29
0 278	8 51	0 312	13 8	0 312	14 5
0·312	12·9	0 363	21 5	0·363	22 8
0 353	18·4	0 396	27 4	0 396	29·6
0·424	32 5	0·465	45 0	0·465	47·8
0·520	56·2	0 511	14 8	0 499	15·9
0·537	19·3	0 520	11·2	0·520	12 15
0·586	15·1	0 586	15·9	0·586	17·8

The slope of the straight line DC gives the index three very approximately, so that the relation between wave-length and the absorption coefficient becomes

$$\tau/\rho = K\lambda^3$$

where the value of K varies for different substances, but is a constant for a given substance over the range of wave-lengths between the absorption bands of that substance. This confirms the relation already obtained by Hull and Rice[*] for the absorption of the end-radiations of a Coolidge tube in aluminium, copper, and lead, and that obtained by Barkla and White[†] in the case of the light elements.

The Wave-length Necessary to Excite the Characteristic Radiation of a Substance.

The point at which the characteristic radiation of the absorber is excited is very definitely marked. The absorption curves show that the characteristic radiation of the absorber is not excited by a radiation of a wave-length equal to that of the longest characteristic wave of the substance. It requires a shorter wave-length to do this. When the wave-length which excites the characteristic radiation is reached, both the α- and the β-lines are emitted together; it does not seem possible to excite the α-radiation without exciting the β-radiation. The approximate values deduced from the absorption curves for the wave-length necessary to excite the characteristic radiations of rhodium, palladium, and silver are given in Table II.

Table II.

Absorber	Longest wave-length necessary to excite the characteristic rays
Rh	0·545 × 10^{-8} cm
Pd	0·508
Ag	0·495

In each case the critical wave-length lies in the neighbourhood of the β-line of the spectrum, which points to the β-line as being the critical wave-length necessary to excite the characteristic radiation of a substance.

Ledoux-Lebard and Dauvillier,[‡] investigating this point by another method, have arrived at the same conclusion. They examined the K-radiations of tungsten, which consist of four lines similar to those in the K series of palladium. Applying different voltages to the tube, they found that a voltage sufficiently high to excite the β-ray had to be reached before the α-characteristic lines were emitted.

Bragg[§] suggests that the characteristic rays of a substance might form a system which can only be excited as a whole. To decide whether it is necessary to excite the γ-ray before the other rays are excited, more observations will have to be taken of the absorption coefficients in the neighbourhood of the critical point This is at present in progress.

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[*] Hull and Rice, 'Phys. Rev.,' vol. 8, 1916, p. 326.

[†] Barkla and White, *loc. cit.*

[‡] Ledoux-Lebard and Dauvillier, 'Comptes Rendus,' December, 1916.

[§] W. H. Bragg, 'Phil. Mag.,' vol. 29, March, 1915.

Summary.

(1) A short account is given of some preliminary experiments carried out with the rays from an ordinary X-ray bulb.

(2) A spectrum of the rays from a palladium anticathode is obtained over a limited range of wave-lengths by reflection in the (111) face of a carborundum crystal The spectrum shows that the bulb emits a continuous band of wave-lengths upon which are superposed the characteristic rays of the metal of the anticathode, and, under the conditions of working in this particular case, the relative intensities of the different wave-lengths in the spectrum remained approximately constant

(3) There is a minimum of intensity in the spectrum corresponding to the wave-length $0\,493 \times 10^{-8}$ cm On the assumption that the minimum is due to the selective absorption of this wave in the crystal, the value $0{\cdot}493 \times 10^{-8}$ cm is assigned to the β-line of the J-series of silicon From the experimental results of Barkla and White on the J-series of the elements Al, C, and O, the approximate values deduced for the β-line of the J-series of oxygen and carbon are $0\,519 \times 10^{-8}$ cm and $0\,559 \times 10^{-8}$ cm respectively

(4) Assuming Bragg's mean value of the α-line of palladium to be $0\,586 \times 10^{-8}$ cm. the following values are obtained for the wave lengths of the β- and γ-lines ·

$$\beta = 0\,520 \times 10^{-8}, \qquad \gamma = 0\,509 \times 10^{-8}\,\text{cm}$$

(5) The absorption coefficients of the rays from the bulb have been measured in rhodium, palladium, and silver The results show that the relation between wave-length and absorption coefficient is expressed by the relation

$$\tau/\rho = K\lambda^3$$

where τ/ρ is the fluorescent coefficient and K is a constant for a given substance over the range of wave-lengths between the absorption bands of that substance

(6) The critical wave-length necessary to excite the characteristic rays of a substance lies in the neighbourhood of the β-ray of that substance. The α-ray is not excited until the β-ray is excited.

(7) It is pointed out that the purity of the characteristic lines emitted by

a bulb and isolated by reflection at a crystal face will depend to a certain extent upon the state of working of the bulb

In conclusion, I wish to express my indebtedness to Prof W. H. Bragg, F R.S, for his kindness in putting his spectrometer at my disposal, without which the work could not have been undertaken Also, my best thanks are due to Prof A W Porter, F R S, for the valuable help he has given me throughout the course of the work

Selenic Acid and Iron.—Reduction of Selenic Acid by Nascent Hydrogen and Hydrogen Sulphide.—Preparation of Ferrous Selenate and Double Selenates of Iron Group.

By A. E H. Tutton, D Sc, M A., F R.S.

(Received February 7, 1918)

Exceptional difficulties have been met with in the preparation of the iron (ferrous) group of double selenates of the monoclinic isomorphous series $R_2M\left(\begin{smallmatrix}S\\Se\end{smallmatrix}O_4\right)_2, 6H_2O$, and some new and unexpected chemical facts have been revealed, which form the subject of this communication. When the author, in pursuing the course of the crystallographic and physical investigation of the double selenates of this grand series on exactly parallel lines to those followed in the completed investigation of the corresponding double sulphates, arrived at the iron group, three difficulties were encountered The first was that of preparing the simple salt ferrous selenate, common to the group The second was the rapid decomposition, with brown deposition, of the aqueous solution of any ferrous selenate obtained, also that of the mixed solution obtained by adding to the latter solution a solution of the molecular equivalent of the alkali selenate, with the object of preparing any one of the four salts of the group,

$$K_2Fe(SeO_4)_2,6H_2O,$$
$$Rb_2Fe(SeO_4)_2,6H_2O,$$
$$Cs_2Fe(SeO_4)_2,6H_2O,$$
$$(NH_4)_2Fe(SeO_4)_2,6H_2O$$

The third difficulty was offered by the fact that no crystals of the potassium ferrous salt were ever obtained at all.

The author's first efforts to obtain these salts were made eight years ago in the cases of the potassium and rubidium salts The result was that, under conditions which were obscure, but undoubtedly included a moderately low temperature, it being winter time, one excellent and permanent crop of rubidium ferrous selenate was obtained, the small quantity employed of the ferrous selenate constituent having been obtained by very prolonged action of a very considerable quantity of moderately dilute selenic acid on much iron wire, the action being very slight All other solutions of the constituents of this salt, and all the solutions of the constituents of potassium ferrous selenate, decomposed with much brown ferric precipitation, and no trace of the potassium salt crystals was ever obtained.

In attempting to prepare ferrous selenate, it is, of course, impossible, on account of the instability of ferrous hydrate, to employ the excellent method used with so much success by the author in the preparation of the zinc group of double selenates, namely, the preparation of the pure precipitated hydrate of the metal, and its subsequent solution in pure, moderately dilute (0 7 grm H_2SeO_4 per cubic centimetre) selenic acid Recourse was therefore made to the apparently obvious method of dissolving metallic iron in more or less dilute selenic acid, using iron wire, keeping the temperature low, and excluding air as one does when preparing ferrous sulphate solution for standardising permanganate, by working in a vessel filled with carbon dioxide or other inert gas But, contrary to what one would expect from text-book statements,* the anticipated reaction, analogous to that with sulphuric acid,

$$Fe + H_2SeO_4 = FeSeO_4 + H_2, \qquad (1)$$

does not occur to any visible extent, no bubbles of gas making their appearance Heating is excluded, as any ferrous salt produced would be oxidised to ferric After some hours in the cold, however, the iron becomes covered with a red film of elementary selenium, which still more retards further action Even after several weeks of time, the greater part of the iron remains undissolved in presence of excess of selenic acid, but is coloured bright red from deposited selenium It would thus appear that the nascent hydrogen liberated as the effect of the very slow operation of reaction (1) reduces selenic acid in accordance with the equation

$$6H + H_2SeO_4 = Se + 4H_2O. \qquad (2)$$

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* 'Watts' Dictionary of Chemistry' (Muir and Morley) states —Dilute H_2SeO_4Aq dissolves many metals, with evolution of hydrogen." Roscoe and Schorlemmer's 'Treatise on Chemistry' states :—"Iron, zinc and other metals dissolve with evolution of hydrogen."

The net result, therefore, of such slow action as occurs between iron and selenic acid may be expressed by the combined equation

$$3Fe + 4H_2SeO_4 = 3FeSeO_4 + Se + 4H_2O \qquad (3)$$

The length of time occupied in obtaining only a small yield of ferrous selenate in solution by this method renders it practically useless, as ferric oxidation supervenes, and also the solution is so dilute that prolonged evaporation, again accompanied by destructive oxidation, is necessary before crystallisation can occur, either of the ferrous selenate or of the double salt desired and attempted to be prepared by the addition of the calculated quantity of the alkali selenate.

The effect of heating the moderately dilute selenic acid standing over iron wire is to produce a little more selenium, the boiling liquid becoming red, and eventually purplish-red, with the finely divided precipitate, which is not large in quantity, as it takes hours to settle No bubbles of hydrogen or other evidence of increased action on the iron are apparent, however

When zinc is placed in moderately dilute selenic acid, action commences in the cold, bubbles of hydrogen being evolved in considerable quantity, and a deposit of selenium, at first yellow, and later orange and red, is produced in small quantity On heating the liquid, the evolution of gas is hastened, and, when the boiling temperature is reached, the coloured deposit almost entirely disappears, no more selenium being deposited, and the original quantity becoming condensed and less visible

When moderately dilute selenic acid is poured on magnesium, rapid and violent action, accompanied by hissing, is produced, the liquid becoming heated almost to boiling, there is a copious evolution of hydrogen and precipitation of red selenium, the magnesium being almost immediately dissolved.

Thus, in the case of zinc, the reaction with selenic acid is almost entirely the simple one, $Zn + H_2SeO_4 = ZnSeO_4 + H_2$, there being only a slight reduction of the selenic acid by nascent hydrogen In the case of magnesium, however, while the main reaction is the intense one, $Mg + H_2SeO_4 = MgSeO_4 + H_2$, the secondary reaction (2), the reduction of selenic acid by nascent hydrogen, occurs also to a considerable extent

Among the salts prepared by Haldor Topsøe, and described by him in his well known dissertation in the year 1870,[*] and used by him in the year 1874 for the still more celebrated optical investigation of the crystals in collaboration with C Christiansen[†] (Professor of Physics at the University of Copenhagen from 1886 to 1912, whose regrettable death on December 28, 1917, has just been announced), was ammonium ferrous selenate, and, from the very meagre details given in Topsøe's dissertation, it would appear that the ferrous selenate was obtained by solution of natural crystals of chalybite (Eisenspath), $FeCO_3$, in selenic acid This is specifically stated to have been so with respect to potassium ferrous selenate, of which Topsøe states that he obtained a few poor crystals, which, however, became dull and decomposed immediately on removal from the mother liquor He gives very rough, merely approximate, measurements of four interfacial angles of the crystals of the potassium salt, but Topsøe and Christiansen make no mention of the salt in the joint paper, so that it is obvious no attempt was made at the optical investigation of the crystals

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[*] 'Krystallogr kem. Unders. over de selensure Salte,' Copenhagen, 1870.

[†] 'Ann. Chim. Phys.,' Series 5, vol. 1, p. 83 (1874)

On attempting to employ this method, using some excellent crystals of chalybite supplied by Mr Gregory, the action of selenic acid, both moderately concentrated and dilute; on the mineral in the cold (heat having to be avoided on account of oxidation) was found to be so excessively slow that, even after the expiration of several weeks, the reaction was only very partial, affording every opportunity for ferric oxidation. The solution of ferrous selenate obtained was excessively dilute, and contained much excess of selenic acid It required so long for evaporation to the crystallising stage, even under very reduced pressure over oil of vitriol, that the greater part had oxidised and deposited brown ferric products before it could afford either ferrous selenate crystals or, when mixed with the concentrated solution of an alkali selenate, the desired crystals of any one of the double selenates

Early in the year 1912 the author enquired of the firm of Merck, of Darmstadt, who had supplied the large stock of selenic acid used in these investigations, whether they were able to supply a specimen of ferrous selenate A reply in the affirmative was received, to the surprise of the author, who at once gave an order After a considerable lapse of time, a further communication was received from the firm, to the effect that they had met with unexpected difficulties in carrying out the order, but hoped shortly to be able to supply the salt The author wrote suggesting that perhaps they would more easily be able to supply ferrous ammonium selenate, and a reply was received that they hoped to supply both However, several months later in the year, a final letter was received from the firm regretting their inability to carry out the undertaking, as their experiments had proved unsatisfactory on account of decomposition of the products obtained

On the completion of the work on the nickel double selenates, described in the author's last communication,* the study of the iron group was taken up once more. Happily, after many experiments and much expenditure of valuable selenic acid, a successful method of preparing ferrous selenate has been found, in the action of selenic acid on ferrous sulphide. The reaction not only proceeds steadily and rapidly in the cold, but affords a ferrous product practically without a trace of ferric oxidation, the sulphuretted hydrogen evolved and partially remaining in the solution preserving by its reducing properties the selenate of iron in the ferrous state. The filtered liquid product may be employed directly for the preparation of the double selenates, by the addition to it of the solution (as concentrated as desirable) of the calculated molecular proportion of potassium, rubidium, cæsium, or ammonium selenate.

The ferrous sulphide should be as free from metallic iron as possible (see later), the best of that supplied as pure, in small cylindrical sticks, for the preparation of sulphuretted hydrogen serving admirably, and the moderately concentrated acid (0·7 grm. H_2SeO_4 per cubic centimetre) is very suitable. The ferrous sulphide was broken into small pellets and coarse powder, and placed in a small glass decomposition vessel, a 50 c.c. weighing bottle being found very convenient, the sulphide of iron occupying about a quarter of its capacity. It was closed by a rubber stopper pierced by two holes, one of which admitted a dropping funnel containing 10–15 c.c. of the selenic acid (of the known strength 0·7 grm. of H_2SeO_4 per cubic centimetre), the exact quantity to be used having been measured from a burette; the other aperture was fitted with an exit tube for leading away the sulphuretted hydrogen gas evolved. Neither tube emerged below the lower level of the stopper.

The selenic acid was dropped in very slowly, with shaking to break up any large bubbles or frothing produced by the immediate and steady liberation of the hydrogen sulphide, and prevent loss of liquid through the exit tube. After all had been added, the last few drops of the selenic acid moistening the dropping funnel were washed in with not more than 2 c.c. of distilled water. It is very undesirable to render the liquid in the decomposition vessel too dilute, for the reason already mentioned that no concentration either by prolonged standing or by warming can be indulged in without loss by ferric oxidation. In order to keep down rise of temperature, which is not inconsiderable if unchecked, the lower two-thirds of the decomposition vessel is with advantage immersed in a dish of cold water. It is also important not to use a larger vessel than indicated, in proportion to the scale of the preparation, in order that as little air may be present as possible; what is unavoidably present at first is rapidly driven out by the evolved hydrogen sulphide.

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* 'Phil. Trans.,' A, vol. 217, p. 199 (1917).

The reaction is mainly the following

$$FeS + H_2SeO_4 = FeSeO_4 + H_2S \tag{4}$$

But almost from the first it is obvious that this is not the only reaction occurring For the liquid is seen to develop a precipitate which is at first yellow, then orange, and latterly dark red, and which contains both sulphur and selenium, and which is produced in accordance with the further equation —

$$3H_2S + H_2SeO_4 = Se + 3S + 4H_2O \tag{5}$$

The total or net result of the operation of both reactions may be expressed by the single equation —

$$3FeS + 4H_2SeO_4 = 3FeSeO_4 + Se + 3S + 4H_2O \tag{6}$$

But reaction (6) is not followed quantitatively, for a considerable amount of hydrogen sulphide unmistakably escapes, so that the operations have to be carried out either in a draught chamber or in the open air

Selenic acid, under these circumstances, is thus reduced to elementary selenium by hydrogen sulphide, contrary to the usual statement in text-books of chemistry, due originally to Mitscherlich,* that sulphuretted hydrogen is without action on selenic acid † The fact that this is an error has also been recently observed independently by E B Benger,‡ who found, for instance, that a 10-per-cent solution of selenic acid at 45° C was completely decomposed by hydrogen sulphide in 14 hours The amount of reduction increases with the temperature and concentration, and is also a function of the time

The reaction is allowed to complete itself for an hour or two, and the contents of the decomposition vessel are poured off the excess of ferrous sulphide on to a filter The clear liquid running from the latter is of a pale green colour, and is allowed to fall into a crystallising dish or small beaker If required for the preparation of one of the double selenates, as in the author's case, the fairly concentrated and recently filtered solution of the selenate of potassium, rubidium, cæsium, or ammonium is added in equal molecular quantity. All these alkali selenates have been prepared by the author in considerable quantities, and stored in the form of excellent crystals, in an efficient desiccator, as they are all more or less deliquescent, and cæsium selenate extremely so They were made by the action of selenic acid on the carbonates of the three alkali metals, and in the case of ammonium selenate by saturating selenic acid with ammonia gas under precautions to keep the liquid cool It is always best to filter before using the solutions of any of these selenates, as a trace of decomposition by light occurs on keeping, a little red selenium being liberated. The crystals of the selenates should, therefore, be kept in the dark when not required

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* 'Ann Phys.,' [1], vol 9, p 629 (1827)

† 'Watts' Dictionary of Chemistry' (Muir and Morley) states that "the acid is not reduced by sulphur dioxide nor by sulphuretted hydrogen;" and a precisely similar statement is given in Roscoe and Schorlemmer's 'Treatise on Chemistry'

‡ 'Journ. Amer. Chem. Soc,' vol 13, p. 2171 (1917) This paper has only just (January, 1918) come into the author's hands, after the author has been employing the reaction for over a year.

Having added the alkali selenate to the solution of ferrous selenate, in the amount only very slightly less than that calculated for equal molecular proportions based on the volume of selenic acid of known strength used in preparing the ferrous selenate (for only a slight amount is lost by reaction (5)), the clear mixed solution is rendered just distinctly acid by the addition of a drop of selenic acid and placed under the receiver of an air pump, after the reduction of the pressure by the working of the pump, the solution is left to crystallise, a vessel of oil of vitriol being also included as usual under the receiver in order to absorb the water vapour evaporated The room should be cold and the receiver protected from draughts and vibrations After a fairly cold night crystals (except in the case of potassium ferrous selenate) will usually be found next morning, and they should be removed, dried on blotting paper, and stored in small stoppered bottles before the temperature rises appreciably with the sun

This mode of preparing ferrous selenate and the double selenates of the iron group has two outstanding advantages, (1) the ease and rapidity with which the reaction proceeds in the cold, and (2) the prevention of oxidation to the higher ferric state by the reducing power of the hydrogen sulphide passing through the liquid, and to a slight extent remaining dissolved in it The presence of a small quantity of metallic iron in the ferrous sulphide is obviously without deleterious effect, as it is practically without action on selenic action in the time employed, especially so long as any ferrous sulphide is present unattacked

If the green solution of pure ferrous selenate obtained after filtration of the contents of the decomposing vessel be itself allowed to crystallise under the receiver of the air pump, the crystals obtained are those of the monoclinic salt $FeSeO_4,7H_2O$, isomorphous with the common form of sulphate of ferrous iron, $FeSO_4,7H_2O$ The crystals are much less stable, however, than those of ferrous sulphate, and rapidly decompose and become opaque, so that no goniometrical work of real value has yet been possible with them In the light of what has been accomplished with potassium ferrous selenate, however, it is hoped after the completion of the present plan of research to do as much for the fugitive crystals of this salt. Topsøe only obtained three roughly approximate angular measurements, which are recorded on p. 33 of his already quoted dissertation. Wohlwill,* who first obtained the salt, was quite unable to obtain any measurements, as his crystals effloresced and decomposed immediately on removal from the mother liquor. Wohlwill also obtained a pentahydrated salt $FeSeO_4,5H_2O$, in a triclinic form resembling that of copper sulphate, $CuSO_4,5H_2O$, but, in this case also, the crystals decomposed so rapidly that work with them was out of the question.

The crystals of rubidium ferrous selenate, cæsium ferrous selenate, and ammonium ferrous selenate, obtained by the author's method now described, have proved excellent for goniometrical and physical investigation, and the results obtained with them are described in a memoir of which the following communication is an abstract, the full paper being published in the 'Philosophical Transactions.'

Happily it has also been possible to include in the memoir a full crystallographic and physical description of potassium ferrous selenate, although this salt is so unstable that the crystals become quite opaque and dull from decomposition in half-a-day's time after removal from the crystallising vessel. Five hours are the limit of time possible for goniometrical, optical, or specific gravity work with them. Immersion under benzene, or painting with balsam dissolved in benzene, immediately after removal from the mother liquor and momentary drying with a soft absorbent handkerchief, preserves their lives possibly for a couple of hours longer, but even then by the end of the day they are opaque and afford no reflection, and by next morning they are like porcelain. A fresh preparation (for rarely can more than one crop be obtained from any one preparation) is consequently required for every day's work on this salt. This has involved immense labour and expenditure of the expensive selenic acid. For, as already mentioned, the greatest difficulty of all has been to produce any crystals whatever of this salt, and many preparations of the solution have been made without success in obtaining the crystals. It has only been at the opening of the present year, 1918, that the author, after attempts spread over eight years, has succeeded at last in preparing them. The former attempts, although made in the winter, had never happened to synchronise with the minimum temperature of a particularly severe winter.

Surmising that the critical upper temperature limit for the existence of this salt, $K_2Fe(SeO_4)_2,6H_2O$, was not more than 2° or 3° C. above the freezing point of water, arrangements for several new preparations were made at the beginning of January, during the most severe cold spell of this winter, which happened also to be a "record" one for Devonshire. Carrying out the first of these preparations exactly as already described, the mixed solution of ferrous selenate and potassium selenate, in as closely equal molecular proportions as possible, and in as concentrated a form as was judged suitable, was left to crystallise under reduced pressure over oil of vitriol one evening in the first week of January, when the temperature outside was well below that of the freezing point of water, and the temperature of the laboratory during the succeeding night fell to within a degree of the freezing point. Next morning the author was delighted to find a most satisfactory crop of clear transparent pale-green crystals of the salt, single individuals of quite suitable goniometrical and optical sizes, and obviously of the expected symmetry, the disposition of faces being exactly that characteristic of the potassium salts of the double sulphate and selenate series Work was carried out with them from the moment they were removed from the mother liquor and superficially dried, and was able to proceed for five to six hours, beyond which the obvious commencement of decomposition rendered it unwise to proceed further The laboratory was unwarmed, so as to prolong the time available as much as possible On a warm day or in a warmed room, decomposition is much more rapid The next night was not quite so cold and the refiltered solution, again set to crystallise under the receiver of the air pump as before, gave no crystals at all. On a third night, which was still less cold, a crop of quite different very minute crystals, useless for goniometry, was obtained, which proved to be those of the dihydrated salt $K_2Fe(SeO_4)_2,2H_2O$, which are referred to by Topsøe as being deposited on warm days and as being of triclinic symmetry

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* 'Über isomorphe Mischungen der selensauren Salze,' Dissertation, Göttingen, 1860, p. 47.

Two days afterwards the very cold weather returned and a further entirely new preparation was made, next morning, after a night of a temperature again closely approaching freezing point in the laboratory, another crop of the unmistakable hexahydrated monoclinic crystals was obtained. Another satisfactory day's work was then carried out on them, lasting about five hours before opacity supervened, after which the filtered solution was again set to crystallise, and, this night proving equally cold if not colder, a fine crop of even larger crystals was found next morning, this being the only occasion on which a second crop was obtained from any one preparation. These were used in further essential measurements as long (about six hours) as the transparency sufficed Warmer weather again returned, but after nearly a week a final spell of exceedingly cold weather supervened, during which a new preparation was made and a fourth crop of suitable and beautifully well-formed and transparent crystals of the desired salt were obtained on the night following, which proved to be the coldest night of the winter Again a full day of intensive work was put in, and all the essential measurements and observations with the salt were, on this day, fortunately completed For the weather again became less severe, and no further crops of the salt have ever been obtained since, all succeeding crops being of the minute crystals of the dihydrated salt, with much ferric decomposition

The intensive work on these four crops of potassium ferrous selenate crystals has fortunately included the measurement of all the interfacial angles on six excellent crystals, the determination of the density of a considerable number from two crops, and the determination of all the optical constants in triplicate with other crystals, so that a full description of the salt is possible The details of the special methods employed in the measurements, and the results, are given in the memoir of which the following communication is an abstract The crystals were in excellent condition when used, and were fine well-developed specimens, and perfectly transparent, so that the results are eminently satisfactory

The extreme instability of potassium ferrous selenate and the relatively high stability of cæsium ferrous selenate, together with the moderate stability of the rubidium salt, afford an excellent example of the progression in stability which, like all the other properties of the salts, follows the order of progression of the atomic weights and atomic numbers of the alkali metals, which by their interchange form the isomorphous group For rubidium ferrous selenate is undoubtedly intermediate in stability between the potassium and rubidium salts, being readily formed at a moderately low temperature, whereas the cæsium salt may be obtained even during a warm night. On the other hand, as we have seen, the potassium salt is only formed on the very coldest nights of a specially hard winter, when the laboratory temperature is almost down to 0° C.

Monoclinic Double Selenates of the Iron Group.

By A. E. H. Tutton, D.Sc., M.A., F.R.S.

(Received February 7, 1918.)

(Abstract.)

In this memoir are described the results of a complete investigation of the crystals of the potassium, rubidium, cæsium, and ammonium salts of the iron group of double selenates of the monoclinic series $\mathrm{R_2M}\left({}^{\mathrm{S}}_{\mathrm{Se}}\mathrm{O_4}\right)_2, 6\mathrm{H_2O}$. It has been rendered possible by the success which has at length attended the author's prolonged attempts to prepare ferrous selenate, and especially potassium ferrous selenate, as described in the preceding communication.

The outstanding result is to confirm in every particular the conclusions drawn from the previous study of three other groups of double selenates (those containing zinc, magnesium, and nickel, as the M-metals), and of the complete set of eight groups of the double sulphates of this grand series.

The general law of progression of the crystallographic properties, with the atomic weight and atomic number of the interchangeable dominating alkali metals which form the group, is obeyed absolutely rigidly by the iron group. The change in the principal (monoclinic axial) angle, the *mean* of the changes in all the 38 different angles measured, and the maximum change of interfacial angle, are all directly proportional to the change in atomic weight or number, to a degree of precision which is remarkable. The change in the dimensions of the structural unit cell of the space-lattice, as indicated by the molecular volumes and topic axial ratios, and the principal measure of the optical refractive power, the molecular refraction, are all properties which show progression with acceleration, with the increase of atomic weight or number.

Crystal Elements.—The direct progression of the axial angle β will be clearly seen from the following Table, which also shows the axial ratios and the fact that those of the rubidium salt are intermediate, corresponding to the intermediate position of rubidium between potassium and cæsium as regards both atomic weight and atomic number. For the atomic weight (84·9) and atomic number (37) of rubidium are the mean of the corresponding potassium and cæsium constants, $\frac{1}{2}(38{\cdot}9+131{\cdot}9) = 85{\cdot}4$, and $\frac{1}{2}(19+55) = 37$.

	Axial angle β		Axial ratios.		
			a	*b*	*c*
	°	′			
KFe selenate .	103	50	0 7490 :	1	0 5044
RbFe selenate .	104	57	0 7424	1 :	0·5000
NH_4Fe selenate	106	9	0 7433	1	0 5019
CsFe selenate .	106	2	0 7308	1	0 4979

The axial ratios of the ammonium salt are close to those of the rubidium salt, and its axial angle is nearly identical with that of the cæsium salt, both being facts agreeing with the supposition of true isomorphism of this salt with the other three salts of the group

Habit—The three characteristic habits for the potassium, rubidium, and cæsium salts of any group of the whole series, especially marked in the progression in the relative sizes of the faces of *c*{001} and *q*{011}, are clearly displayed by the three alkali metallic salts of the iron group, while the ammonium salt shows every gradation of variety between the limits shown by the outer members of the series, the potassium and cæsium salts.

Interfacial Angles—The interfacial angles of the rubidium salt are intermediate in magnitude between those of the potassium and cæsium salts When all the changes in the 38 angles measured are summed and the mean taken, it is found that the average change of angle for the replacement of potassium by cæsium is exactly twice as great as that for the replacement of potassium by rubidium, corresponding exactly to the double change of atomic weight and atomic number The maximum change of angle also follows the same rule The values for the replacement of potassium by ammonium are the same as those for the cæsium interchange These values are set out in the Table which follows, and the relations of the atomic weights and numbers follow the Table

Replacement.	Average change	Maximum change			
	′	′		°	′
K by Rb . .	27	69	=	1	9
K by Cs .	55	139	=	2	19
K by NH_4 .	54	139	=	2	19

	K.	Rb − K.	Rb.	Cs − Rb.	Cs.	Cs − K
Atomic weights ...	38 85	46	84 9	47	131·9	93 = 2 × 46.5
Atomic numbers	19	18	37	18	55	36 = 2 × 18

The ammonium salt of the group is thus isomorphous, but not eutropic (does not follow the law of progression), with the alkali metallic salts, which are eutropic with one another, that is, isomorphous in the strictest sense possible outside the cubic system

Volume—The molecular volumes and topic axial ratios, representing the volumes and axial edge dimensions of the unit cells of the space-lattice, are given in the next Table

	Molecular volume	Topic axial ratios χ	ψ	ω
KFe selenate	210·39	6·2230	8·3085	4·1908
RbFe selenate	220·29	6·3109	8·5006	4·2508
NH_4Fe selenate	220·39	6·3212	8·5048	4·2684
CsFe selenate	233·21	6·3847	8·7366	4·3409

They all show a regular progression from the potassium salt, through the rubidium salt, to the cæsium salt, and the ammonium salt is of almost exactly the same volume and of the same cell-edge dimensions as the rubidium salt. When potassium is replaced by rubidium, there is an increase of volume of 9·9 units, and, when rubidium is replaced by cæsium, of 12·9 units The progress is thus an accelerating one When potassium is replaced by the ammonium radicle NH_4, the increase in volume is exactly 10 units, only 1 per cent different from the increase caused by introducing rubidium in place of potassium

This subject of the isostructure and congruency of the ammonium and rubidium salts of any group was specially dealt with in the author's memoir on "X-Ray Analysis and Topic Axes of the Alkali Sulphates, and their Bearing on the Theory of Valency Volumes"*, the further instance now presented by the iron group of double selenates is so much more evidence in substantiation of the view there expressed that the theory is untenable For it was shown that, if it were founded on fact, in the case of the simple rhombic sulphates, the volume of the space-lattice unit cell of the ammonium salt would be double that of the rubidium salt (24 valency volumes as against 12), whereas the volumes were proved to be almost absolutely identical, as in the case of the double salts now described

The Optical Ellipsoid—The ellipsoid representing the optical properties has its β axis identical in direction with the crystal symmetry axis *b*, and it rotates on this axis when one alkali metal is replaced by another so that the α axis, one of the two rectangular axes x and y which lie in the symmetry plane, is inclined to the vertical crystal axis *c* at the following angles :—

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* 'Roy. Soc. Proc.,' A, vol. 93, p. 72 (1917).

For the ammonium salt	5° 24′
" potassium salt . .	10 27
" rubidium salt . .	13 37
" cæsium salt	21 4

The position of the ellipsoid for the rubidium salt is thus intermediate between the positions for the potassium and cæsium salts, and the rotation is an accelerating one with the atomic weight and number.

Optic Axial Angles.—These have been measured for six wave-lengths of light. The true angles within the crystal $2V_a$ for the three metallic salts progress almost exactly in direct proportion to the change of atomic weight and number, and the value for the ammonium salt is somewhat higher than that for the rubidium salt. These facts may be illustrated by the values for the potassium, rubidium ammonium, and cæsium salts for sodium light, which are, respectively, 64° 18′, 73° 32′, 77° 44′, and 82° 47′.

Refractive Indices and Double Refraction.—The refractive indices have been determined for seven wave-lengths, and show also a progression with atomic weight and number of the alkali metal. The mean index for sodium light, $1/3(\alpha+\beta+\gamma)$, for the potassium, rubidium, ammonium, and cæsium salts respectively is —1·5207, 1·5220, 1·5292, and 1·5357. The double refraction (difference between α and γ indices) for the four salts is —0·0250, 0·0195, 0·0165, and 0·0108, the progression with atomic weight being here a very clear one, the diminution being at an accelerating rate. It is also the cause of the difference of the refractive indices for the potassium and rubidium salts not being more than is shown, for the effect of the progression of the indices, which is well marked in the case of the α values, is counteracted and slightly reversed in the γ values by the opposite kind of progression (a diminution) of the double refraction; when the cæsium salt is reached, however, the progress of all three of the indices, α, β, and γ, has become clearly apparent in spite of the effect of the double refraction change.

The ammonium salt stands between the rubidium and cæsium salts with respect to both properties.

Molecular Optical Constants.—These have been calculated both for the Lorenz and for the Gladstone and Dale formulæ. The molecular refraction is the most important of all the optical constants, and the Gladstone values are given below. The Lorenz values indicate precisely the same facts.

The true progression with atomic weight and atomic number is very clearly shown by these values for the molecular refraction, the rubidium salt standing intermediate between the potassium and cæsium salts, and in a position somewhat closer to the former salt, so that the progression is

	Molecular refraction α	Molecular refraction β	Molecular refraction γ	Mean value $\frac{1}{3}(\alpha+\beta+\gamma)$
KFe selenate . .	106·54	108·38	111·74	108·87
RbFe selenate	112·44	113·89	116·65	114·33
NH_4Fe selenate	114·21	115·62	117·87	115·90
CsFe selenate	122·99	124·11	125·56	124·22

clearly an accelerating one. The ammonium salt possesses almost the same molecular refractive power as the rubidium salt, the value being only very slightly higher, a fact doubtless connected with the interesting volume result, that the ammonium and rubidium salts are almost precisely isostructural.

Concluding Remarks

The key to the interesting crystallographic progression, which has been so strikingly confirmed by the results for this iron group of double selenates, has recently been afforded by the brilliant discovery of Moseley, that the atomic sequence number is the expression of the complexity of the atom, and is itself a fundamental constant, a direct measure of the positive electric charge on the atomic nucleus and of the number of negative electrons clustered around it in electrical equilibrium to form the atom. Atoms so vigorous and dominating as those of the alkali metallic family group of maximum electro-positive character, differing regularly by the number of electrons corresponding to two whole horizontal rows of elements in the Periodic Table, must naturally exhibit a pronounced and similarly regular progressive influence on the structure and physical constants of the crystals of a series of salts in which these elements are the interchangeable constituents. The discovery of Moseley thus renders the results of the author's work over many years, so long without logical explanation, now clearly intelligible and explicable, and indeed only what could be expected from the progressive effect on the crystals of the operation of Moseley's law on the atoms composing them.

On the Numerical Solution of Integral-Equations.

By E. T. WHITTAKER, F.R.S.

(Received December 27, 1917.)

§ 1. *Introductory.*

The present communication is concerned with integral-equations of Abel's type

$$\int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x), \tag{1}$$

and of Poisson's type

$$\phi(x) + \int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x), \tag{2}$$

where $\kappa(x)$ is a given function called the *nucleus*, $f(x)$ is also a given function, and $\phi(x)$ is the unknown function which is to be determined. The object of the work is to obtain solutions of these equations in forms which can be made the basis of numerical calculation.

Theoretical solutions of both these equations, in the form of infinite series, are well known, and have been fully discussed by Volterra* and others. But in these solutions the nth term of the series is a multiple integral involving $(n-1)$ integrations with variable limits: and although such series are valuable for the light they throw on the general properties of the solution, it is obvious that they cannot, except in very special cases, be used in order to compute values of the solution numerically. The only case, so far as I am aware, in which a solution of an equation of one of the above types has been obtained in a form adapted for practical ends is Abel's original special form of equation (1),

$$\int_0^x \frac{\phi(s)\,ds}{(x-s)^p} = f(x), \qquad (0 < p < 1, \qquad f(0) = 0),$$

for which he gave† the solution

$$\phi(x) = \frac{1}{\pi}\sin p\pi \int_0^x \frac{f'(s)\,ds}{(x-s)^{1-p}}.$$

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* 'Torino Atti,' vol. 31, pp. 311, 400, 557, 693 (1896). For fuller references *cf.* H. Bateman, "Report on the Theory of Integral Equations," 'Brit. Assoc. Report,' 1910.

† 'Œuvres,' (ed. 1881), p. 11 (1823) and p. 97 (1826). The fundamental meaning of Abel's result is most clearly seen if the integrals which occur in it are interpreted as in the theory of generalised differentiation. If $\psi(x)$ is written for $\Gamma(1-p)\,\phi(x)$, Abel's formula reduces to the simple statement that if

$$\left(\frac{d}{dx}\right)^{p-1}\psi(x) = f(x),$$

then

$$\psi(x) = \left(\frac{d}{dx}\right)^{-p} f'(x).$$

When $f(x)$ is given, the values of $f'(s)$ and of the integral last written may be obtained without difficulty by the ordinary processes of interpolation and numerical integration [*]

Recently, two friends, one a seismologist and the other an actuary, have enquired of me whether the integral-equations (1) and (2), which had occurred in their researches, could be solved in such a way as to obtain numerical results when the functions $\kappa(x)$ and $f(x)$ are known (tabulated) functions It was under the stimulus of these enquiries that the methods of solution which occupy the following pages were devised It will be seen that I have departed altogether from the customary methods of solution by infinite series whose terms are multiple integrals, and on this account the new solutions, which are formulated in Theorems 1–5 below, and by which the unknown function may be determined numerically, may perhaps be found to be not without interest from the standpoint of pure theory

§ 2 *Solution of Integral-Equations of Abel's Type*

Considering first the generalised Abel's integral-equation,

$$\int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x), \tag{1}$$

we need only consider the case when the nucleus $\kappa(x)$ becomes infinite at $x=0$, for in the simpler case when the nucleus is finite at $x=0$, the equation (1) may be reduced immediately, as we shall see later, to the type (2), and so may be dealt with by the methods which are given subsequently in the paper

We shall, then, suppose $\kappa(x)$ to be such that $x^p\kappa(x)$ is finite and not zero at $x=0$, where p lies between 0 and 1

Now if the nucleus $\kappa(x)$, which is supposed to be given by a numerical Table, has this character, so that it becomes infinite like x^{-p} at $x=0$, but is finite for other values of x within the range of integration, we can in general (by use of Newton's or some other interpolation-formula) represent the function $x^p\kappa(x)$ over the range in question by a polynomial in x, the degree of this polynomial will depend on the nature of $\kappa(x)$ and the order of accuracy to which the work is to be carried We may, then, assume for the nucleus $\kappa(x)$ an analytical expression of the form

$$\kappa(x) = x^{-p}(a_0+a_1x+a_2x^2+\ \ .+a_nx^n), \qquad (0<p<1). \tag{3}$$

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[*] It is possible that the elegant solution in power-series which was given by Sonine, 'Acta Math,' vol 4, p 171 (1884), for his generalised form of Abel's integral-equation, may also be utilised for numerical calculation

Now assume* a solution of the form

$$\phi(x) = \int_0^x f'(s)\,\mathrm{K}(x-s)\,ds, \tag{4}$$

where $\mathrm{K}(x)$ is called the *solving function* We have on eliminating $\phi(x)$ between this equation and (1)

$$f(x) = \int_0^x \kappa(x-s)\left\{\int_0^s f'(t)\,\mathrm{K}(s-t)\,dt\right\}ds,$$

or, inverting the order of integration,

$$\int_0^x f'(t)\,dt = \int_0^x f'(t)\left\{\int_t^x \kappa(x-s)\,\mathrm{K}(s-t)\,ds\right\}dt$$

Since $f'(t)$ is an arbitrary function, this gives

$$\int_t^x \kappa(x-s)\,\mathrm{K}(s-t)\,ds = 1$$

Writing $s = t+u$, this becomes

$$\int_0^{x-t} \kappa(x-t-u)\,\mathrm{K}(u)\,du = 1,$$

or

$$\int_0^a \kappa(a-u)\,\mathrm{K}(u)\,du = 1,$$

where a is arbitrary From this equation the solving function $\mathrm{K}(x)$ is to be determined

By (3) this may be written

$$\int_0^a (a-u)^{-p}\{a_0+a_1(a-u)+a_2(a-u)^2+\dots+a_n(a-u)^n\}\,\mathrm{K}(u)\,du = 1 \tag{5}$$

Now let $\int_0^u \mathrm{K}(u)\,du$ be denoted by $\mathrm{K}_1(u)$, let $\int_0^u \mathrm{K}_1(u)\,du$ be denoted by $\mathrm{K}_2(u)$, and so on Then integrating by parts we have

$$\int_0^a (a-u)^{1-p}\,\mathrm{K}(u)\,du = (1-p)\int_0^a (a-u)^{-p}\,\mathrm{K}_1(u)\,du,$$

$$\int_0^a (a-u)^{2-p}\,\mathrm{K}\,u\,(du) = (2-p)\int_0^a (a-u)^{1-p}\,\mathrm{K}_1(u)\,du,$$

$$= (2-p)(1-p)\int_0^a (a-u)^{-p}\,\mathrm{K}_2(u)\,du$$

and so on. Thus equation (5) may be written

$$\int_0^a (a-u)^{-p}\{a_0\mathrm{K}(u)+(1-p)a_1\mathrm{K}_1(u)+(1-p)(2-p)a_2\mathrm{K}_2(u)+$$

$$+(1-p)(2-p)\dots(n-p)a_n\mathrm{K}_n(u)\}\,du = 1. \tag{6}$$

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* The legitimacy of this may be inferred at once from Volterra's theory

Now if $\int_0^a (a-u)^{-p} h(u)\,du = 1$, where a is arbitrary and $h(u)$ does not involve a, we have by writing $u = as$

$$a^{p-1} = \int_0^1 (1-s)^{-p} h(as)\,ds,$$

and, since a is arbitrary, this shows that $h(u) = Cu^{p-1}$, where C is independent of u

Substituting in the last equation, we have

$$1 = C\int_0^1 (1-s)^{-p} s^{p-1}\,ds = \frac{C\pi}{\sin p\pi}$$

and therefore

$$h(u) = \frac{\sin p\pi}{\pi}\,.\,u^{p-1}$$

Applying this result to equation (6), we have

$$a_0 K(u) + (1-p)\,a_1 K_1(u) + (1-p)(2-p)\,a_2 K_2(u) + \dots$$
$$+ (1-p)(2-p)\dots(n-p)\,a_n K_n(u) = \frac{\sin p\pi}{\pi}\;u^{p-1},$$

or, writing $y(u)$ for $K_n(u)$

$$a_0\frac{d^n y}{du^n} + (1-p)\,a_1\frac{d^{n-1}y}{du^{n-1}} + (1-p)(2-p)\,a_2\frac{d^{n-2}y}{du^{n-2}} + \dots$$
$$+ (1-p)(2-p)\dots(n-p)\,a_n y = \frac{\sin p\pi}{\pi}\,.\,u^{p-1}. \qquad (7)$$

This is a linear differential equation in y, and $K_n(u)$ is that solution of it which vanishes, together with its first $(n-1)$ differential coefficients, when u vanishes

Now let the polynomial

$$a_0 x^n + (1-p)\,a_1 x^{n-1} + (1-p)(2-p)\,a_2 x^{n-2} + \dots + (1-p)(2-p)\dots(n-p)\,a_n$$

be denoted by $F(x)$ and let its n roots (supposed for the present to be distinct) be $\alpha, \beta, \gamma, \dots, \nu$. Then it is known from the general theory of linear differential equations that the solution of (7) which vanishes, together with its first $(n-1)$ differential coefficients, when u vanishes is

$$K_n(u) = \int_0^u \frac{\sin p\pi}{\pi}\,.\,t^{p-1}\,.\,h(u-t)\,dt \qquad (8)$$

where

$$h(x) = \frac{e^{\alpha x}}{F'(\alpha)} + \frac{e^{\beta x}}{F'(\beta)} + \dots + \frac{e^{\nu x}}{F'(\nu)} \qquad (9)$$

From (8) we have by successive differentiation

$$K_{n-1}(u) = \frac{\sin p\pi}{\pi}\int_0^u t^{p-1}h'(u-t)\,dt, \qquad \text{since } h(0) = 0,$$

$$K_{n-2}(u) = \frac{\sin p\pi}{\pi}\int_0^u t^{p-1}h''(u-t)\,dt, \qquad \text{since } h'(0) = 0,$$

. . . .

. . . .

$$K_1(u) = \frac{\sin p\pi}{\pi}\int_0^u t^{p-1}h^{(n-1)}(u-t)\,dt, \qquad \text{since } h^{(n-2)}(0) = 0,$$

$$K(u) = \frac{\sin p\pi}{\pi}\left\{\frac{u^{p-1}}{a_0} + \int_0^u t^{p-1}h^{(n)}(u-t)\,dt\right\}, \quad \text{since } h^{(n-1)}(0) = \frac{1}{a_0}.\text{*}$$

Substituting for $h(x)$ from (9) we have $K(x) = (1/\pi)\sin p\pi\, L(x)$, where

$$L(x) = \frac{x^{p-1}}{a_0} + \frac{\alpha^n}{F'(\alpha)}e^{\alpha x}\int_0^x t^{p-1}e^{-\alpha t}\,dt + \frac{\beta^n}{F'(\beta)}e^{\beta x}\int_0^x t^{p-1}e^{-\beta t}\,dt + \dots$$
$$+ \frac{\nu^n}{F'(\nu)}e^{\nu x}\int_0^x t^{p-1}e^{-\nu t}\,dt$$

Now $$\int_0^x t^{p-1}e^{-\alpha t}\,dt = \alpha^{-p}\int_0^{\alpha x} s^{p-1}e^{-s}\,ds,$$

and thus if the function $$e^x\int_0^x s^{p-1}e^{-s}\,ds$$

(which is well known under the name of the *Incomplete Gamma-Function*) be denoted by $\gamma_p(x)$, we have

$$L(x) = \frac{x^{p-1}}{a_0} + \frac{\alpha^{n-p}}{F'(\alpha)}\gamma_p(\alpha x) + \frac{\beta^{n-p}}{F'(\beta)}\gamma_p(\beta x) + \dots + \frac{\nu^{n-p}}{F'(\nu)}\gamma_p(\nu x).$$

Combining our results, we have

THEOREM 1.—*The solution of the integral-equation*

$$\int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x)$$

where the nucleus $\kappa(x)$ *is supposed to be given numerically and to have been expressed by the ordinary methods of interpolation in the form*

$$\kappa(x) = x^{-p}(a_0 + a_1x + a_2x^2 + \dots + a_nx^n), \qquad (0 < p < 1)$$

is $$\phi(x) = \frac{\sin p\pi}{\pi}\int_0^x f'(s)\,L(x-s)\,ds, \tag{10}$$

where

$$L(x) = \frac{x^{p-1}}{a_0} + \frac{\alpha^{n-p}}{F'(\alpha)}\gamma_p(\alpha x) + \frac{\beta^{n-p}}{F'(\beta)}\gamma_p(\beta x) + \dots + \frac{\nu^{n-p}}{F'(\nu)}\gamma_p(\nu x), \tag{11}$$

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* For $h^{(n-1)}(0)$ = sum of residues of $t^{n-1}/F(t)$ at its singularities $\alpha, \beta, \dots, \nu$ = $-$(residue of this function at ∞) = $1/a_0$.

and where $\alpha, \beta, \ldots, \nu$, are *the roots of the algebraic equation*

$$F(x) \equiv a_0 x^n + (1-p)\, a_1 x^{n-1} + (1-p)(2-p)\, a_2 x^{n-2} + \ldots \\ + (1-p)(2-p) \ldots (n-p)\, a_n = 0, \quad (12)$$

and $\gamma_p(x)$ *denotes the incomplete Gamma-function*

$$\gamma_p(x) = e^x \int_0^x s^{p-1} e^{-s}\, ds$$

This may be regarded as a direct extension of Abel's original formula, which may be derived from it by taking $n = 0$. It expresses the solution of the integral-equation *in a finite form* in terms of the incomplete Γ-function. The incomplete Γ-functions which occur in the solution all have the same parameter p, and are, therefore, really all the same function, with different values of the argument; its values may be tabulated from any of the expansions which were given for it by Legendre,* such as the absolutely convergent expansion

$$\gamma_p(x) = \frac{x^p}{p} + \frac{x^{p+1}}{p(p+1)} + \frac{x^{p+2}}{p(p+1)(p+2)} + \ldots,$$

or (for large positive values of x) the asymptotic expansion

$$\gamma_p(x) = \Gamma(p)\, e^x - x^{p-1} \left\{ 1 + \frac{p-1}{x} + \frac{(p-1)(p-2)}{x^2} + \ldots \right\}.$$

When this function has been tabulated, and the algebraic equation (12) solved,† the function $L(x)$ may readily be tabulated from equation (11), and then the required function $\phi(x)$ is given at once by quadrature from equation (10). In this way Theorem 1 yields a numerical solution of the generalised Abel's equation.

It is obvious that when the polynomial $F(x)$ has a pair of equal roots certain modifications must be made in the above solution, but it does not seem necessary to set these forth in detail here.

It may be remarked that by making the degree n of the polynomial in (3) increase indefinitely, we obtain in the limit the solving-function of the integral equation, with an arbitrary function as nucleus, in the form of an infinite series of incomplete Γ-functions of the same parameter. In connection with such a solution it would be necessary to discuss convergence, etc., and it is not proposed to undertake this in the present paper; but the matter seems worthy to be mentioned in passing, as series of incomplete Γ-functions have not (so far as I know) presented themselves hitherto in analysis. They may evidently be regarded as an extension of *Dirichlet's series*, *i.e.*, series of the type

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* 'Exerc. de Calc. Int.,' vol. 1, pp. 338–343 (1811).

† For this Newton's method will probably be found in most cases the most convenient, if $n > 3$, or if some of the roots are complex, the Lobachevsky-Graeffe method.

$$F(x) = ae^{-\lambda x} + be^{-\mu x} + ce^{-\nu x} + \dots,$$

which have received much attention in recent years.

§ 3. *Solution of Equations of Poisson's Type.*

When the nucleus $\kappa(x)$ of equation (1), instead of being infinite at $x=0$ as we have supposed hitherto, possesses a finite differential coefficient over the range of integration, we have by differentiating equation (1)

$$\phi(x)\kappa(0) + \int_0^x \phi(s)\kappa'(x-s)\,ds = f'(x)$$

which is an integral equation included in Poisson's type (2).* We shall, therefore, now pass on to consider integral-equations of Poisson's type, which we shall take in the form

$$\phi(x) + \int_0^x \phi(s)\kappa(x-s)\,ds = f(x) \qquad (2)$$

Since the nucleus $\kappa(x)$ is supposed to be specified by a table of finite numerical values over the range of values of x considered, we may apply Prony's method of interpolation by exponentials in order to represent it analytically in the form of a sum of μ exponentials

$$\kappa(x) = Pe^{px} + Qe^{qx} + Re^{rx} + \dots + Ve^{vx}, \qquad (13)$$

where $(P, Q, R, \dots, V, p, q, r, \dots, v)$ are constants which are chosen so as to give the closest possible representation of the given numerical values.

Although Prony's method is more than a century old, it does not appear to be widely known or to have found its way into any text book, and, as his original paper is perhaps not accessible to many readers, I may be justified in giving here a brief notice of it.

Suppose that $\kappa(x)$ is given numerically for a certain range of values of x. Take any set of values of x equally spaced within this range, say $x = 0, \omega, 2\omega, 3\omega, 4\omega, \dots$, and let the corresponding values of $\kappa(x)$ be $\kappa_0, \kappa_1, \kappa_2, \kappa_3, \dots$. Now if $\kappa(x)$ could be represented *exactly* in the form of a sum of μ exponentials, say,

$$Pe^{px} + Qe^{qx} + Re^{rx} + \dots + Ve^{vx},$$

then $\kappa(x)$ would satisfy a linear difference-equation of the form

$$A\kappa_{n+\mu} + B\kappa_{n+\mu-1} + C\kappa_{n+\mu-2} + \dots + M\kappa_n = 0,$$

where the roots of the algebraic equation

$$Ax^{\mu} + Bx^{\mu-1} + Cx^{\mu-2} + \dots + M = 0$$

would be

$$e^{p\omega},\ e^{q\omega},\ e^{r\omega},\ \dots\ e^{v\omega}.$$

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* It need scarcely be said that if $\kappa(0)$ vanishes we differentiate again.

Prony's method, which is based on this fact, is to write down a set of linear equations,

$$\begin{aligned}
&A\kappa_{\mu}+B\kappa_{\mu-1}+C\kappa_{\mu-2}+\dots+M\kappa_0=0\\
&A\kappa_{\mu+1}+B\kappa_{\mu}+C\kappa_{\mu-1}+\dots+M\kappa_1=0\\
&A\kappa_{\mu+2}+B\kappa_{\mu+1}+C\kappa_{\mu}+\dots+M\kappa_2=0\\
&A\kappa_{\mu+3}+B\kappa_{\mu+2}+C\kappa_{\mu+1}+\dots+M\kappa_3=0
\end{aligned}$$

where the quantities $\kappa_0, \kappa_1, \kappa_2, \kappa_3, \dots,$ are known, since $\kappa(x)$ is a known tabulated function, and by the ordinary method of Least Squares to find the values of A, B, C, …, M, which most nearly satisfy the equations, then with these values of A, B, C, …, M, to form the algebraic equation

$$Ax^{\mu}+Bx^{\mu-1}+Cx^{\mu-2}+\dots+M=0,$$

and find its roots, these roots will be $e^{p\omega}, e^{q\omega}, e^{r\omega}, \dots e^{v\omega}$, and thus $p, q, r, \dots, v$, are determined. Knowing $p, q, r, \dots, v$, we have a set of linear equations to determine the coefficients P, Q, R, …, V, and these also are to be solved by the method of Least Squares.

Taking then this form (13) *for the nucleus* $\kappa(x)$, *we shall show that the* integral-equation (2) may be satisfied by a solution of the form

$$\phi(x)=f(x)-\int_0^x K(x-s)\,f(s)\,ds, \tag{14}$$

where the solving function $K(x)$ is also a sum of μ exponentials, say

$$K(x)=Ae^{\alpha x}+Be^{\beta x}+Ce^{\gamma x}+\dots+Ne^{\nu x}. \tag{15}$$

To prove this, we remark first that the existence-theorems established by Volterra justify us in assuming for the solution the form (14), where $K(x)$ is now the function to be determined.

In (14) put $\kappa(x)$ for $f(x)$: thus

$$\phi(x)=\kappa(x)-\int_0^x K(x-s)\,\kappa(s)\,ds,$$

which gives the value of $\phi(x)$ corresponding to this value of $f(x)$.

Putting $(x-s)$ for s in the integral, we have

$$\phi(x)=\kappa(x)-\int_0^x K(s)\,\kappa(x-s)\,ds.$$

Comparing this with the integral-equation (2), after replacing $f(x)$ by $\kappa(x)$ in the latter, we have

$$\phi(x)=K(x),$$

and therefore the pair of functions

$$\phi(x)=K(x),\qquad f(x)=\kappa(x),$$

satisfy the integral-equation; that is to say,

$$K(x)+\int_0^x K(s)\,\kappa(x-s)\,ds=\kappa(x). \tag{16}$$

In this equation substitute the value (13) for $\kappa(x)$ and the value (15) for $K(x)$. Thus we have

$$Ae^{\alpha x}+Be^{\beta x}+Ce^{\gamma x}+\dots+Ne^{\nu x}$$

$$+\int_0^x\{Ae^{\alpha s}+Be^{\beta s}+Ce^{\gamma s}+\dots+Ne^{\nu s}\}\{Pe^{px-ps}+Qe^{qx-qs}+\dots+Ve^{vx-vs}\}\,ds$$

$$=Pe^{px}+Qe^{qx}+Re^{rx}+\dots+Ve^{vx}$$

Equating coefficients of $e^{\alpha x}$ on the two sides of this equation, we have

$$\frac{P}{\alpha-p}+\frac{Q}{\alpha-q}+\dots+\frac{V}{\alpha-v}+1=0$$

Similarly by equating coefficients of $e^{\beta x}$

$$\frac{P}{\beta-p}+\frac{Q}{\beta-q}+\dots+\frac{V}{\beta-v}+1=0,$$

and so on these equations show that α, β, γ, ..., ν, are the roots of the algebraic equation in x

$$\frac{P}{x-p}+\frac{Q}{x-q}+\frac{R}{x-r}+\dots+\frac{V}{x-v}+1=0 \tag{17}$$

This enables us to determine α, β, γ, ...

Next equating coefficients of e^{px} on the two sides of the equation, we have

$$\left.\begin{array}{l}
\dfrac{A}{\alpha-p}+\dfrac{B}{\beta-p}+\dfrac{C}{\gamma-p}+\dots+\dfrac{N}{\nu-p}+1=0\\[2ex]
\text{and similarly}\quad \dfrac{A}{\alpha-q}+\dfrac{B}{\beta-q}+\dfrac{C}{\gamma-q}+\dots+\dfrac{N}{\nu-q}+1=0\\[1ex]
\dots\dots\dots\dots\dots\dots\\
\dots\dots\dots\dots\dots\dots\\[1ex]
\dfrac{A}{\alpha-v}+\dfrac{B}{\beta-v}+\dfrac{C}{\gamma-v}+\dots+\dfrac{N}{\nu-v}+1=0
\end{array}\right\} \tag{18}$$

Since $(\alpha, \beta, \gamma, \dots, \nu)$ and $(p, q, r, \dots, v)$ are known, these equations (18) enable us to determine A, B, C, ..., N, and we see that if the constants $(\alpha, \beta, \gamma, \dots, \nu)$ and (A, B, C, ..., N) are determined by equations (17) and (18), the equation (16) is satisfied by the value (15) of $K(x)$.

The value of $K(x)$ may be obtained in a more explicit form in the following manner If we eliminate A, B, C, $\ldots$, N, determinantly from the equations (15) and (18), we have

$$K(x)\begin{vmatrix} \frac{1}{\alpha-p} & \frac{1}{\beta-p} & \cdots & \frac{1}{\nu-p} \\ \frac{1}{\alpha-q} & \frac{1}{\beta-q} & \cdots & \frac{1}{\nu-q} \\ \vdots & & & \vdots \\ \frac{1}{\alpha-v} & \frac{1}{\beta-v} & \cdots & \frac{1}{\nu-v} \end{vmatrix} = -e^{\alpha x}\begin{vmatrix} 1 & \frac{1}{\beta-p} & \cdots & \frac{1}{\nu-p} \\ 1 & \frac{1}{\beta-q} & \cdots & \frac{1}{\nu-q} \\ \vdots & & & \vdots \\ 1 & \frac{1}{\beta-v} & \cdots & \frac{1}{\nu-v} \end{vmatrix} - e^{\beta x}\begin{vmatrix} \frac{1}{\alpha-p} & 1 & \cdots & \frac{1}{\nu-p} \\ \frac{1}{\alpha-q} & 1 & \cdots & \frac{1}{\nu-q} \\ \vdots & & & \vdots \\ \frac{1}{\alpha-v} & 1 & \cdots & \frac{1}{\nu-v} \end{vmatrix} - \ldots$$

The determinants which occur in this equation are of the kind known as *alternants*, and may be factorised by known methods* Performing the factorisation, we have

$$K(x) = -\frac{(\alpha-p)(\alpha-q)(\alpha-r)\cdots(\alpha-v)}{(\alpha-\beta)(\alpha-\gamma)\cdots(\alpha-\nu)}e^{\alpha x} - \frac{(\beta-p)(\beta-q)(\beta-r)\cdots(\beta-v)}{(\beta-\alpha)(\beta-\gamma)\cdots(\beta-\nu)}e^{\beta x} - \ldots - \frac{(\nu-p)(\nu-q)(\nu-r)\ldots(\nu-v)}{(\nu-\alpha)(\nu-\beta)\ldots(\nu-\mu)}e^{\nu x}$$

Combining our results we have

THEOREM 2—*The solution of the integral-equation*

$$\phi(x) + \int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x),$$

where the nucleus $\kappa(x)$ is supposed to be given numerically, and to have been expressed by Prony's method of interpolation in the form

$$\kappa(x) = Pe^{px} + Qe^{qx} + Re^{rx} + \ldots + Ve^{vx},$$

is

$$\phi(x) = f(x) - \int_0^x K(x-s)\,f(s)\,ds,$$

where

$$K(x) = -\frac{(\alpha-p)(\alpha-q)\cdots(\alpha-v)}{(\alpha-\beta)(\alpha-\gamma)\cdots(\alpha-\nu)}e^{\alpha x} - \frac{(\beta-p)(\beta-q)\ldots(\beta-v)}{(\beta-\alpha)(\beta-\gamma)\cdots(\beta-\nu)}e^{\beta x} - \ldots - \frac{(\nu-p)(\nu-q)\ldots(\nu-v)}{(\nu-\alpha)(\nu-\beta)\ldots(\nu-\mu)}e^{\nu x}, \quad (19)$$

and where $\alpha, \beta, \gamma, \ldots, \nu$, are the roots of the algebraic equation in x

$$\frac{P}{x-p} + \frac{Q}{x-q} + \frac{R}{x-r} + \ldots + \frac{V}{x-v} + 1 = 0$$

---

* The evaluation of alternants of this type is due to Cauchy, 'Exercices d'An.,' vol. 2, p 151 (1841).

It is obvious from the last equation that if $p, q, r, \ldots, v$, are arranged in ascending order of magnitude, and, if P, Q, R, ..., V, are all positive, then the lowest root α is less than p, the next root β is between p and q, and so on.

If the number of exponential terms in (13) is supposed to increase indefinitely, the representation of the nucleus $\kappa(x)$ becomes a Dirichlet's series, and by (19) the solving function $\mathrm{K}(x)$ is then also a Dirichlet's series, formed with exponentials $e^{\alpha x}$, $e^{\beta x}$, $e^{\gamma x}$, ..., whose exponents $\alpha, \beta, \gamma, \ldots$, are the roots of equation (17), which now becomes a transcendental equation. A rigorous examination of convergence, etc., in this limit-process would be necessary to establish the theorem which appears to be indicated, namely, that *in the solution of a Poisson's integral-equation whose nucleus is expressible as a Dirichlet's series, the solving-function is also expressible as a Dirichlet's series, but with a different set of exponents for the exponentials.*

§ 4. *An Alternative Solution of Integral-Equations of Poisson's Type.*

Theorem 2 above supplies what I think will be found to be in general the most convenient method of solving integral-equations of Poisson's type numerically. But in certain cases the nucleus $\kappa(x)$ may be given as a polynomial, or it may happen, when $\kappa(x)$ is given in the form of a numerical table, that it is preferred for some reason to apply ordinary interpolation and express $\kappa(x)$ approximately as a polynomial, rather than to apply Prony's method of interpolation and express $\kappa(x)$ as a sum of exponentials.* We shall therefore now consider the problem of integrating an integral-equation of Poisson's type

$$\phi(x)+\int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x) \qquad (2)$$

when the nucleus $\kappa(x)$ is expressed as a polynomial.

The solution of this problem may be deduced as a limiting case from that solved in § 3. For, in equation (13), suppose that $p, q, r, \ldots, v$ (being n in number), each tends to zero, while P, Q, R, ..., V, increase indefinitely in such a way that

$$\left.\begin{aligned} \mathrm{P}+\mathrm{Q}+\mathrm{R}+\ldots+\mathrm{V} &= \kappa_0 \\ \mathrm{P}p+\mathrm{Q}q+\mathrm{R}r+\ldots+\mathrm{V}v &= \kappa_1 \\ \ldots\ldots\ldots\ldots &\ldots\ldots \\ \ldots\ldots\ldots\ldots &\ldots\ldots \\ \mathrm{P}p^{n-1}+\mathrm{Q}q^{n-1}+\mathrm{R}r^{n-1}+\ldots+\mathrm{V}v^{n-1} &= \kappa_{n-1} \end{aligned}\right\} \qquad (20)$$

---

* One great advantage of Prony's method is that each exponential term involves *two* disposable constants (*e.g.*, the term $\mathrm{P}e^{px}$ involves P and p), whereas a term of a polynomial involves only *one* disposable constant (*e.g.*, px^3 involves only p), and therefore it is in general possible to obtain as high a degree of accuracy in approximating with n exponential terms as with a polynomial of $2n$ terms.

where $\kappa_0, \kappa_1, \kappa_2, \ldots, \kappa_{n-1}$, are finite. Then from equation (13) we have

$$\kappa(x) = P\left(1+px+\frac{p^2x^2}{2!}+\ldots\right)+Q\left(1+qx+\frac{q^2x^2}{2!}+\ldots\right)+\ldots$$

$$+V\left(1+vx+\frac{v^2x^2}{2!}+\ldots\right)$$

$$= \kappa_0+\kappa_1 x+\frac{\kappa_2 x^2}{2!}+\ldots+\frac{\kappa_{n-1}x^{n-1}}{(n-1)!},$$

the terms in higher powers of x vanishing.

Moreover, the equation (17), whose roots are $\alpha, \beta, \gamma, \ldots, \nu$, may be written

$$P\left(\frac{1}{x}+\frac{p}{x^2}+\frac{p^2}{x^3}+\ldots\right)+Q\left(\frac{1}{x}+\frac{q}{x^2}+\frac{q^2}{x^3}+\ldots\right)+\ldots+V\left(\frac{1}{x}+\frac{v}{x^2}+\frac{v^2}{x^3}+\ldots\right)+1=0,$$

which by (20) becomes

$$\frac{\kappa_0}{x}+\frac{\kappa_1}{x^2}+\frac{\kappa_2}{x^3}+\ldots+\frac{\kappa_{n-1}}{x^n}+1 = 0,$$

the terms in higher powers of $(1/x)$ vanishing, so that $\alpha, \beta, \gamma, \ldots, \nu$, are now the roots of the algebraic equation

$$x^n+\kappa_0 x^{n-1}+\kappa_1 x^{n-2}+\ldots+\kappa_{n-1} = 0$$

The equation (19) for the determination of the solving function $K(x)$ now becomes

$$K(x) = -\frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)\ldots(\alpha-\nu)}e^{\alpha x}-\frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)\ldots(\beta-\nu)}e^{\beta x}-\ldots$$

$$\ldots-\frac{\nu^n}{(\nu-\alpha)(\nu-\beta)\ldots(\nu-\mu)}e^{\nu x}$$

and thus, collecting our results, we have

THEOREM 3.—*The solution of the integral-equation*

$$\phi(x)+\int_0^x \phi(s)\,\kappa(x-s)\,ds = f(x),$$

where the nucleus $\kappa(x)$ is supposed to be given numerically, and to have been expressed by the ordinary methods of interpolation in the form

$$\kappa(x) = \kappa_0+\kappa_1 x+\frac{\kappa_2 x^2}{2!}+\frac{\kappa_3 x^3}{3!}+\ldots+\frac{\kappa_{n-1}x^{n-1}}{(n-1)!}, \tag{21}$$

is

$$\phi(x) = f(x)-\int_0^x K(x-s)f(s)\,ds, \tag{22}$$

where

$$K(x) = -\frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)\ldots(\alpha-\nu)}e^{\alpha x}-\frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)\ldots(\beta-\nu)}e^{\beta x}-\ldots$$

$$\ldots-\frac{\nu^n}{(\nu-\alpha)(\nu-\beta)\ldots(\nu-\mu)}e^{\nu x}, \tag{23}$$

and where $\alpha, \beta, \gamma, \dots, \nu$, are the roots of the algebraic equation in x

$$x^n + \kappa_0 x^{n-1} + \kappa_1 x^{n-2} + \dots + \kappa_{n-1} = 0 \tag{24}$$

If this equation (24) has a pair of equal roots, terms of the type $xe^{\lambda x}$ will occur in $K(x)$, thus, if

$$\kappa(x) = -2p + p^2 x,$$

we find
$$K(x) = -p(2+px)e^{px}$$

An interesting expansion is obtained by making n increase indefinitely in this theorem We then have the solving problem $K(x)$ of the general Poisson's integral-equation expressed in the form of a Dirichlet's series, or, at any rate, a series of exponentials with real or complex arguments This appears to indicate a new field of analysis, in which Dirichlet's series present themselves naturally. But a thorough investigation of convergence would be necessary for the rigorous establishment of this result, and, indeed, we can show by simple examples that the expression obtained by making n tend to infinity will not necessarily be a Dirichlet's series, even though, for all finite values of n, it is a sum of exponentials For instance, if

$$\kappa(x) = p + p^2 x + \frac{p^3 x^2}{2!} + \frac{p^4 x^3}{3!} + \dots + \frac{p^n x^{n-1}}{(n-1)!},$$

so that $\kappa_0 = p, \quad \kappa_1 = p^2, \quad \kappa_2 = p^3, \quad \dots, \quad \kappa_{n-1} = p^n,$

we see from (23) and (24) that $K(x)$ is equal to the sum of the residues of

$$-\frac{(1-p/t)e^{tx}}{1-p^{n+1}/t^{n+1}}$$

at its n poles, which are the $(n+1)$th roots of unity other than unity itself This sum of residues is evidently a sum of exponentials in x, one corresponding to each pole, so long as n is finite, but, when n increases indefinitely, we have

$$K(x) = \text{coefficient of } 1/t \text{ in } -(1-p/t)e^{tx} = p,$$

so that when the nucleus is $\kappa(x) = pe^{px}$, the solving function is $K(x) = p$, which is not a series of exponentials

§ 5 *A Further Alternative Solution of Integral-Equations of Poisson's Type*

Hitherto we have supposed the nucleus of the integral-equation to be given by a numerical table, and to be represented, by use of the methods of the interpolation theory, as a sum of exponentials, or as a polynomial. If, however, in an equation of Poisson's type

$$\phi(x) + \int_0^x \kappa(x-s)\phi(s)\,ds = f(x)$$

the expansion of the nucleus $\kappa(x)$ as a Taylor's series is known, say

$$\kappa(x) = \kappa_0 + \kappa_1 x + \frac{\kappa_2 x^2}{2!} + \frac{\kappa_3 x^3}{3!} + \ldots, \tag{25}$$

then we shall show that the solving-function $K(x)$ may be written down at once as a Taylor's series. For, the solution of the integral-equation in terms of the solving-function being

$$\phi(x) = f(x) - \int_0^x K(x-s) f(s)\, ds,$$

we have already in equation (16) shown that

$$K(x) + \int_0^x K(s)\, \kappa(x-s)\, ds = \kappa(x).$$

Putting $x = 0$ in this, we have

$$K_0 = \kappa_0, \tag{26}$$

where $K_0, K_1, K_2, \ldots$, are used in order to denote the values of $K(x)$ and its successive differential coefficients at $x = 0$.

Differentiating (16), we have

$$K'(x) + \kappa_0 K(x) + \int_0^x K(s)\, \kappa'(x-s)\, ds = \kappa'(x). \tag{27}$$

Putting $x = 0$ in (27)

$$K_1 + \kappa_0 K_0 = \kappa_1. \tag{28}$$

Differentiating (27)

$$K''(x) + \kappa_0 K'(x) + \kappa_1 K(x) + \int_0^x K(s)\, \kappa''(x-s)\, ds = \kappa''(x). \tag{29}$$

Putting $x = 0$ in (29),

$$K_2 + \kappa_0 K_1 + \kappa_1 K_0 = \kappa_2. \tag{30}$$

If now the linear equations (26), (28), (30), . . ., be solved for $K_0, K_1, K_2, \ldots$, we obtain

$$K_0 = \kappa_0$$

$$K_1 = -\begin{vmatrix} \kappa_0 & 1 \\ \kappa_1 & \kappa_0 \end{vmatrix}$$

$$K_2 = \begin{vmatrix} \kappa_0 & 1 & 0 \\ \kappa_1 & \kappa_0 & 1 \\ \kappa_2 & \kappa_1 & \kappa_0 \end{vmatrix}, \text{ etc.}$$

Let us now consider the convergence of the series

$$K_0 + K_1 x + \frac{K_2}{2!} x^2 + \frac{K^3}{3!} x^3 + \ldots$$

This series will evidently be absolutely convergent for all finite values of x, provided the series

$$K_0 + K_1 x + K_2 x^2 + K_3 x^3 + \ldots$$

converges within a circle of non-zero radius But the equations (26), (28), (30), .., may be comprehended in the single formal equation

$$(\kappa_0+\kappa_1x+\kappa_2x^2+\ldots)(1+\kappa_0x+\kappa_1x^2+\kappa_2x^3+\ldots) \equiv K_0+K_1x+K_2x^2+\ldots$$

or

$$K_0+K_1x+K_2x^2+K_3x^3+\ldots = \frac{1}{x}\left\{1-\frac{1}{1+\kappa_0x+\kappa_1x^2+\kappa_2x^3+\ldots}\right\},$$

and the expression on the right-hand side of this equation represents a holomorphic function within a circle of non-zero radius having the origin as centre, provided the power-series

$$1+\kappa_0x+\kappa_1x^2+\kappa_2x^3+\ldots$$

converges within a circle of non-zero radius and so represents a function whose singularities and zeroes are all at a finite distance from the origin Subject to this last condition, then (which is, as a matter of fact, unnecessarily stringent), the series

$$K_0+K_1x+K_2\frac{x^2}{2!}+K_3\frac{x^3}{3!}+\ldots$$

converges absolutely for all finite values of x We shall assume that it converges, since the computer will not make use of this method unless the convergence of this series is so rapid as to be obvious. Then, combining our results, we have

THEOREM 4.—*The solution of the integral-equation*

$$\phi(x)+\int_0^x\phi(s)\,\kappa(x-s)\,ds=f(x)$$

where the nucleus $\kappa(x)$ *is supposed to be expansible in a Taylor series*

$$\kappa(x)=\kappa_0+\kappa_1x+\frac{\kappa_2x^2}{2!}+\frac{\kappa_3x^3}{3!}+\ldots$$

is

$$\phi(x)=f(x)-\int_0^x K(x-s)f(s)\,ds,$$

where

$$K(x)=\kappa_0-\begin{vmatrix}\kappa_0 & 1\\ \kappa_1 & \kappa_0\end{vmatrix}x+\begin{vmatrix}\kappa_0 & 1 & 0\\ \kappa_1 & \kappa_0 & 1\\ \kappa_2 & \kappa_1 & \kappa_0\end{vmatrix}\frac{x^2}{2!}-\begin{vmatrix}\kappa_0 & 1 & 0 & 0\\ \kappa_1 & \kappa_0 & 1 & 0\\ \kappa_2 & \kappa_1 & \kappa_0 & 1\\ \kappa_3 & \kappa_2 & \kappa_1 & \kappa_0\end{vmatrix}\frac{x^3}{3!}+\ldots \qquad (31)$$

For the benefit of those who may find it convenient to use this solution, it may be well to add some remarks on the computation of the determinantal coefficients which occur in it A numerical determinant should never, or scarcely ever, be evaluated by expanding it, it should be evaluated by reducing it successively to determinants of lower order, without expanding. To do this, we first notice whether any of the elements in the determinant is unity; if not, we reduce one of the elements to unity by dividing the row or column which contains it by that element Having done this, to evaluate, *e.g.*, the fifth-order determinant

$$\begin{vmatrix} 1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix}$$

the rule is Strike out the row and column which contain the unit element, and subtract from each of the other elements the product of the elements which are situated at the feet of the perpendiculars from this element on the deleted row and column Thus the above determinant becomes

$$\begin{vmatrix} b_2-a_2b_1 & b_3-a_3b_1 & b_4-a_4b_1 & b_5-a_5b_1 \\ c_2-a_2c_1 & c_3-a_3c_1 & c_4-a_4c_1 & c_5-a_5c_1 \\ d_2-a_2d_1 & d_3-a_3d_1 & d_4-a_4d_1 & d_5-a_5d_1 \\ e_2-a_2e_1 & e_3-a_3e_1 & e_4-a_4e_1 & e_5-a_5e_1 \end{vmatrix}$$

and in the same way we may reduce this fourth-order determinant to a third-order determinant, and so on

If the unit element is not the leading element, we must multiply the whole determinant by ±1, according as the sum of the row-number of the deleted row and the column-number of the deleted column is even or odd

§ 6 *A Combination of the Solution of* § 4 *with that of* § 5.

A form of solution which in some cases proves useful is obtained by combining the solution of § 4 with that of § 5 This happens when the graph of the nucleus $\kappa(x)$ in the part of it over which the integration takes place, is not very different from that of a polynomial of low degree, so that the first few of the coefficients $\kappa_0, \kappa_1, \kappa_2, \ldots$, are of preponderant importance as compared with those that succeed them In this case it is advantageous to take out of the series (31) the terms which depend solely on these important coefficients, and, by summing them, to obtain a new form for $K(x)$ which can be more readily computed, this, as we shall now show, may be done, the extracted part of $K(x)$ being in the exponential form which was obtained in § 4.

Suppose, for instance, that κ_0 and κ_1 are important, but the succeeding coefficients $\kappa_2, \kappa_3,$ are comparatively small. The terms in (31) which depend only on κ_0 and κ_1 are evidently

$$\kappa_0 - \begin{vmatrix} \kappa_0 & 1 \\ \kappa_1 & \kappa_0 \end{vmatrix} x + \begin{vmatrix} \kappa_0 & 1 & 0 \\ \kappa_1 & \kappa_0 & 1 \\ 0 & \kappa_1 & \kappa_0 \end{vmatrix} \frac{x^2}{2!} - \begin{vmatrix} \kappa_0 & 1 & 0 & 0 \\ \kappa_1 & \kappa_0 & 1 & 0 \\ 0 & \kappa_1 & \kappa_0 & 1 \\ 0 & 0 & \kappa_1 & \kappa_0 \end{vmatrix} \frac{x^3}{3!} + \ldots \qquad (32)$$

Now if u_n denote the determinant

$$(-)^{n-1}\begin{vmatrix} \kappa_0 & 1 & 0 & 0 & 0 \\ \kappa_1 & \kappa_0 & 1 & 0 & 0 \\ 0 & \kappa_1 & \kappa_0 & 1 & 0 \\ 0 & 0 & \kappa_1 & \kappa_0 & 1 \\ & & & & \end{vmatrix} \quad (n \text{ rows})$$

we have by expanding in terms of the elements of the first row

$$u_n = -\kappa_0 u_{n-1} - \kappa_1 u_{n-2}.$$

The solution of this difference-equation is

$$u_n = A\alpha^n + B\beta^n$$

where α and β are the roots of the quadratic $x^2+\kappa_0 x+\kappa_1 = 0$, and A and B are independent of n. Moreover, since

$$u_1 = \kappa_0 = -(\alpha+\beta) = -\frac{\alpha^2-\beta^2}{\alpha-\beta},$$

and
$$u_2 = -\kappa_0^2+\kappa_1 = -(\alpha^2+\alpha\beta+\beta^2) = -\frac{\alpha^3-\beta^3}{\alpha-\beta},$$

we see that
$$A = -\frac{\alpha}{\alpha-\beta}, \qquad B = \frac{\beta}{\alpha-\beta},$$

and therefore
$$u_n = -\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$$

The expression (32) may thus be written

$$-\frac{\alpha^2-\beta^2}{\alpha-\beta}-\frac{\alpha^3-\beta^3}{\alpha-\beta}x-\frac{\alpha^4-\beta^4}{\alpha-\beta}\frac{x^2}{2!}-\frac{\alpha^5-\beta^5}{\alpha-\beta}\frac{x^3}{3!}+ \ldots,$$

or
$$-\frac{\alpha^2 e^{\alpha x}-\beta^2 e^{\beta x}}{\alpha-\beta}$$

Thus, we have

THEOREM 6.—*In Theorem 4 the formula* (31) *may be replaced by*

$$K(x) = -\frac{\alpha^2 e^{\alpha x}}{\alpha-\beta}-\frac{\beta^2 e^{\beta x}}{\beta-\alpha}+\kappa_2\frac{x^2}{2!}+(\kappa_3-2\kappa_0\kappa_2)\frac{x^3}{3!}+\ldots,$$

where α and β are the roots of the quadratic $x^2+\kappa_0 x+\kappa_1 = 0$. This form of $K(x)$ is preferable for purposes of computation when κ_0 and κ_1 are of preponderant importance as compared with κ_2, κ_3, ...

We may in this way replace the terms of (31) to any extent by exponentials, and it is evident that when the terms thus replaced are those which involve $\kappa_0, \kappa_1, \ldots \kappa_{n-1}$ without involving $\kappa_n, \kappa_{n+1}, \ldots$, the exponential terms which are obtained are precisely those which would be obtained as the value of $K(x)$ in Theorem 3, if $\kappa(x)$ were the polynomial

$$\kappa_0+\kappa_1 x+\kappa_2\frac{x^2}{2!}+\ldots+\kappa_{n-1}\frac{x^{n-1}}{(n-1)!}.$$

The Parent of Actinium

By FREDERICK SODDY, F.R.S., and JOHN A. CRANSTON, B.Sc., Carnegie Research Scholar in the University of Aberdeen

(Received December 28, 1917.)

It has been recognised for more than a decade that actinium cannot be a primary radio-element, but hitherto all attempts to obtain evidence of its continuous production, in the same way as has been established for polonium and for radium, have met with no success. As our knowledge of radioactive change has become precise, the conceivable modes in which actinium could originate have been narrowed down and, one by one, experimentally eliminated without the parent of actinium having been discovered. In the present paper an account is given of experiments which have been successful in separating from uranium minerals preparations initially free from actinium but producing that element in the course of years. In 1913, a minute production of actinium, almost too small to be confident of, was recorded by one of us in the old preparations of uranium-X separated from 50 kgrm. of uranyl nitrate in 1909.* The subsequent history of these preparations has confirmed this minute growth, and placed it beyond all doubt, although even now, after eight years, it is still very small in comparison with the growth in the later preparations separated from uranium minerals.

Before dealing with the newer experiments, the opportunity will be taken of giving a fairly full account of the long and involved history of the problem, and of correlating and bringing up to date the evidence that can be derived from these old uranium-X preparations. The experiments were undertaken when the course of the disintegration of uranium and its connection with radium was quite obscure, but the experiments afford valuable data, which has never been properly discussed in the light of recent discoveries, on the possible modes of origin of actinium. With regard to the new work, in the absence of one of us on military service since 1915, the experiments were continued for a time by Miss Ada Hitchens, B.Sc., Carnegie Research Scholar, until she also left to engage in war duties. Her valuable assistance has contributed very materially to the definiteness of the conclusions that it has been possible to arrive at.

History of the Problem.

Although discovered in 1899, by Debierne, actinium still remains probably the least known of all the radio-elements. The statement does not apply to its numerous family of disintegration products, which are quite as well known as those of radium and of thorium, and the existence of which established beyond all doubt the claim of actinium to be considered a new radio-element. The earlier attempts at the chemical and radioactive characterisation of actinium are of little present significance, for the doubt remains as to how far they refer to actinium itself rather than to its products, radioactinium and actinium-X, or even to ionium, with which the earlier preparations were undoubtedly admixed. Nothing can be said definitely about its period, except that it must be at least of the order of a decade, but may be almost indefinitely longer. The observations by Mme. Curie as to a marked decay occurring in the β-radiation of an old preparation of actinium, to the extent of some 10 per cent. in three years, have not unfortunately been completed or confirmed. They indicated a period of only some 30 years for this element, which it is difficult to accept, as in this case one would have expected evidence of the production of actinium to have been obtained fairly easily.

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* F. Soddy, 'Chem. News,' vol. 107, p. 97 (1913).

The researches of Boltwood[*] on the relative proportions of the total α-activity of uranium minerals were the first to throw light on the genetic relations of actinium. The proportionality between the quantities of uranium and actinium which he found, and which has been confirmed by others, indicated that uranium must be the ultimate parent of actinium as well as of radium. But of the total α activity of the mineral only some 6 per cent. was contributed by the whole actinium series, whereas the radium series contributes some 65 per cent., and the uranium and ionium the remaining 29 per cent. It was pointed out by Rutherford in 1906[†] that on this account actinium could not be an intermediate member of the uranium-radium-polonium disintegration series, for in this case the proportion of the α-radiation contributed by its products would be somewhat greater than that contributed by the radium series. He concluded that it was not a lineal descendant of uranium in the same sense as radium is, but was derived from uranium in much less amount than the products of the radium series. To account for the simultaneous production of two products in different proportions he suggested that, in the rearrangement of the parts of the atom after the expulsion of the α-particle, more than one fairly stable arrangement may be possible, so that two or more products would result, which, though of equal atomic mass, would exhibit differences in chemical properties and be capable of separating from one another, but which would not necessarily be formed in equal amounts.

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[*] B. Boltwood, 'Amer. J. Sci.,' IV, vol. 25, p. 269 (1908).

[†] 'Radioactive Transformations,' 1906, pp. 170 and 176.

A slightly different explanation was offered somewhat later,* which ascribed the formation of the branch actinium series to the simultaneous existence of two types of instability in the same atom at the same time, resulting in two modes of disintegration proceeding simultaneously as if each alone were in progress according to the law of simple disintegration The ratio between the quantities of the two products resulting is, then, that between the periods of the two separate simultaneous changes, whereas the radiation resulting from both modes of disintegration decays exponentially according to the same period, which is the sum of the two separate periods Definite cases of the branching of a disintegration series were established later by the work of Fajans, and of Marsden and Darwin, for the cases of the C-members of the radium and the thorium series, and it has been proved that the C-member disintegrates dually in two ways In the one mode an α-particle is expelled, followed by a β-particle in the next change, while in the other mode the order is reversed All attempts to separate the C-member into two different substances responsible for the two modes of change have failed and both the α- and β-rays expelled in the dual disintegration decay with the same period Hence, it is not sufficient to suppose that the C-member is a mixture of isotopes, but also that each must have the same period. This is tantamount in its experimental consequences to regarding the C-member as homogeneous, disintegrating dually according to the theory of multiple disintegration

The simple generalisation that in an α-ray change the element shifts its position in the Periodic Table by two places in the direction of diminishing mass, and in a β-ray change by one place in the opposite direction, put the problem of the origin of actinium in a very definite form In the first place, it showed that if two products were simultaneously produced in an α-ray change, they would be isotopic and chemically inseparable, so that they would only be distinguishable by a difference of period. On the other hand, the scheme of disintegration adopted for the C-members, in agreement with the theory of multiple disintegration, was, so far as it could be tested, in conformity with the generalisation Fleck showed that the C-members are isotopic with bismuth, whereas thorium-D formed from thorium-C in the α-mode is isotopic with thallium, but radium-D, formed from radium-C in the β-mode, followed by a subsequent α-change, is isotopic with lead. This confirmation made it possible to apply the theory of multiple disintegration to the problem of the origin of actinium with considerable confidence

With regard to actinium itself, it had long been expected that it would occupy the hitherto vacant place in the Periodic Table between radium and thorium, on account of its resemblance in chemical character to the rare-earth family of elements, and in particular, as Giesel showed, to lanthanum Fleck proved this by showing that mesothorium-II is isotopic with actinium As mesothorium-II is formed in a β-ray change from mesothorium-I, an isotope of radium, it, and also actinium, must occupy the place indicated.*

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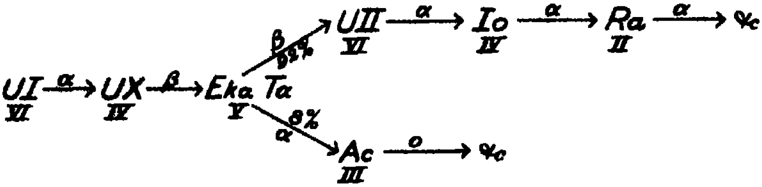
* F. Soddy, 'Phil. Mag' [6], vol. 18, p. 739 (1909).

Being in Group III of the Periodic Table, the parent of actinium must be either an isotope of radium, if it is formed in a β-ray change, or, if formed in an α-ray change, its parent must occupy the vacant place in Group V between thorium and uranium The nearest analogue of this missing element is tantalum, and hence the parent of actinium in this case must be a long-lived, α-ray-giving, "eka-tantalum," still undiscovered The generalisation further showed that the place in question cannot be vacant, but must be occupied by the then unknown product of uranium-X, which is an isotope of thorium giving β-rays The first suggestion made is shown below. It was supposed that this unknown product in the VA family, like the C-members in the VB family, disintegrated dually, the main product being uranium-II in a β-ray change, and the branch product being actinium in an α-ray change †



The discovery soon after, by Fajans and Gohring,‡ that uranium-X consisted of two successive products, uranium-X_1 and uranium-X_2, both giving β-rays, and that uranium-X_2 was a very short-lived element, with period 1·65 minutes, responsible for the more penetrating β-rays of uranium-X, disproved this view Being the first occupant of the vacant place between uranium and thorium to be discovered, uranium-X_2 was given the distinctive name "Brevium."

The alternative suggestion that actinium was the product of radium, or an isotope of radium, in a β-ray change was then definitely disproved by the examination of a specimen of radium bromide, containing 13·2 mgrm. of radium element, prepared by Giesel 10 years before, and untouched chemically since preparation Not the least trace of radioactinium could be detected in this preparation, though the isotopic radiothorium, present in minute amount because of the existence in Joachimsthal pitchblende of an

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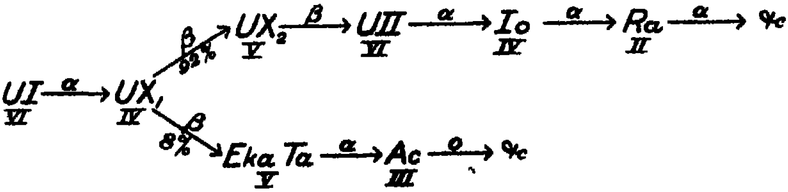
[*] A. Fleck, 'Trans. Chem. Soc.,' vol. 103, pp. 388 and 1052 (1913).

[†] F Soddy, 'Chem. News,' vol. 107, p. 97 (1913).

[‡] Fajans and Göhring, 'Physikal. Zeitsch.,' vol. 14, p. 877 (1913).

infinitesimal amount of thorium, was clearly detected* This has been confirmed by Paneth and Fajans with a preparation containing 180 mgrm of radium, six years old†

The remaining possibility that eka-tantalum was the direct parent of actinium was revived in a new form by Hahn and Meitner‡ Hitherto, it had been supposed that, as in the case of the C-members, the radiations accompanying the dual modes were different, being α-rays in the one case and β-rays in the other. Hahn and Meitner suggested that the change of uranium-X_1 might be dual, both modes giving β-rays, so that there would be two isotopes of eka-tantalum produced, one the known uranium-X_2, or brevium, the other giving actinium in an α-ray change This suggestion is shown below



This scheme still has to be regarded as a possible, if not the most probable, mode of origin of actinium, always provided that the isotope of eka-tantalum producing it has a very extended period of life For, if true, there must be a growth of α-radiation accompanying the decay of the β-radiation of the uranium-X. A laborious investigation on this very point was carried out by one of us in 1909§ No such growth could be detected in the several preparations of uranium-X separated from 50 kgrm. of uranyl nitrate, so that, if it occurred, it must be minute. The preparations always possessed an appreciable α-activity, which remained constant, while the β-rays decayed, and subsequently during the eight years since preparation. Making allowance for the fact that only 8 per cent of the atoms of uranium-X_1 produce this supposed parent of actinium, a re-examination of the data of these old experiments leads to the definite conclusion that, for Hahn and Meitner's scheme to be in accordance with them, the period of average life of the supposed parent of actinium cannot be less than 10,000 years. The growth of α-radiation, during the decay of the β-rays, would certainly have been detectable if the period had been less than this as a minimum.

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* F Soddy, 'Nature,' vol. 91, p. 634 (1913).

† Referred to by C Göhring, 'Physikal. Zeitschr.,' vol. 15, p. 642 (1914).

‡ Hahn and Meitner, 'Physikal. Zeitschr.,' vol. 14, p. 756 (1913).

§ F. Soddy, 'Phil Mag.' [6], vol. 18, p. 858 (1909), vol. 20, p. 342 (1910)

Thus, if λ_1 is the fraction changing per year of uranium-I, and λ_2 that of the supposed eka-tantalum, and if 0·92 of the atoms follow the radium and 0·08 the actinium mode of disintegration, the quantity of eka-tantalum produced per year is $0{\cdot}08\,\lambda_1$ times that of the originating uranium. Its α-radiation is $\lambda_2/1{\cdot}92\,\lambda_1 \times 0{\cdot}08\,\lambda_1$, or $0{\cdot}045\,\lambda_2$ times that of the originating uranium. The uranium-X in equilibrium with uranium is 35·5 days' production, and the α-radiation of the eka-tantalum resulting is $35{\cdot}5/365 \times 0{\cdot}045\,\lambda_2$, or $0{\cdot}0044\,\lambda_2$ times that of the originating uranium. But the uranium-X preparation last separated in September, 1909, weighed 78 mgrm., and initially contained the uranium-X in equilibrium with 5·05 kgrm. of uranium. Its α-activity is one-third of that of a similar surface of uranium oxide, and a change of 10 per cent., equal to the α-rays of 2·3 mgrm. of uranium, during the decay of the β-rays, would have been with certainty detected. That is to say, $0{\cdot}0044\,\lambda_2$ is less than $2{\cdot}3/5050000$, and therefore $1/\lambda_2$, the period of average life of the supposed eka-tantalum, is certainly greater than 10,000 years.

The subsequent growth of actinium by these preparations will be dealt with later.

The scheme which recommends itself as the most probable mode of origin of actinium, and which has been the starting point of the present investigation, follows that of Hahn and Meitner in regarding both modes of disintegration to be accompanied by the same type of radiation, but puts the branching-point still one member further back, and the rays emitted as α- instead of β-rays. In 1911 Antonoff* found that certain preparations of uranium-X, separated from uranium by addition of an iron salt and precipitating it by boiling, showed an abnormality in the soft β-radiation emitted, which decayed more rapidly than the penetrating β-rays during the first week from preparation, whereas uranium-X, separated by adsorption with barium sulphate, did not show this peculiarity. The effect he attributed to the existence of a new product of uranium, which he named uranium-Y, with a period of average life of 1·5 days, giving soft β-rays on disintegration, which are somewhat more penetrating than those given by uranium-X_1. He suggested that uranium-Y is probably the first member of the actinium series, the ratio between the radiations of uranium-X and uranium-Y being of the expected order for members in the radium and actinium branches respectively. Great difficulty was experienced in repeating these results, but they were subsequently confirmed, as regards the existence of uranium-Y and its probable connection with the actinium series.† It was shown that the chemical method of separating the uranium-X from the uranium is of no significance, but that the essential condition for the decay of uranium-Y to be observed is that the time of reaccumulation of the uranium-X after separation from the uranium must be short. Uranium-Y is isotopic with uranium X_1, and can only be detected by the difference in period. To account for its relation to uranium on the one hand, and uranium-X on the other, the following schemes were suggested.*

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* G. N. Antonoff, 'Phil. Mag.' [6], vol. 22, p. 419 (1911); vol. 26, p. 1068 (1913).

† F. Soddy, 'Phil. Mag.' [6], vol. 27, p. 215 (1914).

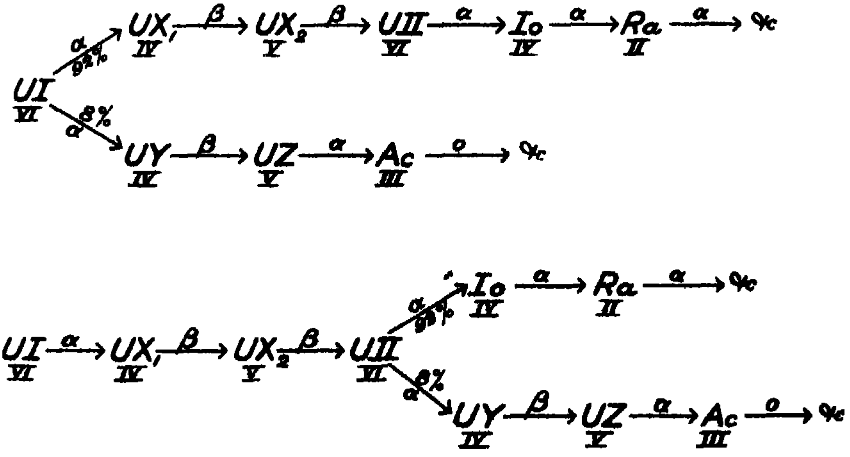

Fig. 1.

The two schemes shown (fig. 1), in which uranium-I in one case and uranium-II in the other are regarded as the parent of uranium-Y, are experimentally indistinguishable at present, and may therefore be regarded as one, but on general grounds the first seems the more probable. The new member, termed uranium-Z in the diagram, to indicate that it is the product of uranium-Y, is the supposed missing direct parent of actinium, occupying the eka-tantalum place in the Periodic Table, isotopic with uranium-X_2 or brevium, and distinguished from it by its long period and by its giving α- instead of β-rays.

According to this scheme no growth of α-rays during the decay of the β-rays of uranium-X is to be expected, for the uranium-Y is so short lived that, in the experiments referred to, hardly any would survive the purification process, after the separation from the uranium, and before the first measurements were made. On the other hand, the constant α-radiation of the preparations, assuming initial α-ray-giving impurities to have been all removed in former separations, would be due to the ionium formed between separations, since ionium is isotopic with uranium-X. The determination of the period of ionium as 100,000 years* enables the α-radiation from this source to be calculated. If λ_3 is the constant of ionium, 10^{-5} (year)$^{-1}$, the α-rays due to the ionium, in comparison with those of the originating uranium, should be $(0{\cdot}92/1{\cdot}92)\lambda_3 T$, where T is the time of accumulation in years. This is $5 \times 10^{-6}T$. For the preparation referred to T was 0·25 year, and the found α-activity, that of 22 mgrm. of uranium, was $1{\cdot}1 \times 10^{-6}$ of the originating uranium, 20 kgrm. Hence the α-radiation is almost exactly of the magnitude to be expected if it was due to ionium alone.

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* Compare also 'Chemical Society's Annual Reports,' vol. 10, p. 268 (1913).

But this exact agreement must be accidental, for all the ionium formed could not have been separated from the large quantity of uranium. Indeed, since uranium-X_1 is isotopic with ionium, it is possible to calculate fairly closely what proportion was separated, and this proportion amounts to one-third of that present. But the experiment indicates satisfactorily that the α-radiation of this uranium-X preparation from the fourth and last separation carried out, is not much greater than can be accounted for by the ionium alone, which somewhat limits the possibilities to be considered. As soon as the opportunity occurs, it is proposed to repeat these uranium-X separations, when it is to be expected that more definite data as to the periods in the early part of the uranium disintegration series will be thus obtained.

Quite recently, a new suggestion has been made as to the origin of actinium,† namely, that it is derived, through uranium-Y and eka-tantalum, from a third isotope of uranium, to which the name actino-uranium is given, which does not belong to the uranium-radium family, but is a primary radio-element, of atomic weight greater than that of uranium, viz., 240, that of uranium-I being 238. This explains the figure 238·16, found for the atomic weight of uranium, since $0{\cdot}08 \times 240 + 0{\cdot}92 \times 238 = 238{\cdot}16$, and also the integral values of the atomic weight of radium, 226, and of uranium-lead, 206. Assuming that there is constancy of proportionality between the uranium and actino-uranium in minerals, as the constancy of proportionality between uranium and actinium seems to indicate, the suggestion involves the further assumption, not considered by the author, that the periods of uranium-I and actino-uranium must be the same, and a mixture of isotopes in constant proportions, both having the same period, but giving different products, is indistinguishable experimentally from a single substance disintegrating dually. So far as the present problem is concerned, the suggestion may be regarded as a third possibility giving the same consequences as the two shown in fig. 1. At the same time, the atomic weight evidence and, also, as pointed out by the author, the difference in the value of the constants in the Geiger-Nuttall relation for the radium and actinium family are in favour of this suggestion, that the actinium and radium families may be quite distinct.[*] It is very desirable that the accepted conclusion as to the constancy between uranium and actinium in minerals should be more rigorously tested.

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* F. Soddy and Miss A. Hitchins, 'Phil. Mag.' [6], vol. 30, p. 209 (1915).

† A. Piccard, 'Arch. Sci. Phys. Nat.' [4], vol. 44, p. 161 (1917).

Growth of Actinium in Old Uranium-X Preparations.

As recorded,[†] actinium was present in the first uranium-X preparation separated in 1909 from 50 kgrm. of uranyl nitrate in quite obvious and unmistakable quantity, and in the second preparation separated it could be detected, but it was only one-fifteenth as great as in the first, whereas none could be detected in separate tests on each of the three preparations of the third separation or in the single preparation of the fourth separation. A year from preparation, a combined test, on the four latter together, showed a just detectable actinium active deposit. This increased in amount, until at the end of 1912 it was clearly detectable, though very small, and it has since then rather more than doubled in amount, and now (December, 1917) it causes an increase in the leak of the rather insensitive instrument at present used of about 1.5 divisions per minute, the natural leak being 1.0 d.p.m.

This was the first growth of actinium observed, and it is probably to be ascribed to a minute amount of the parent of actinium existing as an impurity in the uranium rather than to a growth of this element, since the preparation was separated from the uranium. The earlier preparations also show a slight increase in the amount of actinium present, but the increase is not much, if any, greater than in the later ones.

The Separation of Eka-tantalum from Pitchblende.

Early attempts to separate the supposed parent of actinium from pitchblende, based on its expected analogy in chemical character to tantalum, led to no positive result, probably because only small quantities of pitchblende were available. Solutions of the mineral in nitric acid were shaken with small quantities of tantalic acid, and the latter then kept under observation for the growth of actinium. Similar negative results have been recorded by O. Göhring[‡] in the same field. A comparison of the properties of the compounds of niobium and tantalum, and the methods of separating these elements, led to the following process being tried, first on uranium-X, to ascertain whether it was capable of separating uranium-X_2, the isotope of eka-tantalum, from uranium-X_1, the isotope of thorium When this was found to be successful, the method was tried first on small and then on large amounts of pitchblende The specimen of pitchblende used was from a very fine block of practically pure Indian pitchblende, found to consist of 86 per cent U_3O_8, 11 9 per cent PbO, 1·9 per cent ThO_2, and 0 6 per cent SiO_2. The figure for the uranium may be too high, and has still to be checked by a radium estimation

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[*] F. Soddy, 'Chemistry of Radio-elements,' Part II, p. 29 (1915).

[†] F. Soddy, 'Phil. Mag.' [6], vol. 20, p. 344 (1910); 'Chem. News,' vol. 107, p. 97 (1913).

[‡] O. Göhring, 'Physikal. Zeitsch.,' vol. 15, p. 642 (1914).

The material was heated in a quartz tube in a current of dried air, charged with the vapours of carbon tetrachloride by passing it through that liquid, kept a few degrees below its boiling-point It was expected that, under these circumstances, an element resembling tantalum would volatilise at a lower temperature than the other pre-emanation disintegration products in the mineral In a trial with uranium oxide and tantalic acid, placed side by side in two boats in the tube, it was found that, at a temperature in the neighbourhood of visible red-heat, at which the tantalic acid volatilised freely, the uranium did not volatilise at all The oxide, in fact, gained weight through absorption of chlorine

Using a preparation of uranium-X, it was found that, even at a temperature below a visible red-heat, a considerable part of the uranium-X_2 volatilised The boat containing the uranium-X preparation was removed from the furnace, and measurements of the penetrating β-rays started as rapidly as possible. The quartz boat was chilled externally by water after removal from the furnace The curves shown in fig 2 indicate the growth of

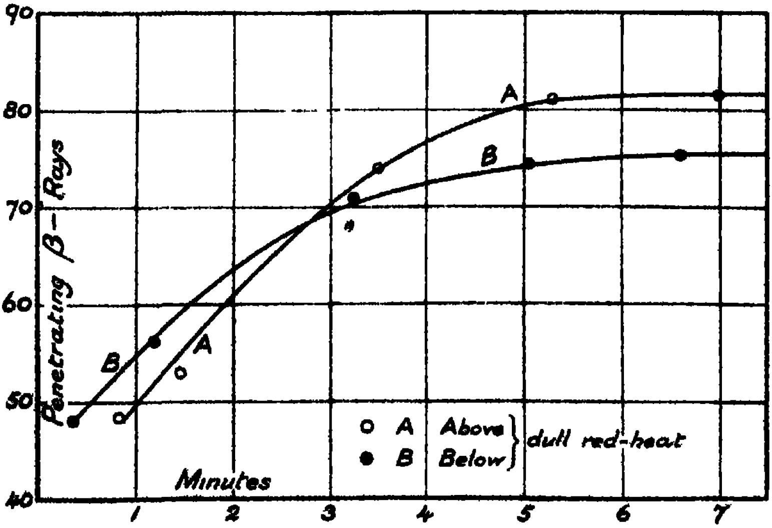

Fig 2.

the penetrating rays, due to uranium-X_2, after removal from the furnace For the curve marked A the temperature was just visible red-heat, and for that marked B just below visible red-heat The results indicate that, even in the very short period of uranium-X_2, a large proportion of it is removed by this treatment It follows, therefore, *à fortiori*, that the method should be capable of removing the predicted element, since it is isotopic with uranium-X_2, and must have a very extended period of average life

In the experiments with pitchblende, it was found that only a very small part of the material volatilised at temperatures in the neighbourhood of visible red-heat. The sublimate had a high α- and β-activity, the former being for the most part due to polonium, as shown by its subsequent history, and the latter to radium-E, as shown by its absorption coefficient But none of the known pre-emanation members could be detected in the sublimate, so that there was nothing present which would interfere with the active deposit test for the presence of actinium

In the main experiment 470 grm of finely powdered pitchblende were used. It was heated in the carbon tetrachloride stream just below visible red-heat for 22 hours, the boat containing the material being several times removed during the sublimation, and the contents well mixed before it was replaced In this operation the material gained 25 grm in weight through absorption of chlorine, and the sublimate obtained weighed 5 grm The sublimate was put back in an empty boat, and re-sublimed, as before, for four hours The second sublimate weighed 0 28 grm It was washed into a platinum capsule with hydrochloric acid, and dried on the water-bath It will be termed Preparation I, the date of preparation being March 18, 1915

The part not sublimed in the two processes described was re-sublimed, as before, at the same temperature, but for a further period of 90 hours. The considerable amount that sublimed was re-sublimed as before, and a second preparation, weighing 2·45 grm, obtained It was dried on a crucible lid and mounted in a shallow lead tray, a platinum wire being fixed to make contact between the preparation and the lead tray It is termed Preparation II, the date of preparation being May 10, 1915.

It was thought that Preparation I would contain most if not all, of the eka-tantalum if present in the mineral, and that Preparation II would give an indication whether the separation in the first operation had been complete or not. After the foregoing treatment, a third preparation was obtained by Miss Hitchins from the same material, by carrying out the sublimation at a much higher temperature, in order to make sure that all the eka-tantalum present was removed Heating was continued for six or seven hours in all, and the temperature attained was a bright red heat. A considerable quantity of bright yellow sublimate was so obtained, which, very surprisingly, was found to be free from uranium and to consist mainly of lead chloride. The absence of uranium was surprising, as uranium oxide had been found to sublime at much lower temperature than here employed, and the point is under further investigation.

A small part of the sublimate could not be made to dissolve and was mounted separately on a copper tray. It weighed 0·5 grm., and constitutes Preparation III C. The lead from the solution was precipitated with sulphuric acid and the filtrate precipitated by ammonia in two fractions. Both were dried on copper trays, the first precipitate, III B, weighed 2·4, and the second, III A, weighed 0·4 grm., the date of preparation being December 14, 1915.

Thus from 470 grm. of Indian pitchblende three preparations were obtained by sublimation in carbon tetrachloride vapour, the I and II at below visible red-heat for 22 and 90 hours respectively, and III at a very much higher temperature.

Method of Testing for the Presence of Actinium.

The preparations were kept in a desiccator, and tests for the presence of actinium carried out at intervals. Owing to the exigencies of the time, the intervals elapsing between the tests were rather irregular, and in one case over a year elapsed between successive tests. The preparation was covered with a metal plate, insulated by a rubber or mica washer and connected to the negative pole, the plate being connected with the positive pole of the 220 V. direct current mains. Equilibrium for the actinium active deposit is attained practically in four or five hours, and the exposure was usually overnight. In a few cases of shorter exposure the necessary correction was supplied from the table of the decay of the actinium active deposit given in Makower and Geiger's 'Practical Measurements in Radioactivity,' p. 147. After exposure the plate was removed as quickly as possible to a simple α-ray electroscope, and readings commenced as soon as possible after the disturbance caused by the opening of the electroscope had subsided and continued for four and five hours. The decay curve over this period was plotted and the constant determined from the curve. If the activity is due to actinium the activity decays exponentially,—after an initial somewhat slower rate of decay extending over the first few minutes caused by actinium-C of period 3·1 minutes,—with the period of actinium-B, 51 minutes. The character of curve shows at once whether it is due to actinium, for although the period of radium-B, 38·5 minutes, is not very greatly different from that of actinium-B, the active deposit could not possibly be mistaken for that of radium, since the latter would show the large rapid initial decay due to radium-A

Observations on the Growth of Actinium

Preparations I and II, when prepared, gave completely negative results when tested as described during the period covered by the first few weeks from preparation Not the slightest detectable growth of actinium has yet occurred in Preparation I, in which it was thought that the main quantity of eka-tantalum would be concentrated, though over 2·5 years have elapsed since preparation Neither has there been any growth in Preparation III, a minute initial activity in III C having, if anything, decreased But in Preparation II there has been a continuous growth of actinium, so that now, after 2·6 years from preparation, the leak in the electroscope due to the active deposit of actinium amounts initially to over 10 divisions per minute This is 20 times as much as could be with certainty detected, and 10 times as much as could be with certainty characterised as due to actinium

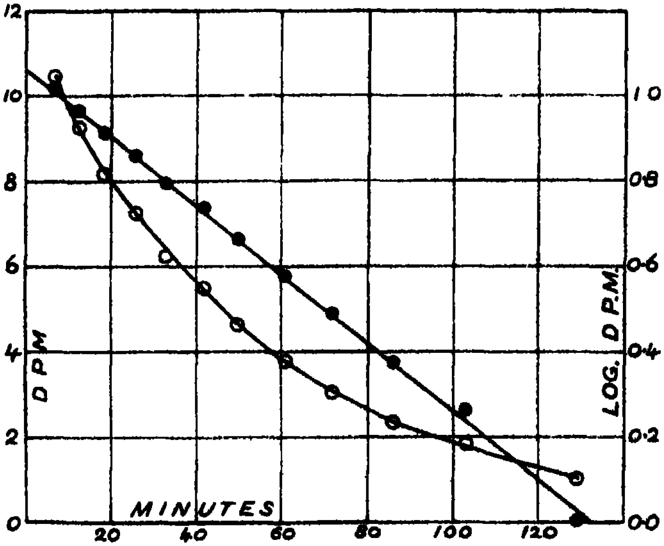

Fig 3

The decay curve of a recent measurement of this preparation, 2·6 years from preparation, is shown in fig 3. The value of the period as calculated from the slope of the logarithmic decay curve is 53·5 minutes, that of actinium-B being 52·1 minutes, and the curve is that to be expected of a pure actinium active deposit.

Fig 4 shows the growth of actinium with time for this preparation. It indicates the expected initial period of retardation due to the members radioactinium of period 28 days, and actinium-X of period 16·4 days, and after that a regular increase with time. The quantitative estimation of actinium

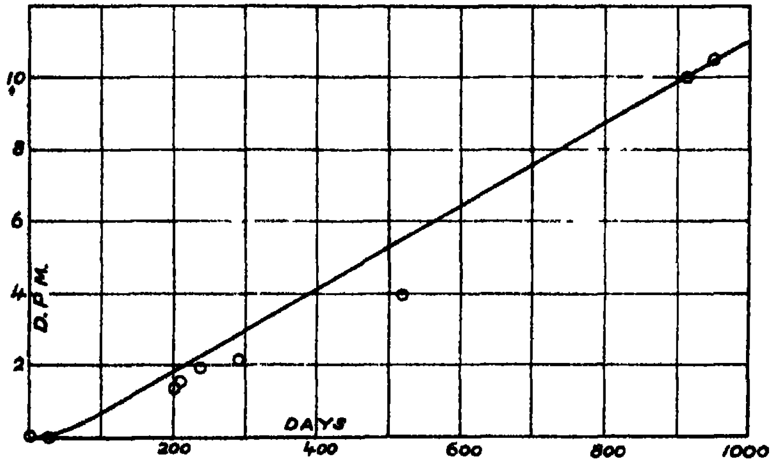

FIG. 4.

is of course very much more uncertain and difficult than that of radium, and, indeed, there has been little or no work practically attempted yet in this field, and it may be doubted whether a really accurate method applicable to very minute quantities is yet possible. No great consistency is to be expected from the method described, for the solid preparation probably varies somewhat in emanating power. The moisture of the atmosphere on the day of testing is, as will be shown later, an important factor. For this reason it would be premature to build much upon the precise form of the curve, and the indication that it gives of an increase in the rate of growth of actinium with the lapse of time, and an increase of slope in the later measurements. Not enough measurements have been taken, and the method is too uncertain to justify the conclusion that this indicates the existence of intermediate bodies between the eka-tantalum and the actinium. For we know, from the α- and β-ray change rules, that three, if any, such intermediate bodies must intervene, and this is not very probable.

Thus, contrary to expectation, instead of the main part of the parent of actinium being in the first sublimate obtained, it is, so far as can be seen, contained wholly in the second, and neither the first nor third preparations contain a detectable amount. The second preparation was obtained by merely continuing the process of sublimation at the same temperature as the first for a further 90 hours after an initial 22 hours, so that it is surprising that the whole of the eka-tantalum should appear to be in the second preparation. The probable explanation is that the eka-tantalum does not commence to volatilise until the material has become saturated to a certain point with chlorine In the case of the large mass of material used this preliminary to the process would of necessity take a considerable time.

A Separation of Eka-tantalum from Joachimsthal Pitchblende

A similar growth of actinium to that in Preparation II has occurred in another preparation obtained differently In 1903, a kilogramme of very fine selected Joachimsthal pitchblende, probably containing over 60 per cent of uranium, was dissolved in nitric acid, and the solution filtered The clear solution gradually deposited a small white precipitate, and subsequent examination showed that almost all the radium had precipitated from the solution This spoiled the material for the purpose for which it had been prepared, and the solution and precipitate were left together in the bottle till 1908 Probably sulphides in the mineral had been oxidised to sulphate by the nitric acid, accounting for the precipitation of the radium In May, 1908, the precipitate was filtered off, the filtrate evaporated to dryness, redissolved in water and again filtered, and the total insoluble material, after thorough digestion with strong nitric acid, was washed, dried, and preserved. It weighed 8 grm, and its α-activity was recorded as about five times as great as that of uranium oxide In May, 1914, about one-quarter of the material was subjected to the carbon tetrachloride sublimation, under similar conditions to those described for Preparations I and II. It was thought that any element in the pitchblende resembling tantalum in chemical properties would most probably be contained in this precipitate, and this view was supported by the fact that in 1914 it contained an easily detectable quantity of actinium, which one would hardly have expected to have been present initially, but which had probably grown since preparation

The preparation obtained by sublimation gave an actinium active deposit initially (May, 1914) of 2 to 2·5 divisions per minute, which had increased in August, 1914, to rather more than 3 d.p.m. At present (November, 1917) it gives an active deposit of about 12 d.p.m., slightly greater than that given by Preparation II In age it is about 100 days older. Some of the original material was spread as a film on a copper tray, and its α-activity compared with that of a similar film of uranium oxide. In May, 1914, the activity was about three times as great as the uranium, but, recently tested, the activity was found to have increased to about 5·6 times. This is probably of not much present significance, as such an increase is to be expected if radium D is present, as it is almost certain to be.

The results with this preparation are thus confirmatory to those described, but the history of the material is more involved and the exact amount of mineral from which it was derived less definite than in the former case. There are certain lacunæ to be filled up, and this cannot be done till after the war.

Theory of the Growth of Actinium from Eka-tantalum.

If the scheme assumed (fig. 1) is correct, and actinium is the direct product of eka-tantalum, it follows that the growth of actinium should proceed linearly with the time, if both substances are of long life in comparison with the time of observation. In this scheme let A, B, C, D be the quantities respectively of eka-tantalum, actinium, radioactinium and actinium-X, their radioactive constants being, λ_1, λ_2, λ_3, λ_4. The amount of active deposit obtained should vary as D if the emanating power of the preparation remains constant.

The growth of actinium being given by

$$dB/dt = \lambda_1 A - \lambda_2 B,$$

it follows, if both actinium and its parent are long-lived, that the first term may be regarded as a constant and the second term neglected in comparison with it. Hence, if t is the time

$$B = \lambda_1 A t$$

The growth of radioactinium, initially absent, is given by

$$dC/dt = \lambda_2 B - \lambda_3 C,$$

so that $$C = (\lambda_1\lambda_2/\lambda_3)\,A\,\{t - 1/\lambda_3 - (1/\lambda_3)\,e^{-\lambda_3 t}\}$$

Similarly, the quantity of actinium-X is given by

$$D = (\lambda_1\lambda_2/\lambda_4)\,A\,[t - 1/\lambda_4 - 1/\lambda_3 + \{\lambda_4/\lambda_3(\lambda_4-\lambda_3)\}\,e^{-\lambda_3 t} - \{\lambda_3/\lambda_4(\lambda_4-\lambda_3)\}\,e^{-\lambda_4 t}]. \quad (1)$$

The last two terms within the bracket become zero as t increases, and may be neglected after the first six months, when what Rutherford terms "transient" equilibrium, of the active deposit products with the actinium, is attained. After that initial period the growth of active deposit is given by a straight line cutting the zero axis at a time $1/\lambda_3 + 1/\lambda_4$, or 44·5 days, from the start. The curve in fig 4 represents a theoretical curve according to expression (1) above, and the departure from this curve is not sufficiently great to justify yet any conclusion that the production of actinium from eka-tantalum is not direct.

If eka-tantalum is the direct parent of actinium and both are long-lived, the growth of actinium is in every respect analogous to the growth of radium from ionium, studied by Boltwood. Just as Rutherford and Boltwood were able to deduce from the rate of growth of radium the period of radium, knowing nothing about the period of ionium except that it was very long, so it should be possible to find the period of actinium from its rate of growth without knowing anything as to the period of eka-tantalum; this can be done if Preparation II contains the whole of the eka-tantalum originally in the mineral, and as none can be detected in the other preparations from it, this may be provisionally assumed. Then the $\lambda_1 A$ of the above equations is equal to and may be replaced by $\lambda_4 D_0$, where D_0 is the equilibrium quantity of actinium-X in 470 grm. of the original mineral. After transient equilibrium is reached

$$D/D_0 = \lambda_2 (t - 1/\lambda_4 - 1/\lambda_3)$$

or

$$1/\lambda_2 = D_0/D\,(t - 44{\cdot}5 \text{ days}) \qquad (2)$$

The period of actinium is thus given by the ratio between the actinium-X present in the 470 grm. of mineral, and that in the preparation separated from it, multiplied by the age of the preparation less 45 days.

Provisional Determination of the Period of Actinium.

What is therefore needed is to compare the actinium active deposit, now given by Preparation II, with that given by a known weight of the Indian pitchblende under the same conditions. The proportion of the emanation generated in the preparation contributing to the measurements of the active deposit in fig. 3 was quite unknown, and, owing to the very short period, cannot even be guessed. The problem to be solved is a difficult one, and although, in the following attempt, the estimate of the period arrived at is considered a probable one on the specified theoretical assumptions made, it must not be regarded as entirely free from the possibility of serious experimental error. But it seemed to be worth while making the attempt at the present stage, even at the risk which, unfortunately, has actually been incurred, of interrupting irrevocably the sequence of observations on the growth of actinium in this preparation, and possibly losing some of the eka-tantalum it contains.

The preparation was found, on chemical examination, to be, as was to be expected, a very heterogeneous material. It appeared to consist principally of lead and uranium chloride, and could not be got completely into solution in the small volume of water that had to be used for the test. In the first direct comparison with pitchblende, the material was transferred to a weighing bottle, in the stopper of which tubes for leading through a current of gas were fused. The volume of the bottle was 17 c.c. A little pure nitric acid was added, and most of it evaporated off, then water, making a final volume of 8 c.c. Solution was only partial, and lead chloride crystallised out on cooling. In the test there was a considerable amount of solid with the solution. For the comparison standard, 1 grm. of the original pitchblende in nitric acid solution was made up to the same volume in a precisely similar weighing bottle.

The apparatus used for the comparison is shown in fig. 5. It was designed to secure as large a proportion of the active deposit of actinium, whilst suppressing that due to radium and thorium in the mineral A constant stream of air, of 430 c c a minute, was sent through the solution, from a water-dropper delivering air into a reservoir under constant pressure, secured by causing an excess of air always to escape through the water at the bottom of the reservoir. The air stream was led into a glass cylinder of 250 c.c. volume, lined with wire gauze, connected with the positive main, and along the axis of the cylinder was hung a strip of aluminium foil connected with the negative main The strip was doubled, and the two halves pressed together in close contact and clamped by a binding screw attached to the cork The air stream was continued for an hour, the aluminium strip opened out, cut in pieces, and introduced into the electroscope active side uppermost, measurements being taken as usual.

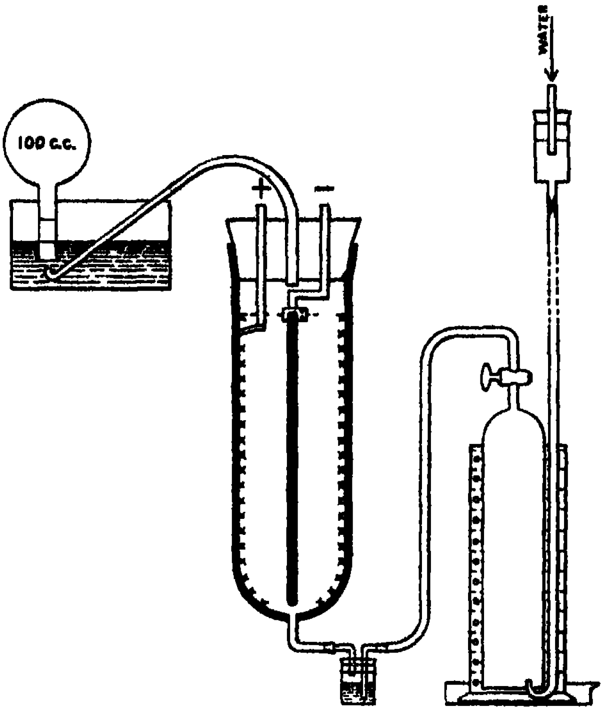

Fig 5

Under these circumstances, the actinium active deposit obtained from Preparation II was only one-quarter of that obtained from the same preparation in the solid state, and it was equal to that given by 0·25 grm. of the original pitchblende in solution. In the case of the latter, the effect due to the thorium present could be detected, but it was small in the case of an hour's run, and could be very readily corrected for. The actinium active deposit given by the pitchblende solution was only from one-sixth to one-fourth of that which it was calculated it should have given, if all the actinium emanation generated had been effective. A delay of only 8–10 seconds between the time of formation of the emanation by the disintegration of actinium-X and its arrival in the testing vessel would account for this difference. The delay in the passage through the dead space above the liquid in the weighing bottle was only 1 7 seconds, so that it appears that, in solution, the greater part of the actinium emanation disintegrates before it can be swept out of the liquid phase. This is very unfortunate, as it means that, for accurate measurements by the solution method, some four to six times the quantity of raw material must be worked up, or four to six times the period waited for the effects than would theoretically be necessary The fact that Preparation II gave some four times as much active deposit in the solid state as in solution indicates that the emanating power of the solid preparation was surprisingly high, and not much short of the maximum possible. This conclusion is consistent with all the subsequent results obtained with it.

Further attempts were then made to get the whole of Preparation II in solution. The chlorine was first removed, by dissolving as much as possible in boiling water, adding ammonium carbonate to convert the lead into carbonate, filtering and washing the precipitate, and re-dissolving it in nitric acid. The filtrate, which deposited ammonium uranate on evaporation, was cautiously ignited, to expel ammonium chloride, and the residue re-dissolved in the nitric acid solution of the precipitate. A small insoluble residue, weighing rather less than 0 1 grm., was filtered off, and the filtrate got back in the weighing bottle as a clear solution. Comparison of this with a new 0·25 grm. pitchblende standard nearly confirmed the previous result with the partial solution, and showed the amount of actinium-X to be that in 0 20 grm. of pitchblende.

The small insoluble residue was treated with hydrofluoric acid, when very nearly all went into solution without effervescence, leaving a jelly on evaporation. To remove hydrofluoric acid, the material was treated with ammonia, evaporated, and, thoughtlessly, ignited to expel ammonium fluoride, overlooking the possibility that this procedure might cause some of an element resembling tantalum in properties to volatilise. The ignited residue weighed only 0·0175 grm. The loss may have been entirely silica, but it is feared some of the eka-tantalum may also have been volatilised. The minute residue, however, was extremely active, giving an α-radiation corresponding with that of some 60 cm.^{2} surface of uranium oxide. But in this state an active deposit test failed to show the presence of actinium. So far as can be guessed from its expected chemical analogy with tantalum, this small insoluble residue is likely to contain the eka-tantalum, but only its future history can disclose this.

It thus appears from the foregoing comparison that the amount of actinium-X grown in Preparation II from 470 grm. of pitchblende, in 2·55 years + 45 days is equal to the amount of actinium-X in about 0·25 grm. of the original material. The period of average life of actinium is thus about 5000 years on the assumptions made, viz.: (1) that actinium is the direct product of the element producing it in the preparation; (2) that the preparation contained all of this direct parent that was present in the mineral; (3) that the comparison with pitchblende did not involve any serious error, owing to the chemical dissimilarity of the two materials.

The Effect of Moisture on the Emanating Power of Preparations containing Actinium.

After the comparisons recorded, the solution of Preparation II was evaporated to dryness on a crucible lid, ignited for an hour in an air-bath at 250° and an active deposit test carried out in the ordinary way, after the preparation had been exposed for some hours to the air of the laboratory. Under these conditions the amount obtained was the same as that given by the solution, only about a fourth of that originally given. The preparation was then put in a closed vessel containing water, and after some hours' exposure to the humid atmosphere, an active deposit test was made within the vessel in air saturated with moisture. Now the active deposit given was increased three times, and was very nearly equal to that originally given before the chemical treatment. Similarly it was found that the minute insoluble residue, in which no actinium could be detected dry, after exposure in the humid atmosphere, gave an active deposit nearly a third as great as that given by the preparation in the test last described. The whole series of tests is consistent with the view that the emanating power of the original preparation was near the maximum, and the actinium was that which corresponds with about 0·25 grm. of the original pitchblende. On further exposure to moisture the emanating power of the insoluble part diminished, and notable deliquescence was found to have occurred. The emanating power of the preparation from Joachimsthal pitchblende was increased 1·8 times by moisture, but further exposure also in this case caused a diminution. Evidently the problem of finding a quantitative radiometric method for the estimation of actinium, which shall be suited for these minute quantities, and the long term of years over which the measurements must be conducted, will not prove an easy one to solve, but, until it is solved, it will not be possible to determine with absolute certainty the exact form of the growth-curve, and to settle the question whether the growth is linear with the time, and therefore direct

Summary

(1) In a full historical introduction, the data obtained in 1909 relative to the rays and products of uranium-X are discussed in so far as they throw light on the various possible modes of origin of actinium

(2) The minute growth of actinium previously put on record in 1913 as having been observed in the old uranium-X preparations has been confirmed by their later history and is now established beyond doubt.

(3) Uranium-X_2 can be separated from uranium-X_1 by sublimation in a current of air charged with vapours of carbon tetrachloride at a temperature below visible red-heat

(4) 470 grm. of a very pure Indian pitchblende were similarly treated, in the expectation of removing eka-tantalum, isotopic with uranium-X_2 and giving actinium in an α-ray change of long period.

(5) The preparations so obtained were initially free from actinium, but one of them has produced it continuously with the lapse of time.

(6) A direct comparison of the amount of actinium in this preparation after the lapse of 2 5 years with that in the original pitchblende showed that it was equal to that in about 0 25 grm.

(7) On the assumptions that eka-tantalum and actinium are both long-lived, that no intermediate members intervene between them, and that the preparation contained the whole of the parent of actinium in the original mineral, the period of average life of actinium is calculated to be 5000 years. Nothing can yet be said definitely as to the period of the parent.

(8) A second preparation separated from Joachimsthal pitchblende, the treatment of which commenced in 1903 and ended in 1914 with the carbon tetrachloride sublimation, has given a similar growth of actinium.

(9) The work was undertaken to test and confirms the view that the parent of actinium occupies the eka-tantalum place in the Periodic Table, and gives actinium in an α-ray change of long period, itself being formed as the product of uranium-Y, discovered by Antonoff, who suggested that it was the first member of the actinium series. But this mode of origin of actinium, though at present the most probable, has not yet been conclusively established to the exclusion of all the other possible modes of origin, discussed in the historical introduction.

Critical Loading of Struts and Structures.

By W. L. Cowley, A.R.C.Sc., D.I.C., Wh.Sc., and H. Levy, M.A., B.Sc., F.R.S.E.

(Communicated by Sir Richard Glazebrook, F.R.S. Received January 2, 1918.)

The main points of practical importance in the following paper are:—

(*a*) Discussion of stability of a prismatic homogeneous strut on simple supports, loaded laterally and longitudinally—criterion for failure.

(*b*) Equation of three moments generalised to include the case where the central axis is constrained to pass through $n+1$ non-collinear points, and where lateral loading and longitudinal end thrusts exist.

(*c*) Application to the case of n bays, and the calculation of the moments, reactions, etc.

(*d*) Case where one or more bays of Euler's critical length are present in the system. It is here shown that in general failure does not result.

(*e*) Failure will not in general take place if $n-1$ bays are of Euler's critical length, and the remaining one less.

(*f*) Case where one or more bays are twice Euler's critical length.

(*g*) General condition of failure for a structure of n bays.

(*h*) Conditions of strengthening or weakening by the addition of an extra bay to a structure critically loaded.

(*k*) Extension of the previous work to include the external loading due to wires in tension attached to the beam in the region of the supports.

(*l*) Clamped supports.—Determination of bending moments, etc., and condition of failure.

(*m*) Strengthening effect of clamped supports.

Failure in the above is understood to imply instability of geometrical configuration of the structure, under the critical loading.

§ 1. Many interesting questions of considerable practical importance relating to the failure of a structure under compression have been brought to light by a study of the problems of the strength of aeroplane frameworks. Unfortunately, however, very little advance has been made in this question on the theoretical side in its direct application to aeronautics, although a large number of investigations exist dealing with the general theory of elastic stability. It is extremely difficult to translate such results to make them directly useful in dealing with aeroplane frameworks, and, in fact, the general theory as ordinarily presented does not appear to contain within it certain particular forms of failure that arise in this connection

The present paper is an attempt to solve, as far as possible, problems relating to the strength of such a construction as a beam under end thrusts, and supported at intermediate points.

In a sense there are two types of failure of a structure On the one hand, the material of a member may rupture on account of the yield stress having been exceeded during the course of the loading, while, on the other, the loading may be such that the geometrical configuration previously existing can no longer be maintained, even approximately It is the second type of failure to which attention in this paper will be chiefly directed.

§ 2 The ground will be cleared initially by a simple discussion of an elementary case that may arise, such as, for example, when a prismatic, homogeneous, rectilinear uniform strut with simple supports at the ends is loaded both laterally and longitudinally

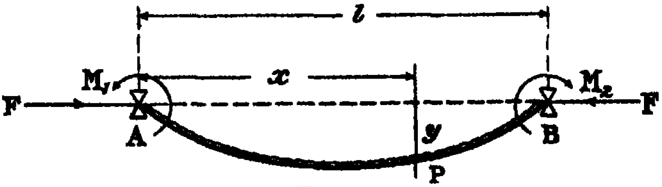

Fig 1

Let AB be the strut, simply supported at the ends A and B, and let w lb. per unit length be the intensity of lateral loading at P, of co-ordinates x and y Let M_1 and M_2 be externally applied bending moments at the ends. Taking the origin at A, and AB as the axis of x, consider the equilibrium of the beam. The bending moment at P is given by

$$F \cdot y - M_2 + (M_2 - M_1)(l-x)/l + H(x),$$

where $H(x)$ is the bending moment at P due to the lateral loading alone Equating this to the resisting moment, viz., $-EI$ (curvature) and assuming the deflections small, we find

$$-EI\frac{d^2y}{dx^2} = F \cdot y - M_1 + (M_1 - M_2)x/l + H(x), \tag{1}$$

where $H(x)$ does not involve M_1 and M_2.

This equation may be written

$$\frac{d^2y}{dx^2} + \lambda^2 y = M_1\lambda^2/F + x\lambda^2(M_2 - M_1)/Fl - \lambda^2 H(x)/F, \tag{2}$$

where

$$\lambda^2 = F/EI \tag{3}$$

Integrating, we obtain

$$y = C_1 \cos \lambda x + D_1 \sin \lambda x + M_1/F + x(M_2 - M_1)/Fl - \frac{\lambda^2}{F} \cdot \frac{H(x)}{D^2+\lambda^2}, \tag{4}$$

where the operator $1/(D^2+\lambda^2)$ has its usual significance, and the expression $-\lambda^2/F \, . \, H(x)/(D^2+\lambda^2)$ is easily evaluated in general by expanding $H(x)$ in a Fourier series For convenience, represent this as a known function, $X(x)$, independent of M_1 and M_2. In the two special cases, firstly, in which there is no lateral loading, and, secondly, in which this loading is uniform, $X(x)$ becomes zero and W/λ^2F respectively

To determine C_1 and D_1 we note that

$$y = 0, x = 0,$$

$$y = 0, x = l,$$

therefrom from (4)

$$0 = C_1 + M_1/F + X(0),$$

and
$$0 = C_1 \cos \lambda l + D_1 \sin \lambda l + M_2/F + X(l)$$

Hence

$$C_1 = -M_1/F - X(0). \tag{5}$$

$$D_1 = (M_1 \cos \lambda l - M_2)/F \sin \lambda l + [X(0) \cos \lambda l - X(l)]/ \sin \lambda l \tag{6}$$

The bending moment at any point is

$$\text{B.M} = -EI\frac{d^2y}{dx^2} = \left[\left\{ \frac{M_1 \cos \lambda l - M_2}{\sin \lambda l} \right\} + \left\{ \frac{FX(0) \cos \lambda l - FX(l)}{\sin \lambda l} \right\} \right] \sin \lambda x$$

$$- \{M_1 + FX(0)\} \cos \lambda x + H(x) + FX(x) \tag{7}$$

Since all the terms on the right-hand side of this equation in general must be finite, with the exception of that involving $\sin \lambda l$, the bending moment at any point can only become infinite when

$$\sin \lambda l = 0, \tag{8}$$

unless at the same time

$$M_1 \cos \lambda l - M_2 + F\{X(0) \cos \lambda l - X(l)\} \longrightarrow 0 \tag{9}$$

Equation (8) indicates that when the length of the strut is given by

$$\lambda l = \pi,$$

i.e.
$$l = \pi\sqrt{(EI/F)}, \tag{10}$$

bending moments and deflections become excessively large compared with those allowable by the analysis. Equation (10) is the commonly recognised expression for "Euler's strut"

It should be particularly noted that under these conditions equation (9), reducing to

$$M_1 + M_2 = -F\{X(0) + X(l)\}, \tag{11}$$

furnishes a relation between the externally applied bending moments which may prevent the strut failing in the Eulerian sense, but this will be returned to later For the moment it suffices to suggest that, in a structure in which M_1 and M_2 are determined by the loading and configuration of the system, it may be possible to have present a strut of Euler's length without failure resulting Such cases will shortly be discussed

§ 3 It appears, then, that in the simple case here treated, a critical loading of the beam is essentially associated with the production, at all points, of bending moments and deflections excessively large in comparison with those allowable by the assumptions upon which the analysis is based This immediately suggests that failure in a corresponding sense may take place in a more complicated structure, and that these criticals will be furnished from the expressions determining the bending moments, on the assumption of small deflection, by inserting the condition that the former become infinite.

In the case of the simple strut, as treated by Euler, it has been shown that, up to the first critical loading given by

$$F = \pi EI/l^2,$$

the straight position of equilibrium is stable to small disturbances, but that, for loadings above this, the system deflects from this position to a new position of equilibrium, where the axis takes up the form of an elastica, the geometry of the structure being unstable. It will be assumed in the present discussion that, in the more complicated structures here to be treated, the lowest critical position furnished by the criterion already stated is one of instability with respect to the geometrical configuration, that, in fact, a loading greater than that stated here as critical will involve such deflections as would destroy the shape of the structure

§ 4 *Case of a Uniform Beam under End Thrusts where the Central Axis is constrained to pass through* $n+1$ *Fixed Points*—Let 0, 1, 2, 3, ..., n, be the $n+1$ supports whose positions are given, and let their distances from the datum line $0n$ be small compared with the lengths of the n bays 01, 12, 23, etc., and suppose E_1I_1, E_2I_2, ..., E_nI_n, are the flexural rigidities of the corresponding bays 01, 12, ..., $(n-1)n$. Let δ_1, δ_2, δ_3, etc., be the distances of 1, 2, 3, etc, from 02, 13, 24, etc., then δ_1, δ_2, δ_3, etc., will be termed the

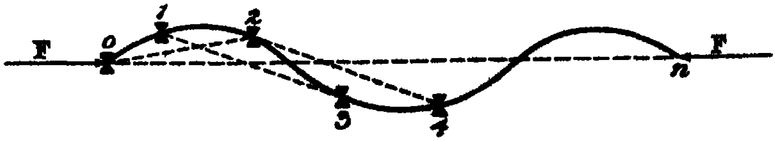

Fig 2.

deflections at the supports It is clear that, as long as the beam is not otherwise constrained by wires inclined to the datum line On, at any intermediate points the longitudinal force F will be constant and equal to F along each bay as far as quantities of the second order Modifications resulting from the attachment of such wires will receive special treatment later. It is proposed, in the first place, to find an expression for the bending moment at each point of the beam.

§ 5 Consider the rth and $(r+1)$th bays, ABC (fig. 3), laterally loaded according to some given law, and of lengths l_r and l_{r+1} respectively. The remainder of the beam to the left of the $(r-1)$th support, together with the support itself, may be replaced, since the deflections are supposed small, by forces F and f_{r-1}, parallel and perpendicular respectively to AC, and a bending moment M_{r-1} Similarly, the beam to the right of C and the $(r+1)$th support itself may be replaced by F, f'_{r+1}, and M_{r+1}

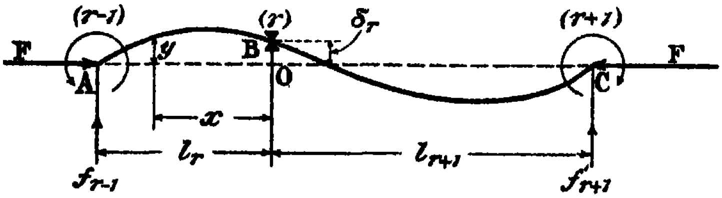

FIG 3

Taking the origin of co-ordinates at O, the bending moment at any point P (x, y) is

$$M_{r-1}-f_{r-1}(l_r-x)+F\,.\,y+\psi_r(x),^*$$

where $\psi_r(x)$ is the moment about P of the lateral loading. Equating this to the restoring moment $-E_rI_r\,d^2y/dx^2$, y being small, it is found on investigation that

$$y = C_r\cos\lambda_r x+D_r\sin\lambda_r x-\{M_{r-1}-f_{r-1}(l_r-x)\}/F+\chi_r(x), \qquad (12)$$

where

$$\lambda_r = \sqrt{(F/E_rI_r)} \qquad (13)$$

and

$$\chi_r = -\frac{\lambda_r^2}{F}\cdot\frac{\psi_r(x)}{D^2+\lambda_r^2}$$

similar to the simple case already treated C_r and D_r are to be obtained from the end conditions, viz,

$$x = 0, \qquad y = \delta_r,$$

$$x = l_r, \qquad y = 0.$$

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* $\psi_r(x)$ is now the moment of the lateral force only on the portion of AB to the left of P, about the point P {*cf* equation (1)}.

In addition it is to be remembered that

$$\begin{aligned} M_r &= \text{bending moment at B} \\ &= M_{r-1}+F\delta_r-f_{r-1}l_r+\psi_r(0). \end{aligned} \tag{14}$$

Accordingly

$$C_r = M_r/F-\{\psi_r(0)+F\chi_r(0)\}/F, \tag{15}$$

$$D_r = \{(M_{r-1}-M_r\cos\lambda_r l_r)+\cos\lambda_r l_r\,(\psi_r(0)+F\chi_r(0)\}/F\sin\lambda_r l_r-F\chi_r(l_r). \tag{16}$$

If θ_r be the inclination of the strut at the rth support to AC, then

$$\begin{aligned} \tan\theta_r &= \left(\frac{dy}{dx}\right)_0 \\ &= \lambda_r D_r+\chi_r'(0)-\{M_{r-1}-M_r+F\delta_r+\psi_r(0)\}/Fl_r, \\ &= A_r M_{r-1}+B_r M_r-\delta_r/l_r+K_r, \end{aligned} \tag{17}$$

where

$$A_r = \frac{l_r}{E_r I_r}\cdot\frac{1}{\lambda_r^2 l_r^2}\cdot\left\{\frac{\lambda_r l_r}{\sin\lambda_r l_r}-1\right\} = \frac{l_r}{E_k I_r}\quad \alpha_r,\ \text{say} \tag{18}$$

$$B_r = \frac{l_r}{E_r I_r}\cdot\frac{1}{\lambda_r^2 l_r^2}\quad\left\{1-\frac{\lambda_r l_r}{\tan\lambda_r l_r}\right\} = \frac{l_r}{E_r I_r}\cdot\beta_r,\ \text{say}, \tag{19}$$

$$K_r = -B_r\psi_r(0)+\chi_r'(0)+\frac{\lambda_r}{\sin\lambda_r l_r}\{\chi_r(0)\cos\lambda_r l_r-\chi_r(l_r)\}. \tag{20}$$

In the same way, for the portion BC between the rth and $(r+1)$th supports, still maintaining the origin at O, similar equations are easily derived, giving as the gradient of the beem at the rth support

$$-\tan\theta_r = A_{r+1}M_{r+1}+B_{r+1}M_r-\delta_r/l_{r+1}+K_{r+1}'.^* \tag{21}$$

This is easily obtained from equation (17) by remembering that the $(r+1)$th support now replaces the $(r-1)$th, the rth support remaining the same, and the $(r+1)$th bay takes the place of the rth

By addition of equations (17) and (21) we find

$$A_r M_{r-1}+(B_r+B_{r+1})M_r+A_{r+1}M_{r+1} = \delta_r(1/l_r+1/l_{r+1})-(K_r+K_{r+1}') \tag{22}$$

A_r, B_r, etc, are defined in terms of the dimensions of the bays, their flexural rigidities, and the longitudinal loading as shown by equations (18) and (19), and K_r and K_{r+1}' in addition by the lateral loading. This is the extended form for the equation of three moments, or Clapeyron's theorem when the axis of the beam is constrained to pass through given points, and when both

---

[*] The dashes on the K's indicate that these refer to the lateral load on the bay to the right of the point considered, so that K_r' has the same form as K_r, but the appropriate ψ's, etc., must be used. If the loading in a bay is uniform, $K_r = K_r'$ for that bay.

lateral and longitudinal loading exist. A slightly more general form of the equation will shortly be obtained to include the case where external loading of the nature of wires in tension and inclined to the axis exists.

§ 6. If there is no lateral loading, $K_r = 0$. If the beam be uniformly loaded with intensity w lb per unit length, then it can easily be verified that

$$K_r = \frac{w_r l_r^3}{2E_rI_r} \cdot \frac{1}{\lambda_r^2 l_r^3} \cdot \left\{1 - \frac{2}{\lambda_r l_r} \tan \frac{\lambda_r l_r}{2}\right\} = -\frac{w_r l_r^3}{2} \cdot \frac{l_r}{E_rI_t} \cdot \gamma_r, \text{ say} \qquad (23)$$

When there are no end thrusts, $F = 0$, and consequently $\alpha \longrightarrow 1/6$, $\beta \longrightarrow 1/3$, $\gamma \longrightarrow 1/12$, and equation (22) becomes a simple extension of Clapeyron's theorem for the case where the supports are not collinear.

The non-dimensional functions α, β, and γ will be found very convenient for actual calculation in practical applications, and they are easily plotted for variations of the non-dimensional expression ϕ where

$$\phi = \lambda l = l\sqrt{(F/EI)}$$

Equation (22) can now be applied directly to the solution of the problem under discussion where there are $n+1$ supports.

§ 7. *Case of* n *Bays.*—Let there be $n+1$ supports, figured 0, 1, 2, . . , n, and spaced as in the diagram, so that the rth bay appears to the left of the support numbered r. Let the bending moments at these supports be

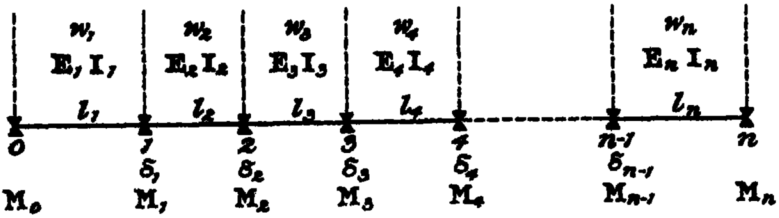

FIG. 4.

$M_0, M_1, \ldots, M_n$, the deflection as already defined $\delta_1, \delta_2, \ldots, \delta_{n-1}$, and suppose the lateral loading considered uniform along each bay, $l_1, l_2, \ldots, l_n$, is $w_1, w_2, \ldots, w_n$. $E_1I_1, E_2I_2, \ldots, E_nI_n$, are as before the flexural rigidities of the various bays.

Applying the generalised form of the equation of three moments derived above, (22), to the three supports of every pair of contiguous bays, we obtain the following system of equations

Central support of trio

1 $$M_0A_1+M_1(B_1+B_2)+M_2A_2=\delta_1(1/l_1+1/l_2)-(K_1+K_2), \qquad (24_1)$$

2 $$M_1A_2+M_2(B_2+B_3)+M_3A_3=\delta_2(1/l_2+1/l_3)-(K_2+K_3), \qquad (24_2)$$

$$\cdot\cdot \qquad \cdot\cdot \quad \cdot\cdot$$

r $$M_{r-1}A_r+M_r(B_r+B_{r+1})+M_{r+1}A_{r+1} = \delta_r(1/l_r+1/l_{r+1})-(K_r+K_{r+1}), \qquad (24_r)$$

$n-1$ $$M_{n-2}A_{n-1}+M_{n-1}(B_{n-1}+B_n)+M_nA_n = \delta_{n-1}(1/l_{n-1}+1/l_n)-(K_{n-1}+K_n), \qquad (24_{n-1})$$

where A_r, B_r, and K_r have the significance already attached to them in equations (18), (19), and (20)

These constitute $n-1$ equations from which, if the bending moments at any two supports, usually 0 and n, be known, the remaining bending moments in general are uniquely determined The equations are fortunately in a simple form for rapid solution, in sharp contrast to the method of strain energy, where generally all the variables appear in all the equations. Once the bending moments at all the supports have been evaluated, the determination of the bending moment at any point in each bay is a simple matter.

From equation (12), by differentiating twice, it is easily found that the bending moment at any point in the bay to the left of the rth support at distance x from that support is given by

$$\begin{aligned} \mathrm{B\,M} &= -E_rI_r\frac{d^2y}{dx^2} \\ &= E_rI_r\lambda_r^2\{C_r\cos\lambda_r x+D_r\sin\lambda_r x\}-E_rI_r\chi_r''(x) \\ &= F\{C_r\cos\lambda_r x+D_r\sin\lambda_r x\}-\{\psi_r(x)-F\chi_r(x)\}, \end{aligned} \qquad (25)$$

where C_r and D_r are furnished by equations (15) and (16)

For this case where each bay is uniformly loaded

$$\psi_r(x) = \tfrac{1}{2}w_r\,.\,(l_r-x)^2, \qquad (26)$$

therefore $$\chi_r(x) = -\frac{\lambda_r^2}{F}\cdot\frac{\psi_r(x)}{D^2+\lambda_r^2}$$

$$= -\psi_r(x)/F+w_r/\lambda_rF, \qquad (27)$$

therefore $$\psi_r(x)-F\chi_r(x) = w_r/\lambda_r^2, \qquad (28)$$

therefore $$C_r = M_r/F-w_r/\lambda_rF, \qquad (29)$$

$$D_r = \{M_{r-1}-M_r\cos\lambda_rl_r-(1+\cos\lambda_rl_r)\,w_r/\lambda_r^2\}/F\sin\lambda_rl_r. \qquad (30)$$

The reactions at the supports are likewise easily evaluated, as follow:—

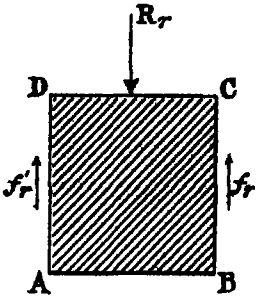

Fig 5.

Consider a portion ABCD of the beam, enclosing the rth support, and let the reaction of the support upon it be R_r acting downwards, while f_r and f_r' are the shearing forces to the right and left respectively of the support, then

$$R_r = f_r + f_r'$$

From equation (14)

$$f_r = M_r - M_{r+1} + F\delta_{r+1} + w_{r+1}l_{r+1}^2/2$$

By taking moments about the $(r-1)$th support for forces to the right of that point we find

$$f_r' = M_r - M_{r-1} + F\delta_{r-1} + w_r l_r/2$$

From these R_r is at once derived Using the notation given in fig 4, and measuring δ as in fig 3, the bending moments when positive will tend to make the beam concave on the under surface, and the reaction of the support upon the beam will be positive when directed downwards

§ 8 In general, equations (24_1), (24_2), , (24_{n-1}), will suffice to determine the values of the bending moments at the supports, and consequently also at any point of the beam, according to the method outlined above It has already been seen that the type of critical failure here investigated is associated with the production of infinite bending moments, but before proceeding to the general treatment of this question, it will be worth while to enquire into the suggestion already thrown out, that the presence of a bay of Euler's critical length would not of itself involve failure of the type here considered. In common engineering practice failure is often assumed to be the case.

§ 9. *Case where One or More of the Bays are of Euler's Critical Length.*—Suppose

$$\lambda_r l_r = \pi,$$

i.e.

$$F = \pi^2 E_r I_r / l_r^2.$$

Such a bay, unassociated with the remainder of the structure, would collapse under the load. It is proposed to enquire whether or not the presence of the remainder of the structure can exert a strengthening effect. The expression for the shape of the rth bay is given by equation (12) in the form

$$y = C_r \cos \lambda_r x + D_r \sin \lambda_r x + \chi_r(x) - \{M_{r-1} - f_{r-1}(l_r - x)\}/F, \qquad (12)$$

where f_{r-1} satisfies equation (14), viz.,

$$f_{r-1} = (M_{r-1} - M_r)/l_r + F\delta_r/l_r + \psi_r(0)/l_r. \qquad (14)$$

Remembering that

$$x = 0, \qquad y = \delta_r,$$
$$x = l_r, \qquad y = 0,$$

and

$$\lambda_r l_r = \pi,$$

and inserting these in (12) modified by (14) we find

$$0 = C_r - M_r/F + \psi_r(0)/F + \chi_r(0),$$
$$0 = -C_r - M_{r-1}/F + \chi_r(l_r).$$

Therefore

$$C_r = -M_{r-1}/F + \chi_r(l_r) \qquad (31)$$

and

$$M_r + M_{r-1} = \psi_r(0) + F\{\chi_r(0) + \chi_r(l_r)\}. \qquad (32)$$

For no lateral load (32) takes the simplified form

$$M_r + M_{r-1} = 0, \qquad (33)$$

and for uniform load

$$M_r + M_{r-1} = 2w_r/\lambda_r^2 = 2w_r l_r^2/\pi^2. \qquad (34)$$

The constant D_r can at once be determined in terms of the M's by the statement that the gradient at the $(r-1)$th support for the rth bay is equal to the gradient at the same point for the $(r-1)$th bay, the expression for the latter involving no indeterminateness if it is not itself of Euler's length. If, however, the $(r-1)$th bay is also of Euler's length, then that bay must be treated in the same way as the rth above, with reference to the $(r-2)$th. It follows that all the coefficients are determinate, provided that one bay is not of Euler's length. Corresponding with each Euler's bay, there will, of course, exist a condition of the form given in (34), and under these circumstances the structure will not fail provided it can be shown that none of the bending moments become infinite for any value of the longitudinal load F up to that here considered. This will come out from a discussion of the equations for bending moments.

It will be necessary to investigate the limiting forms which the coefficients A_r and B_r assume in this case. It is clear, of course, that both these quantities become in the limit infinitely great, but

$$\lim_{\lambda_r l_r = \pi} (A_r - B_r) = \lim_{\lambda_r l_r = \pi} \left\{ \frac{l_r}{E_r I_r} \quad \frac{1}{\lambda_r^2 l_r^2} \left(\frac{\lambda_r l_r}{\sin \lambda_r l_r} + \frac{\lambda_r l_r}{\tan \lambda_r l_r} - 2 \right) \right\}$$

$$= \lim_{\lambda_r l = \pi} \left\{ \frac{l_r}{E_r I_r} \left(\frac{1 + \cos \lambda_r l_r}{\sin \lambda_r l_r} - \frac{2}{\lambda_r l_r} \right) \right\} \frac{1}{\lambda_r l_r}$$

$$= \lim_{\lambda_r l_r = \pi} \left\{ \frac{l_r}{E_r I_r} \quad \frac{1}{\lambda_r l_r} \left(\frac{1}{\tan \frac{1}{2} \lambda_r l_r} - \frac{2}{\lambda_r l_r} \right) \right\}$$

$$= -\frac{2 l_r}{\pi^2 E_r I_r} \tag{35}$$

Equations (24_1) to (24_{n-1}) determine the bending moments, but (24_{r-1}) and (24_r) require special treatment on account of the limiting forms of A_r, B_r, and K_r. Setting out these two equations we obtain

$$M_{r-2}A_{r-1} + M_{r-1}B_{r-1} + (M_{r-1}B_r + M_r A_r + K_r)$$

$$= \delta_{r-1}(1/l_{r-1} + 1/l_r) - K_{r-1}, \tag{36}$$

$$(M_{r-1}A_r + M_r B_r + K_r) + M_r B_{r+1} + M_{r+1}A_{r+1} = \delta_r(1/l_r + 1/l_{r+1}) - K_{r+1} \tag{37}$$

The groups of terms enclosed in the brackets to the left-hand side of these equations clearly require modification The following relations have been found to hold for uniform loading —

$$\lim (M_{r-1} + M_r) = \frac{2w_r}{\lambda_r^2} \tag{38}$$

and
$$\lim (A_r - B_r) = -\frac{2l_r}{\pi^2 E_r I_r} \tag{39}$$

Subtracting (36) from (37) we find

$$(M_{r-2}A_{r-1} + M_{r-1}B_{r-1}) - (M_r B_{r+1} + M_{r+1}A_{r+1}) + (M_{r-1} - M_r)(B_r - A_r)$$

$$= \delta_{r-1}(1/l_{r-1} + 1/l_r) - \delta_r(1/l_r + 1/l_{r+1}) - K_{r-1} + K_{r+1}$$

The only term which adopts an indeterminate form is

$$\lim_{\lambda_r l = \pi} (M_{r-1} - M_r)(B_r - A_r) \longrightarrow \frac{2l_r}{\pi^2 E_r I_r} (M_{r-1} - M_r)$$

Equations (24_{r-1}) and (24_r) must now be replaced by

$$M_{r-1} + M_r = \frac{2 w_r l_r^2}{\pi^4}, \tag{40}$$

and
$$M_{r-2}A_{r-1} + M_{r-1}\left(B_{r-1} + \frac{2l_r}{\pi^2 E_r I_r}\right) - M_r\left(B_{r+1} + \frac{2l_r}{\pi^2 E_r I_r}\right) - M_{r+1}A_{r+1}$$

$$= \delta_{r-1}(1/l_{r-1} + 1/l_r) - \delta_r(1/l_r + 1/l_{r+1}) - K_{r-1} + K_{r+1} \tag{41}$$

The two equations (24_{r-1}) and (24_r), containing coefficients which become infinite or indeterminate, can now, therefore, be replaced by (40) and (41), in which all the bending moments occurring in the previous two are still present, but the coefficients are all finite and known It follows that in general the system of equations (24_1), (24_2), .., (24_{r-2}), (40), (41), . . (24_{n-1}), will determine finite bending moments for all the supports, and, consequently, also for every point in the beam Only under special conditions about to be treated will failure occur due to buckling

Summing up, then, a beam supported at any number of simple supports will not fail, in general through the bending moments becoming excessive, even if some of the bays are of Euler's critical length, provided at least one bay is not of this length Whether or not this will correspond with a stable loading of the structure will depend on whether or not the actual longitudinal load, if any, that would produce infinite bending moments is greater or less than Euler's load for the bays in question It appears then that the presence of bays of Euler's length is not in itself either a necessary or sufficient criterion for instability An accurate investigation of the conditions under which failure will take place will be given and a comparison between the crippling load, thus determined, and Euler's load will decide whether instability exists at the latter position

§ 10 *Cases where One of the Bays is twice Euler's Length*—The equation for the rth bay is

$$y = \mathrm{C}_r \cos \lambda_r x + \mathrm{D}_r \sin \lambda_r x - \{\mathrm{M}_{r-1} - f_{r-1}(l_r - x)\}/\mathrm{F} - \chi_r(x), \tag{12}$$

when $x = 0, \quad y = \delta_r,$

$$x = l_r, \quad y = 0,$$

and, remembering that $2\pi = \phi_r = \lambda_r l_r$, we find

$$\delta_r = \mathrm{C}_r - (\mathrm{M}_{r-1} - f_{r-1} l_r)/\mathrm{F} + \chi_r(0),$$

$$0 = \mathrm{C}_r - (\mathrm{M}_{r-1})/\mathrm{F} + \chi_r(l_r),$$

therefore $\delta_r = \dfrac{f_{r-1} l_r}{\mathrm{F}} + \delta_r(0) - \chi_r(l_r)$

Now $$f_{r-1} l_r/\mathrm{F} = \delta_r + (\mathrm{M}_{r-1} - \mathrm{M}_r)/\mathrm{F} + \chi_r(0)/\mathrm{F}, \tag{14}$$

therefore $$\mathrm{M}_{r-1} - \mathrm{M}_r = \mathrm{F}\{\chi_r(l_r) - \chi_r(0)\} - \psi_r(0) \tag{42}$$

This is the modified equation for this case corresponding with that given in equation (32)

Proceeding by the same method adopted there the equations that ought to furnish the bending moments for the rth bay are

$$\mathrm{M}_{r-2}\mathrm{A}_{r-1} + \mathrm{M}_{r-1}(\mathrm{B}_{r-1} + \mathrm{B}_r) + \mathrm{M}_r\mathrm{A}_r + \mathrm{K}_r = \left(\frac{1}{l_{r-1}} + \frac{1}{l_r}\right)\delta_{r-1} - \mathrm{K}_{r-1}, \tag{43}$$

$$\mathrm{M}_{r+1}\mathrm{A}_{r+1} + \mathrm{M}_{r-1}\mathrm{A}_r + \mathrm{M}_r(\mathrm{B}_r + \mathrm{B}_{r+1}) + \mathrm{K}_r = \delta_r\left(\frac{1}{l_r} + \frac{1}{l_{r-1}}\right) - \mathrm{K}_{r+1} \tag{44}$$

In these equations, however, when $\phi_r = 2\pi$, A_r and B_r become infinite It remains to show that the bending moments under these circumstances must also become infinite From the last two equations, by subtraction, we find

$$M_{r+1}A_{r+1} + M_r(B_r + B_{r+1} - A_r) + M_{r-1}(A_r - B_{r-1}) - M_{r-2}A_{r-1}$$

$$= \delta_r\left(\frac{1}{l_r} + \frac{1}{l_{r-1}}\right) - K_{r+1} - \delta_{r-1}\left(\frac{1}{l_{r-1}} + \frac{1}{l_r}\right) + K_{r-1}, \quad (45)$$

therefore

$$(M_r - M_{r-1})(B_r - A_r) - M_{r-2}A_{r-1} - M_{r-1}B_{r-1} + M_rB_{r+1} + M_{r+1}A_{r+1} = \text{etc.} \quad (46)$$

Now $M_r - M_{r-1}$ is already found in terms of the loading and in general is not zero, while

$$\operatorname*{Lim}_{\phi \to 2\pi}(B_r - A_r) \to \infty$$

Hence this equation, giving a relation between the bending moments M_{r-2}, M_{r-2}, M_r, and M_{r+1}, contains an infinite term, and since the coefficients of these bending moments and the remaining terms are all finite, it follows that one at least of the bending moments must become infinite, and the structure must fail This is, however, not the lowest critical loading

§ 11. *Conditions of Failure of a Supported System* —The criterion of failure which will be utilised in the present discussion is, as already explained, the production of infinite binding moments at the supports Associated with this will be the assumption already known to be valid in the case of a single strut, that for values of the longitudinal force greater than the least that will produce these infinite bending moments, the geometry of the structure will not be maintained, the failure, in this sense, thus corresponding with instability It remains, therefore, to write down the mathematical expression which must be satisfied when the bending moments are infinite and to determine the least root, regarding it as an equation in F, the longitudinal force

§ 12 *Condition of Failure for the Case of n Bays, i.e., n+1 Supports* (fig 4) —The equations (24_1), .. , (24_{n-1}) are sufficient in general to determine the bending moments at the supports. These equations being linear, each of the bending moments may be expressed as the ratio between two determinants whose constituents are functions of the lengths of the bays, deflections, and the lateral and longitudinal loading. The bending moments will become infinite when, and only when, the denominator vanishes. This condition, assuming the bending moments at the end supports zero, expressed in determinantal form is —

$$
0 = \begin{vmatrix}
B_1+B_2, & A_2, & 0, & 0, & \cdots & 0, & 0 \\
A_2, & B_2+B_3, & A_3, & 0, & & 0, & 0 \\
0, & A_3, & B_3+B_4, & A_4, & & 0, & 0 \\
0, & 0, & A_4, & B_4+B_5, & & 0, & 0 \\
0, & 0, & 0, & A_5, & & 0, & 0 \\
 & & \cdot\cdot & & & & \cdot \\
 & \cdot & & & & & \cdot \\
\cdot & & & & & & \\
 & & & & A_{n-3}, & 0, & 0 \\
0, & 0, & 0, & A_{n-3}, & B_{n-3}+B_{n-2}, & A_{n-2}, & 0 \\
0, & 0, & 0, & 0, & A_{n-2}, & B_{n-2}+B_{n-1}, & 0 \\
0, & 0, & 0, & 0, & 0, & A_{n-1}, & B_{n-1}+B_n
\end{vmatrix} \tag{47}
$$

since

$$M_1 = 0, \qquad M_{n+1} = 0$$

$$A_n = \frac{l_n}{E_n I_n} \cdot \frac{1}{\phi_n^2}\left(\frac{\phi_n}{\sin\phi_n} - 1\right), \qquad B_n = \frac{l_n}{E_n I_n} \frac{1}{\phi_n^2}\left(1 - \frac{\phi_n}{\tan\phi_n}\right) \tag{48}$$

where

$$\phi_n = \lambda_n l_n$$

§ 13 For any particular case this symmetrical determinant can be very easily evaluated by the following scheme

Writing the value of this determinant for the case bays of Nos 1, 2, etc, as Δ_1, Δ_2, etc, we obtain—

		No of bays
1	$= \Delta_1$	1
$(B_1+B_2)\,\Delta_1$	$= \Delta_2$	2
$(B_2+B_3)\,\Delta_2 - A_2^2\Delta_1$	$= \Delta_3$	3
$(B_3+B_4)\,\Delta_3 - A_3^2\Delta_2$	$= \Delta_4$	4
$(B_4+B_5)\,\Delta_4 - A_4^2\Delta_3$	$= \Delta_5$	5
. .	.	
$(B_{n-1}+B_n)\,\Delta_{n-1} - A_{n-1}^2\Delta_{n-2}$	$= \Delta_n$	n

(49)

When the value of ϕ for any bay reaches π, A and B become infinite and accordingly the determinant in general also becomes infinite To avoid this it would be convenient to divide throughout by the product of the A's, so that instead of taking $\Delta_n = 0$ as the determinantal equation the condition

$$D_n = \frac{1}{A_1 \,.\, A_2 \,.\, A_3 \,\ldots A_n} \Delta_n \tag{50}$$

may be used

Two special cases need comment corresponding with any one of the A's becoming either zero or infinite In the former this can only happen when l is made indefinitely small, but, as can be seen, in neither case is any real difficulty involved in the calculations It is interesting to note that $D_1 = 1/A_1 = 0$, the criterion of failure for one bay, gives the result obtained by Euler

§ 14 The determinantal condition (53) involves $\phi_1, \phi_2, \phi_3, \ldots$, which are all known functions of F, the end thrust, and is independent of the lateral loading Thus

$$\phi_r^2 = l_r^2 F/E_r I_r,$$

and a solution of the problem is obtained immediately the smallest value of F satisfying the condition (53) is found It has been assumed that this longitudinal thrust is practically constant along the whole length of the beam, since the reactions at the joints acting perpendicular to the central axis can only modify F by second-order quantities The analysis, however, can be applied more generally, with very slight modification, to the more practical case—as far as aeroplane frameworks are concerned—by supposing wires under tension fixed to the beam at the supports with a component force directed along the central axis A simple resolution of forces suffices to show that the bays are now under end thrusts, not necessarily all equal The rth bay may then be represented as being under a longitudinal force F_r, expressible of course in terms of the force in one particular ϕ_1, ϕ_2, etc, now take the more generalised form

$$\phi_1 = l_1 \sqrt{\frac{F_1}{E_1 I_1}}, \qquad \phi_r = l_r \sqrt{\frac{F_r}{E_r I_r}}, \text{ etc}$$

and the analysis proceeds as before The determinantal condition (47), being an expression containing all the ϕ's, can then be easily expressed as a function of the ϕ for any particular bay When the point of attachment of the wires is not on the neutral axis, a bending moment of known magnitude is introduced at the corresponding support, and the appropriate modification must then be inserted in the equations for three moments

§ 15 *Clamped Supports*—Suppose at any point in the beam the central axis besides being constrained to pass through a particular point is, by means of a clamp of some nature, compelled to have its central axis in a given direction at the point Such a support will be termed a clamped support It is proposed to consider what effect the introduction of such constraints has upon the strength of the structure by the determination of the positions of the criticals in comparison with those produced when the same support is unclamped. For the present purpose, a clamped support is in effect equivalent to two simple supports infinitely close together, and the discussion consequently centres round the limiting forms of the conditions already obtained when one or more of the terms l_r tends to zero.

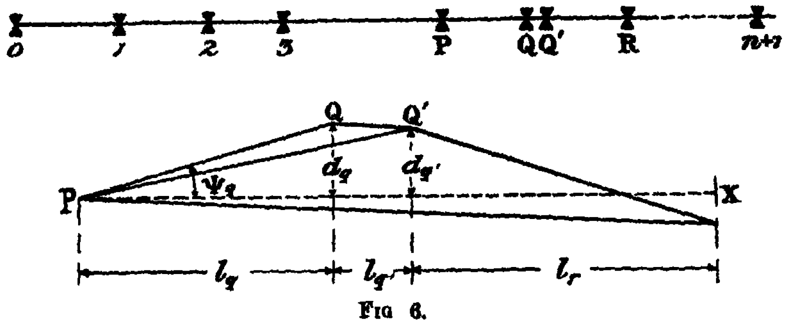

FIG 6.

Let 0, 1, , $n+1$ be a beam supported at $n+1$ points by simple supports, and at QQ' by two neighbouring supports, to constitute a clamped support Consider a small section, PQQ'R, and let the heights of Q and Q' above the datum line PX be d_q and d_q' respectively, then

$$\begin{aligned}\delta_q &= \text{the distance of Q above PQ'}\\ &= d_q - d_q' \frac{l_q}{l_q + l_q'} \text{ approx}\\ &= d_q - d_q'(1 + l_q'/l_q)^{-1} = d_q - d_q'(1 - l_q'/l_q)\end{aligned}$$

Hence $\delta(1/l_q + 1/l_q') = \{d_q - d_q'(1 - l_q'/l_q)\}(1/l_q + 1/l_q')$

In the limit, when Q and Q' coincide,

$$\operatorname*{Lim}_{l_q'=0} \delta_q'(1/l_q' + 1/l_r) = \theta_q - \Psi_r, \tag{51}$$

where Ψ_q is the inclination of PQ to the datum line and θ_q the inclination of the tangent to the same line, measuring angles in an anti-clockwise direction

Similarly, δ_q' = distance of Q' from QR, and hence

$$\operatorname*{Lim}_{l_q'=0} \delta_q'(1/l_q' + 1/l_r) = \theta_q - \Psi_r \tag{52}$$

In the light of these, the equations of three moments become modified by the substitution of the expression found in (56) and (57) for the deflection terms, and the analysis proceeds as before It is evident, of course, both physically and mathematically, that the system of equations is separated at the points of double support into series of self-inclusive sets so that each portion of the beam between the clamped supports may be treated separately It remains, therefore, to discuss only the cases where the beam is clamped at either or at both ends

§ 16 *Beam Supported at $n-1$ Intermediate Points and Clamped at Both Ends*

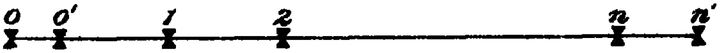

FIG 7

Let 0, 0′, 1, . . , n', n be the beam supported simply at 1, 2, $(n-1)$, and clamped at 0′0 and $n'n$ Let 0′0 and $n'n$ be l_0' and l_n' respectively, where these limit to zero Then if δ_0' = deflection at 0′ and δ_n = deflection at n,

$$\delta_0'\left(\frac{1}{l_0}+\frac{1}{l_1}\right) = \theta_0-\Psi_1$$

and

$$\delta_0'\left(\frac{1}{l_{n-1}'}+\frac{1}{l_n'}\right) = -\theta_n+\Psi_n$$

Remembering that

$$A_0 = 0, \quad B_0 = 0, \quad A_n' = 0 \text{ and } B_n' = 0,$$

the system of equations for the moments at the supports takes the form

$$M_0B_1+M_1A_1 = \theta_0-\Psi_1-K_1,$$

$$M_0A_1+M_1(B_1+B_2)+M_2A_2 = \delta_1\left(\frac{1}{l_1}+\frac{1}{l_2}\right)-(K_1+K_2),$$

$$M_{n-2}A_{n-2}+M_{n-1}(B_{n-2}+B_{n-1})+M_nA_{n-1} = \delta_{n-1}\left(\frac{1}{l_{n-1}}+\frac{1}{l_n}\right)-(K_{n-1}+K_n),$$

$$M_{n-1}A_{n-1}+M_nB_{n-1} = -\theta_n+\Psi_n-K_n \qquad (53)$$

These provide $n+1$ equations for the $n+1$ bending moments which now include M_0 and M_n as unknowns

The determinantal condition corresponding with critical loading is of course

$$\begin{vmatrix} B_1, & A_1, & 0, & 0, & & 0 \\ A_1, & B_1+B_2, & A_2, & 0, & & 0 \\ & & & & & \\ \cdot & \cdot & & & & \\ 0, & & & A_{n-2}, & B_{n-2}+B_{n-1}, & A_{n-1} \\ 0, & \cdot\ \cdot & & 0, & A_{n-1}, & B_{n-1} \end{vmatrix} = 0 \qquad (54)$$

§ 17 The case where the double support is at one end only may likewise be simply treated

The effect upon the strength of the structure of the introduction of the double support may, qualitatively at least, be seen from simple considerations without reference to the determinant of failure It has already been shown that if a structure of any number of simple supports be so spaced as to be critical under the loading, then the addition of one more bay at the end will have a strengthening effect, provided that the length of the bay is not greater than the length of Euler's strut Remembering that the substitution of a clamped for a simple support is ultimately merely the addition of another bay of infinitely small length, it follows that clamping one or both ends is equivalent to a direct addition of strength The exact determination of the magnitude of this increased safety is, of course, to be obtained by finding the smallest root of the determinant regarded as an equation in F In fact, the whole problem of the most efficient disposition of the supports to provide the strongest structure, in the sense that the critical longitudinal force is a maximum among those for all possible dispositions, can be treated with comparative simplicity by a discussion of the determinants already set down

The authors are indebted to Prof Love for his helpful criticisms and valuable suggestions regarding the notation

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The Electromagnetic Inertia of the Lorentz Electron.

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(Communicated by Prof J W Nicholson, F R.S Received January 18, 1918)

1 In a recent paper[*] on "The Effective Inertia of Electrified Systems moving with High Speed," G W Walker extends his previous investigation to the case of a perfectly conducting oblate spheroid, whose axis lies in the direction of motion, for the time being, of its centre, and whose eccentricity is equal to k, where k denotes the ratio of the speed of the centre to that of light, also for the instant under consideration He finds (Note 1) that the longitudinal and transverse masses are equal respectively to

$$2e^2(1-\tfrac{1}{5}k^2)/3ac^2(1-k^2)^{3/2} \quad \text{and} \quad 2e^2(1+\tfrac{1}{60}k^2)/3ac^2(1-k^2)^{1/2}, \tag{1}$$

instead of

$$2e^2/3ac^2(1-k^2)^{3/2} \quad \text{and} \quad 2e^2/3ac^2(1-k^2)^{1/2}, \tag{2}$$

the values given by the mass formulæ of Lorentz and required by the Principle of Relativity

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[*] 'Roy. Soc. Proc,' A, vol 93, p. 448 (1917)

The method used by Walker obviates the objections raised to the assumption of quasi-stationary motion, which is necessary in the ordinary energy method of calculating the masses, but it requires special assumptions to be made at the outset both as to the constitution of the electron and as to the boundary conditions at its surface whenever definite information is needed These assumptions obviously limit the generality of the investigation, so that it is open to the objection that Walker's model of the electron is not a suitable one, precisely because it does not agree with the Principle of Relativity

2 About ten years ago I developed a method of calculating the electromagnetic masses of an electron from the fundamental equations of the Electron Theory, without making any assumptions respecting either the structure of the electron or the nature of its motion, whether quasi-stationary or otherwise, until the conclusion of the general part of the investigation This formed the basis of Chapter XI of my Adams Prize Essay, and was published in full in my book on 'Electromagnetic Radiation,' Appendices C and D (Note 2)

In this method the usual retarded potentials due to an element of charge moving in any assigned manner are developed in series, of which the first terms are the potentials of the element due to uniform motion with the velocity at the instant considered, whilst the succeeding terms are derived from them by definitely assigned operations, and involve the accelerations of various orders The electric, magnetic, and mechanical forces on a second element of charge are derived by the usual operations, and the resultant mechanical force on the electron is obtained by a double integration over all the pairs of elements of charge in the form of a series proceeding according to ascending powers of a length a, which measures the linear dimensions of the electron—the radius for the Abraham electron, the transverse radius for the Lorentz electron The first two terms of this series for the internal electromagnetic force **F** on the electron are of the form

$$\mathbf{F} = -\frac{d\mathbf{G}}{dt} + \mathbf{K}\,\text{*} \qquad (3)$$

G may be identified with the electromagnetic momentum of the electron, it is of the order a^{-1}, and depends only on the configuration of the electron, on the relative motion of its parts, and on the velocity of its centre, all for the instant under consideration. It gives rise to the principal term in the expression for the resultant internal electromagnetic force, the only term found in the ordinary energy method.

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* *Loc. cit.*, p 246, equation (341)

K represents the reaction due to radiation, it does not depend on the configuration of the electron, nor on the relative motion of its parts, but only on the velocity, acceleration, and acceleration of the second order of its centre It is of order zero in a, and is very small compared with the principal term, except when the speed is nearly equal to that of light and the accelerations are very large On p 183 of my book I have estimated the error arising from neglecting it in the case of a β-particle moving in an electrostatic field of 1,000,000 volts per centimetre With a minimum speed of 0 999 of that of light I find it to be less than two parts in 10,000,000,000 For an electron moving in an orbit of atomic dimensions, the error would be much greater, but still quite unimportant, unless the speed were very nearly equal to that of light

The next term of the series is of order a and depends on the velocity and the accelerations of the first, second, and third orders, and so on These higher terms could be calculated if it were worth while, but in all practical cases we can obtain a sufficient idea of the rate of convergence of the series from its first two terms It is important to notice that, according to my results, the error made in assuming quasi-stationary motion arises from neglecting **K** and the higher terms, but cannot affect the principal term

3 By applying this method to the Lorentz electron* I obtain precisely the two Lorentz mass formulæ, in full agreement with the ordinary energy method, but differing from Walker's result The discrepancy is all the more serious in that it occurs in the principal term, and for this reason casts grave doubts on the accuracy of the most fundamental investigations of the Electron Theory

In looking for the cause of this discrepancy one is struck at once by the fact that Walker assumes k (his symbol for the ratio of the speed of the electron to that of light, usually denoted by β) to be constant On p 452 of his paper he writes "It might be argued that I ought to have supposed that the contracted electron should alter in shape as the acceleration proceeds But the difficulty is to know how it alters There is no correspondence between accelerated motions at different speeds such as can be proved for uniform speeds, for the transformation entirely breaks down if the velocity is not constant One might, as a pure hypothesis which has no *a priori* justification, assume that the surface is deformed according to the same law as holds for uniform velocity This could be worked out, and might lead to a first order term (in acceleration) for longitudinal inertia, but clearly would lead to only second order terms for transverse inertia But this appears to me too speculative to be of any value at present."

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* *Loc cit.*, p. 257

4. It appears to me that it is of the very essence of the Lorentz electron that its shape should alter as its acceleration proceeds, and that in such a manner as to correspond to its speed at each instant; at any rate, that is the assumption tacitly made in my own investigation and, I believe, in all others except Walker's. At the same time, I have thought it worth while to follow out the consequences of Walker's assumption of a constant k by my own method, as I had the material for such an investigation ready to hand. On p. 260 of 'Electromagnetic Radiation' I have given expressions for the electromagnetic masses of a deformed Lorentz electron, which can be utilised for this purpose. The deformation there contemplated is supposed to be small and constant, whilst that of Walker's electron increases in proportion to the time t measured from the instant at which the speed is kc. But t is to be made zero ultimately, so that the smallness creates no difficulty. The electromagnetic momentum **G**, on the other hand, depends on the relative velocities of the parts of the electron, so that the variability of the deformation of Walker's electron from Lorentz's form might be expected to cause difficulty. This, however, is not so, for I have shown on p. 247 of my book that the relative motion term in the electromagnetic momentum vanishes identically for an electron which, like Lorentz's electron, is symmetrical with respect to the centre.

5. In using the expressions in question we must bear in mind that the actual electron at time t, which is approximately of Lorentz's form, is for convenience of calculation transformed by stretching in the direction of motion in the ratio $\sqrt{(1-\beta^2)}:1$, where β denotes the ratio of the speed of the electron at time t to that of light, a process which changes the Lorentz electron with axes $a\{\sqrt{(1-\beta^2)}, 1, 1\}$ into a sphere of radius a. The electron thus transformed is assumed to be given by the equation of its surface

$$r = a + \epsilon P_2(\mu). \tag{4}$$

The quantity ϵ is so small that its square and higher powers can be neglected; the axis of the spherical harmonic is supposed to make an angle ψ with the direction of motion of the electron measured in the osculating plane of its orbit towards the principal normal. The co-ordinates ξ, η, ζ, of an element of the transformed electron are referred to the tangent, principal normal, and bi-normal of the orbit as axes. The tangential and normal components of the electromagnetic momentum are given by the equations

$$G_\xi = \frac{2e^2v}{3c^2a\sqrt{(1-\beta^2)}}\left\{1+\frac{\epsilon}{10a}\left(3\cos^2\psi-1\right)\right\}, \tag{5}$$

$$G_\eta = \frac{e^2v\epsilon}{5c^2a^2}\sin\psi\cos\psi \tag{6}$$

Here $v = c\beta$ and denotes the speed of the electron at the time t, and the expressions apply to a surface charge, of uniform density on the transformed Lorentz electron, but of density proportional to the perpendicular from the centre on the tangent plane of the actual Lorentz electron before deformation

6 We shall first apply these expressions to Walker's first case of an electron moving in a straight line with acceleration μ (in his notation) At time t the speed is given by $\beta = k+\mu t/c$, while ψ is zero From (4) we find that the tangential and transverse radii of the transformed electron are equal to $a+\epsilon$ and $a-\frac{1}{2}\epsilon$ respectively, hence those of the actual electron are equal to $(a+\epsilon)\sqrt{(1-\beta^2)}$ and $a-\frac{1}{2}\epsilon$ respectively But the same values for Walker's electron are $a_0\sqrt{(1-k^2)}$ and a_0 respectively, where the suffix is used for the moment to distinguish the two electrons Equating corresponding values we find to the first order in μ

$$a\sqrt{(1-\beta^2)} = a_0\sqrt{(1-k^2)}\{1-2k\mu t/3(1-k^2)c\}, \quad \epsilon/a = 2k\mu t/3(1-k^2)c.$$

Substituting these values in (5) we find, putting $\psi = 0$, and a for a_0,

$$G_t = \frac{2e^2}{3c^2a\sqrt{(1-k^2)}}\left\{ck+\frac{1-\frac{1}{6}k^2}{1-k^2}\mu t\right\},$$

while G_n obviously vanishes Using (3) and differentiating on the supposition that k is constant we obtain for the resultant internal electromagnetic force on the electron

$$F_t = -\frac{dG_t}{dt} = -\frac{2e^2(1-\frac{1}{6}k^2)}{3c^2a(1-k^2)^{3/2}}\mu, \tag{7}$$

whilst F_n vanishes.

The coefficient of the acceleration μ in (7) is the longitudinal electromagnetic mass and agrees with the first of Walker's expressions (1)

7 Walker's second case is more difficult to investigate So far as I understand his procedure from the very brief description given in his paper, he seems to treat the electron as a rigid perfect conductor, which coincides at the instant $t = 0$ with the Lorentz electron as before, but moves without rotation, so that at the time t its axis deviates from that of the Lorentz electron by the small angle ωt, measured in the osculating plane towards the outside of the orbit Here ω denotes the angular velocity of the electron in its orbit, so that the acceleration μ is equal to ωv, where v is constantly equal to ck Thus the equation of the surface of Walker's electron referred to the tangent, principal normal, and binormal of the orbit at time t as axes becomes on this supposition

$$\frac{(x-y\omega t)^2}{1-k^2}+(y+x\omega t)^2+z^2 = a^2 \quad \text{or} \quad \frac{x^2}{1-k^2}+y^2+z^2-\frac{2k^2\omega t}{1-k^2}xy = a^2$$

to the first order in the small angle ωt. Applying the inverse Lorentz transformation $x = \xi\sqrt{(1-k^2)}$, $y = \eta$, $z = \zeta$, we find that the equation of the transformed Walker's electron is

$$\xi^2+\eta^2+\zeta^2-\frac{2k^2\omega t}{\sqrt{(1-k^2)}}\xi\eta = a^2$$

Two principal axes lie in the osculating plane and make angles $\psi_1 = 45°$ and $\psi_2 = 135°$ with the tangent, whilst the lengths of the corresponding semi-axes are equal to $a\{1 \pm k^2\omega t/2\sqrt{(1-k^2)}\}$ respectively

On the other hand we may write the equation in the form

$$r = a + \epsilon_1 P_2(\mu_1) + \epsilon_2 P_2(\mu_2) \tag{8}$$

analogous to (4), where μ_1 is measured from the axis ψ_1 and μ_2 from ψ_2 Putting $\mu_1 = 1$, $\mu_2 = 0$, and $\mu_1 = 0$, $\mu_2 = 1$, alternately we find that the two principal semi-axes are $a+\epsilon_1-\frac{1}{2}\epsilon_2$ and $a-\frac{1}{2}\epsilon_1+\epsilon_2$ respectively Comparing the two pairs of values we obtain

$$\epsilon_1 = -\epsilon_2 = ak^2\omega t/3\sqrt{(1-k^2)} \tag{9}$$

Owing to the smallness of ϵ_1 and ϵ_2, the principle of superposition holds and (5) and (6) each contain two terms corresponding to ϵ_1 and ϵ_2 In the first the two terms clearly cancel each other, in the second they are equal Bearing in mind that, in the present case, β and k are identical, we find

$$G_\xi = \frac{2e^2v}{3c^2a\sqrt{(1-k^2)}}, \qquad G_\eta = \frac{e^2k^2v\omega t}{15c^2a\sqrt{(1-k^2)}}$$

In using (3) we must bear in mind that the axes are rotating with the angular velocity ω, hence we find that the resultant internal electromagnetic force on the electron at the time $t = 0$ is given by

$$F_\xi = -\frac{dG_\xi}{dt}+\omega G_\eta = 0, \qquad F_\eta = -\frac{dG_\eta}{dt}-\omega G_\xi = -\frac{2e^2(1+\frac{1}{10}k^2)v\omega}{3c^2a\sqrt{(1-k^2)}} \tag{10}$$

The value of F_η differs from that found by Walker by having a term $\frac{1}{10}k^2$ inside the bracket in the numerator in place of $\frac{1}{80}k^2$.

8 One difficulty remains which requires examination In the three preceding sections we have assumed the surface density of the transformed electron to be uniform, or, what amounts to the same thing, that of the actual electron to be proportional to p, the perpendicular from the centre on the tangent plane. But the surface density of Walker's electron is not proportional to p simply, in the first case it involves a small term, of the first order in the acceleration, proportional to px, and, in the second case, similar small terms proportional to py, as we see from his expressions for σ given on p. 450 and on p. 451 respectively. I shall now prove that these small additional terms in the surface density do not affect the expressions

(7) and (10) so long as we confine our investigation to terms of the first order The proof depends upon the following propositions, which are stated as briefly as possible —

(1) Suppose that a slightly deformed Lorentz electron of uniform volume density is transformed by the Lorentz transformation $x = \xi\sqrt{(1-\beta^2)}$, $y = \eta$, $z = \zeta$, so that the equation of the bounding surface of the transformed electron becomes of the form $r = a + \Sigma\epsilon P_i(\mu)$

Then the only terms which influence the electromagnetic masses are those for which $i = 2$

This proposition is proved on p 260 of 'Electromagnetic Radiation'

(2) Let the equation of the transformed surface be of the form

$$f(\xi, \eta, \zeta) = r - a - \Sigma\epsilon Z_i = 0, \tag{11}$$

where Z_i is a homogeneous function of ξ, η, ζ of degree i, for instance, the solid harmonic associated with P_i, and ϵ is a small coefficient as before

Then the perpendicular from the origin on the tangent plane ϖ is equal to the radius r, squares and products of the ϵ's being neglected.

In fact, if the direction cosines of ϖ be λ, μ, ν, we have in succession,

$$\frac{\partial f}{\partial \xi} = \frac{\xi}{r} - \Sigma\epsilon\frac{\partial Z_i}{\partial \xi}, \text{ etc }, \qquad \sqrt{\left\{\left(\frac{\partial f}{\partial \xi}\right)^2 + \left(\frac{\partial f}{\partial \eta}\right)^2 + \left(\frac{\partial f}{\partial \zeta}\right)^2\right\}} = 1 - \Sigma\epsilon\frac{iZ_i}{r},$$

$$\lambda = \frac{\partial f}{\partial \xi}\Big/\sqrt{\left\{\left(\frac{\partial f}{\partial \xi}\right)^2 + \left(\frac{\partial f}{\partial \eta}\right)^2 + \left(\frac{\partial f}{\partial \zeta}\right)^2\right\}} = \left\{\frac{\xi}{r} - \Sigma\epsilon\frac{\partial Z_i}{\partial \xi}\right\}\left\{1 + \Sigma\epsilon\frac{iZ_i}{r}\right\}, \text{etc},$$

$$\varpi = \lambda\xi + \mu\eta + \nu\zeta = \{r - \Sigma\epsilon i Z_i\}\left\{1 + \Sigma\epsilon\frac{iZ_i}{r}\right\} = r + \text{terms of order } \epsilon^2$$

(3) Consider a homœoid of uniform volume density A, bounded by the surfaces with radii r' and $(1+\alpha)r'$, where

$$r' = r\{1 + \gamma S_n\} = a + \Sigma\epsilon Z_i + a\gamma S_n \tag{12}$$

A is infinitely large and α infinitely small, but so that $A\alpha$ is finite and equal to $e/4\pi a^3$ ultimately S_n is a homogeneous function of ξ, η, ζ of degree n, for instance, a spherical harmonic of order n, and γ is a small coefficient of the order of ϵ

Then the homœoid is equivalent to a surface distribution of density τ on the surface (11), where

$$\tau = er\{1 + \gamma S_n\}/4\pi a^3 = e\varpi\{1 + \gamma S_n\}/4\pi a^3 \tag{13}$$

In fact, the thickness of the homœoid is equal to $\alpha r'$, and the surface density to the limit of $A\alpha r'$, whilst $r = \varpi$ to the first order by proposition (2)

(4) Let us regard (11) as a transformed electron, to which corresponds an

actual electron with surface density σ, and perpendicular on the tangent plane p Then the surface densities σ and τ are connected by the equation

$$\sigma = \tau p/\varpi \sqrt{(1-\beta^2)} \tag{14}$$

Apply the Lorentz transformation, bearing in mind that it leaves elements of charge unaltered, but changes elements of volume in the ratio $\sqrt{(1-\beta^2)}$ 1 If corresponding surface elements be $d\mathrm{S}$ and $d\Sigma$, then corresponding elements of charge are $\sigma d\mathrm{S}$ and $\tau d\Sigma$, while corresponding elements of volume are $pd\mathrm{S}$ and $\varpi d\Sigma$ Hence we have

$$\sigma d\mathrm{S} = \tau d\Sigma, \qquad pd\mathrm{S} = \varpi d\Sigma\sqrt{(1-\beta^2)},$$

whence the result follows

(5) Replacing Z_i and S_n in (11), (12), and (13) by $r^i\mathrm{P}_i$ and $r^n\mathrm{P}_n$ respectively, where the P's denote zonal harmonics as usual, we may write a in place of r in the small terms hence it follows that the total charge of the homœoid in (3) is equal to e

Then we may express the result of the last section as follows —

Let the uniform homœoid, with total charge e, whose inner surface is given by (12), where Z_i and S_n now denote spherical harmonics, be transformed by the Lorentz transformation appropriate to the motion for the time being of its centre, viz, $\xi = r/\sqrt{(1-\beta^2)}$, $\eta = y$ $\zeta = z$ Then the transformed homœoid is equivalent to a charge of density σ distributed over the surface whose equation is (11), where

$$\sigma = ep\{1+\gamma\mathrm{S}\}/4\pi a^4\sqrt{(1-\beta^2)} \tag{15}$$

9 In order to apply this result to Walker's electrons, for which σ includes terms proportional to x and y respectively, we need only take Z_i proportional to the zonal harmonics P_i used in (4) and (8) respectively, and S_n proportional to x and y respectively, so that n is unity in each case

Then the corresponding transformed electrons are equivalent to uniform homœoids, whose inner boundaries are given by equations of the type (12) with $n = 1$, and $i = 2$ as before

By proposition (1) the first small deviation from the Lorentz electron does not influence the electromagnetic masses, whilst the second gives the expressions (7) and (10) respectively as before

10 It has been already stated in Section 4 that the electromagnetic momentum **G** depends on the relative motion of the parts of the electron, as well as on its configuration, and it might be thought that the relative motion term might account for the discrepancy between (10) and Walker's expression for the transverse mass, but this is not so

The relative motion term in question is given by (343) on p 247 of 'Electromagnetic Radiation', it is of order zero in a, but even if it were of order -1, it would disappear. For in the Lorentz electron there is rotation, but there is also symmetry with respect to the centre; on the other hand, in Walker's electron there is no symmetry with respect to the centre, but, if I understand Walker aright, there is no rotation and therefore no relative motion.

A clue is afforded by the presence of the last term, $+\frac{2}{3}Bk^2(1-k^2)^{-1}vpa^{-3}$, in the expression for $4\pi\sigma$ on p. 451 of Walker's paper, for this term when multiplied by the principal term in Q and integrated over the electron accounts for the discrepancy; but I have not been able to trace it to its source from the very brief indications given. (Notes 1 and 3.)

Be that as it may, the general agreement (apart from this discrepancy) between the results obtained by such widely different methods must be regarded as a welcome confirmation of the accuracy of the calculations, and in my opinion leaves little doubt that the differences between Walker's and Lorentz's electromagnetic masses are after all due to Walker's assumption of a contracted electron, which does not alter in shape as the acceleration proceeds (Note 4). Hence his results do not appear to me to constitute an argument against the Lorentz mass formulæ, or the Principle of Relativity, at any rate so long as his resulting mass formulæ do not afford an agreement with experiment so decidedly superior as to outweigh theoretical considerations.

I think that it will be pretty generally admitted that the mass formulæ of Lorentz have the advantage of all others from the point of view of pure theory, in so far as they are consistent with the Principle of Relativity and in addition admit of the possibility of a mechanical explanation of the electron, as shown in 'Electromagnetic Radiation,' Appendices D and E, whatever this possibility may be worth in the future. But, what is perhaps more important from the practical point of view, they afford the basis for a comparatively simple and workable mechanics of the electron, as I have shown in 'Electromagnetic Radiation,' Appendices F and G. This advantage of simplicity, which is almost essential from the point of view of economy of thought, is not shared by the more complicated formulæ, such as those of Abraham and Walker. Thus the superiority of any one of the latter over the formula of Lorentz in respect of agreement with experiment would require to be extremely marked before it could be adopted in preference.

11. When the experimental evidence is examined no marked difference is observed between the two formulæ for the transverse mass, and indeed all that Walker claims is that his formula is as good as that of Lorentz in this respect, and that the experiments so far performed cannot distinguish between them. Walker does not appear to be acquainted with the very careful and

extensive series of experiments of Neumann.* These were made by Bucherer's method and confirmed his results fully In all 55 plates were obtained, including all preliminary attempts, of which 26 were good enough to be measured Of these 22, taken between $\beta = 0{\cdot}4$ and $\beta = 0{\cdot}7$, gave an agreement with the Lorentz formula with a probable error in the specific charge, e/m_0, of 0·15 per cent, the remaining 4, taken between $\beta = 0{\cdot}7$ and $\beta = 0{\cdot}8$, indicated that the specific charge, e/m_0, increased faster than it should according to the Lorentz formula, by perhaps 2 per cent. The last result is regarded as doubtful by Neumann on account of the small number of experiments and the large experimental errors to which measurements at such high speeds are liable, he proposes to extend his experiments in this direction. If the result should be confirmed it would hardly be likely to make much difference in the relative performances of the two formulæ, although it might require the addition of some proportion of nonelectromagnetic mass to the formula of Lorentz. This is a possibility that I also have had in mind for some years in connection with the effect due to the second order terms in the internal electromagnetic force, those included in the radiation pressure **K**, this investigation, however, is not yet completed Nevertheless, the results obtained so far indicate that further experiments on the specific mass of the electron for speeds from $\beta = 0{\cdot}7$ upwards are very urgently needed For my own part I doubt whether experiments at these high speeds on the old lines can be expected to yield very good results, owing to the smallness of the deflections to be measured and the comparatively great width and diffuse character of the trace obtained on the photographic plate. In 'Electromagnetic Radiation,' p 300, a method is given which would appear to have some advantages, but it could only be performed on a scale beyond the resources of most laboratories and possibly only in institutions possessing an equipment such as that of the National Physical Laboratory.

Notes added April 24, 1918.

(1) On repeating Walker's calculations for the transverse acceleration, I find that the sign of the discrepant term in the expressions for (X, Y, Z) and $4\pi\sigma$ on p. 451 of his paper should be positive In consequence, the factor $1+\frac{11}{30}k^2$ in the expression for the transverse inertia on p. 452 must be replaced by $1+\frac{1}{30}k^2$.

(2) My expression for the internal electromagnetic force on the electron is based on three assumptions: (*a*) the retarded potentials of a point charge,

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* 'Ann. d. Phys.,' Folge 4, Band 45, S. 529 (1914).

according to Liénard and Wiechert, (*b*) the usual expression for the mechanical force on a moving electric charge, together with the usual rule of composition of forces, and (*c*) the assumed convergence of my expansions for the potentials and electric, magnetic, and mechanical forces. Few will cavil at (*a*) and (*b*), but (*c*) has not been, and I doubt whether it ever will be, proved for the most general type of motion In certain cases the convergence of the series can be demonstrated, as I have proved elsewhere, for a uniform longitudinal acceleration,* they converge almost certainly in all cases where the acceleration is not too great, and the speed of the electron not too close to that of light With this proviso, my expression for the electromagnetic momentum can be applied to any electron, whatever its constitution, form, state of strain or internal motion, and motion as a whole may be, provided that the relative velocities of its parts are small enough

(3) I have now succeeded in tracing the causes of the discrepancy referred to in the text by applying Walker's method, but on the basis of the principles used by Lorentz and myself Consistently with his assumption of a perfectly conducting electron and consequent zero force inside it, Walker calculates the total mechanical force on it by means of his expressions $\frac{1}{2}\int\sigma\mathrm{P}d\mathrm{S}$ and $\frac{1}{2}\int\sigma\mathrm{Q}d\mathrm{S}$ in the two cases respectively Lorentz and myself, on the other hand, postulate the truth of the electromagnetic equations inside the charge as well as outside, and consequently admit the possibility of force inside the electron, so that on our assumptions the correct expressions for the total force on it are $\frac{1}{2}\int\sigma(\mathrm{P}+\mathrm{P}_i)d\mathrm{S}$ and $\frac{1}{2}\int\sigma(\mathrm{Q}+\mathrm{Q}_i)d\mathrm{S}$ respectively, where P_i and Q_i denote the forces just inside For the longitudinal acceleration of Walker's rigid electron P_i vanishes, but for the transverse acceleration Q_i differs from zero

In order to obtain a convenient formula for calculating the inside force we proceed as follows. With the usual notation the electromagnetic equations written in vector form are

$$\mathrm{C}\ \mathrm{curl}\ \mathbf{h} = \frac{\partial\mathbf{d}}{\partial t} + 4\pi\rho\mathbf{v} = \frac{\mathrm{D}\mathbf{d}}{\mathrm{D}t} - (\mathbf{v}\nabla)\mathbf{d} + 4\pi\rho\mathbf{v}, \qquad \mathrm{div}\ \mathbf{h} = 0,$$

$$\mathrm{C}\ \mathrm{curl}\ \mathbf{d} = -\frac{\partial\mathbf{h}}{\partial t} = -\frac{\mathrm{D}\mathbf{h}}{\mathrm{D}t} + (\mathbf{v}\nabla)\mathbf{h}, \qquad \mathrm{div}\ \mathbf{d} = 4\pi\rho,$$

where D/D*t* denotes differentiation following the motion of the element of charge, whose volume density is ρ and velocity $\mathbf{v}$ Apply these equations to an infinitesimal element of the infinitely thin surface layer of surface density σ, which constitutes the moving electron. Since the surface discontinuity travels with the element, the differential coefficients $\mathrm{D}\mathbf{d}/\mathrm{D}t$ and

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* 'Ann d Phys.,' vol 25, p. 77, § 9 (1908).

$\mathrm{D}\mathbf{h}/\mathrm{D}t$ remain finite, whilst the others are generally infinite Integrating throughout the volume of the surface element, we obtain in the usual way

$$\mathrm{C}\{[\mathbf{Nh}]\} = 4\pi\sigma\mathbf{v} - (\mathbf{Nv})\{\mathbf{d}\}, \qquad \{(\mathbf{Nh})\} = 0,$$
$$\mathrm{C}\{[\mathbf{Nd}]\} = (\mathbf{Nv})\{\mathbf{h}\}, \qquad \{(\mathbf{Nd})\} = 4\pi\sigma,$$

where **N** denotes a unit normal vector drawn from the inside to the outside of the surface, and brackets { } indicate that the excess of the value outside above that inside the surface is to be taken for the quantity enclosed in the brackets Solving these equations we find

$$(\mathrm{C}^2-(\mathbf{Nv})^2)\{\mathbf{d}\} = 4\pi\sigma(\mathrm{C}^2\mathbf{N}-(\mathbf{Nv})\mathbf{v}), \qquad (\mathrm{C}^2-(\mathbf{Nv})^2)\{\mathbf{h}\} = -4\pi\sigma\mathrm{C}[\mathbf{Nv}] \tag{16}$$

Hence we find for the mechanical force **f** on unit charge

$$(\mathrm{C}^2-(\mathbf{Nv})^2)\{\mathbf{f}\} = (\mathrm{C}^2-(\mathbf{Nv})^2)\{\mathbf{d}+[\mathbf{vh}]/\mathrm{C}\} = 4\pi\sigma(\mathrm{C}^2-\mathbf{v}^2)\mathbf{N} \tag{17}$$

Knowing the surface density σ, the velocity **v**, normal **N** and the value of **f**, components (P, Q, R), just outside the surface, we can use this formula to calculate the values of (P_i, Q_i, R_i) just inside We shall apply it to each of Walker's cases in turn, neglecting squares of μt throughout

(a) *Walker's Longitudinal Case*—Denoting the components of **N**, the direction cosines of the outward normal at (x, y, z) by (l, m, n), we have

$$(l, m, n) = (xg^{-2}, y, z)pa^{-2}, \qquad \text{where } g^2 = 1-k^2,$$
$$\mathbf{v} = (\mathrm{C}k+\mu t, 0, 0), \qquad \mathrm{C}^2-\mathbf{v}^2 = \mathrm{C}^2g^2(1-2k\mu t\mathrm{C}^{-1}g^{-2}),$$
$$\mathrm{C}^2-(\mathbf{Nv})^2 = \mathrm{C}^2(1-k^2l^2-2l^2k\mu t\mathrm{C}^{-1})$$

Hence we find by means of (17)

$$\{\mathrm{P}\} = 4\pi\sigma g^2l(1-k^2l^2)^{-1}(1-2k\mu t\mathrm{C}^{-1}g^{-2}(1-l^2)(1-k^2l^2)^{-1})$$
$$= 4\pi\sigma xp^{-1}(1-2k\mu t\mathrm{C}^{-1}g^{-2}(1-x^2g^{-2}a^{-2})) \tag{18}$$

This is identical with the value of P given by Walker on p 450, when his value of σ is inserted, hence P_i is zero, and Walker's calculation of the force on the electron is correct It is easy to verify, by (16), that $l\mathrm{X}_i+m\mathrm{Y}_i+n\mathrm{Z}_i$ vanishes at the surface, so that Walker's σ is correct

(b) *Walker's Transverse Case*—Here we have, using (17),

$$\mathbf{v} = (\mathrm{C}k, \mu t, 0), \quad \mathrm{C}^2-\mathbf{v}^2 = \mathrm{C}^2g^2, \quad \mathrm{C}^2-(\mathbf{Nv})^2 = \mathrm{C}^2(1-k^2l^2-2lmk\mu t\mathrm{C}^{-1}),$$
$$\{\mathrm{Q}\} = 4\pi\sigma g^2m(1-k^2l^2)^{-1}(1+2k\mu t\mathrm{C}^{-1}lm(1-k^2l^2)^{-1})$$
$$= 4\pi\sigma g^2yp^{-1}(1+2k\mu txy\mathrm{C}^{-1}g^{-2}a^{-2})$$
$$= \mathrm{A}ya^{-3}(g^2+2k\mu txy\mathrm{C}^{-1}a^{-2}-2\mu(1+\tfrac{1}{10}k^2)y\mathrm{C}^{-2}+\tfrac{1}{5}\mu k^2y\mathrm{C}^{-2}),$$

by inserting the value of $4\pi\sigma$ given by Walker on p 451 with the sign of the discrepant last term corrected. Comparing with his value of Q we obtain

$$\mathrm{Q}_i = \mathrm{Q}-\{\mathrm{Q}\} = -\tfrac{1}{3}\mathrm{A}\mu k^2y^2\mathrm{C}^{-2}a^{-3}$$

Thus Q_x differs from zero inside the electron owing to the presence of the discrepant term, and hence we must add to Walker's value for the force on the electron a correction, which amounts to

$$\tfrac{1}{2}\int\sigma Q_x dS = -(1/24\pi)A^2\mu k^2C^{-2}a^{-6}\int py^2dS = -\tfrac{1}{18}\mu e^2k^2C^{-2}a^{-1}g^{-1}.$$

Adding this to the corrected value of Walker's force on p 452, we obtain

$$-\tfrac{2}{3}\mu e^2(1+\tfrac{1}{10}k^2)C^{-2}a^{-1}g^{-1},$$

which is precisely the value obtained in the text, § 7, equation (10).

Hence the discrepancy is completely accounted for

Here also it is easy to verify that $lX_x+mY_x+nZ_x$ vanishes at the surface on account of (16), so that Walker's σ is correct

(4) *Longitudinal Acceleration of the Lorentz Electron*—I have now succeeded in verifying the supposition in the text for the longitudinal acceleration by adapting Walker's solution to the case of the continually deformed electron, due account being taken both of the relative motion and of the internal force

At time t the eccentricity is $k+\mu tC^{-1}\equiv\beta$, as in § 6 Putting

$$x = \xi\sqrt{(1-\beta^2)},$$

we notice that ξ is constant for the Lorentz electron, so that there is a relative velocity $x = -\beta\beta x(1-\beta^2)^{-1} = -\mu kxC^{-1}g^{-2}$ to the first order. By interaction with the external magnetic force $Aka^{-3}(0, -z, y)$ at the surface this produces an additional external mechanical force $A\mu k^2C^{-2}g^{-2}a^{-3}(0, xy, xz)$ To neutralise its tangential component we may add to Walker's χ on p 450 a term

$$\chi' = \tfrac{1}{6}A\mu k^2a^2xC^{-2}g^{-4}\rho^{-3}(1-a^2\rho^{-2}+\tfrac{5}{3}a^2x^2g^{-2}\rho^{-4})$$

This gives rise to an additional surface density σ', given by

$$4\pi\sigma' = A\mu k^2pxC^{-2}g^{-4}a^{-3}(\tfrac{2}{3}-\tfrac{4}{3}x^2g^{-2}a^{-2})$$

Altogether we obtain an additional external normal mechanical force (P', Q', R'), such that at the surface we have

$$P' = A\mu k^2x^2C^{-2}g^{-4}a^{-3}(\tfrac{7}{6}-\tfrac{7}{3}x^2g^{-2}a^{-2})$$

These quantities, σ', P', are to be added to σ, P, given by Walker on p. 450, but in using the latter we must bear in mind that they refer to an electron moving with speed $Ck+\mu t$ and of the shape appropriate to this speed; hence, in the principal terms, the quantities A and p must be taken for the eccentricity β In other words, if A and p are still to refer to eccentricity k as before, we must, in the principal terms, replace them by

$$A+\mu tC^{-1}dA/dk = A(1+k\mu tC^{-1}g^{-2}),$$

and
$$p+\mu tC^{-1}dp/dk = p\{1-k\mu tC^{-1}g^{-2}(1+k^2)l^2\},$$

where $l = pxg^{-2}a^{-2}$ in the small term.

With these changes in Walker's solution we obtain

$$\begin{aligned}\mathrm{P}+\mathrm{P}' &= \mathrm{A}xa^{-3}\{1-k\mu t\mathrm{C}^{-1}g^{-2}(1-2x^2g^{-2}a^{-2})\\ &\qquad -2\mu x\mathrm{C}^{-2}g^{-4}(1-\tfrac{9}{10}k^2+\tfrac{7}{8}k^2x^2g^{-2}a^{-2})\},\\ 4\pi(\sigma+\sigma') &= \mathrm{A}pa^{-3}\{1+k\mu t\mathrm{C}^{-1}g^{-2}(1-(1+k^2)l^2)\\ &\qquad -2\mu x\mathrm{C}^{-2}g^{-4}(1-\tfrac{3}{5}k^2+\tfrac{4}{5}k^2x^2g^{-2}a^{-2})\}\end{aligned}$$

In order to find $\{\mathrm{P}+\mathrm{P}'\}$, we must use (16) with

$$\mathbf{v} = (\mathrm{C}k+\mu t-\mu kx\mathrm{C}^{-1}g^{-2},\ 0,\ 0)$$

Thus, we may use (18), provided that we replace t by $t-kx\mathrm{C}^{-1}g^{-2}$, σ by $\sigma+\sigma'$, and p by the value belonging to eccentricity β Hence we find

$$\begin{aligned}\{\mathrm{P}+\mathrm{P}'\} &= \mathrm{A}xa^{-3}\{1-k\mu t\mathrm{C}^{-1}g^{-2}(1-2x^2g^{-2}a^{-2})\\ &\qquad -2\mu x\mathrm{C}^{-2}g^{-4}(1-\tfrac{3}{5}k^2+\tfrac{4}{5}k^2x^2g^{-2}a^{-2})\}\end{aligned}$$

Thus we obtain

$$\mathrm{P}_i+\mathrm{P}_i' = \mathrm{P}+\mathrm{P}'-\{\mathrm{P}+\mathrm{P}'\} = -\mathrm{A}\mu k^2x^2\mathrm{C}^{-2}g^{-4}a^{-3}(1-x^2g^{-2}a^2)$$

Hence the mechanical force just inside the electron differs from zero, and the total force on it is given by

$$\begin{aligned}&\tfrac{1}{2}\int(\sigma+\sigma')(\mathrm{P}+\mathrm{P}'+\mathrm{P}_i+\mathrm{P}_i')d\mathrm{S}\\ &\quad = -(1/2\pi)\mu\mathrm{A}^2\mathrm{C}^{-2}g^{-4}a^{-6}\int(1-\tfrac{3}{5}k^2+\tfrac{3}{5}k^2x^2g^{-2}a^{-2})x^2pd\mathrm{S} = -\tfrac{2}{3}\mu e^2\mathrm{C}^{-2}a^{-1}g^{-3},\end{aligned}$$

all other terms being odd functions of x and disappearing by symmetry This is precisely the expression of Lorentz

As before, we must verify that the density is unaltered by the presence of internal force. The same process gives for $\rho = a$

$$\begin{aligned}&\{\mathbf{d}+\mathbf{d}'\} =\\ &\mathrm{A}a^{-3}\{1+k\mu t\mathrm{C}^{-1}g^{-2}(1+2x^2g^{-2}a^{-2})-2\mu x\mathrm{C}^{-2}g^{-4}(1-\tfrac{3}{5}k^2+\tfrac{4}{5}k^2x^2g^{-2}a^{-2})\}(x,y,z)\\ &\qquad -\mathrm{A}k\mu x\mathrm{C}^{-1}g^{-2}a^{-3}(2t-2kx\mathrm{C}^{-1}g^{-2})(1,0,0),\end{aligned}$$

$$\begin{aligned}&\mathbf{d}+\mathbf{d}' =\\ &\mathrm{A}a^{-3}\{1+k\mu t\mathrm{C}^{-1}g^{-2}(1+2x^2g^{-2}a^{-2})-2\mu x\mathrm{C}^{-2}g^{-4}(1-\tfrac{3}{5}k^2+\tfrac{7}{5}k^2x^2g^{-2}a^{-2})\}(x,y,z)\\ &\qquad -\mathrm{A}k\mu x\mathrm{C}^{-1}g^{-2}a^{-3}(2t-kx\mathrm{C}^{-1}g^{-2})(1,0,0)\end{aligned}$$

Hence we find

$$\mathbf{d}_i+\mathbf{d}_i' = \mathrm{A}\mu k^2x^3\mathrm{C}^{-2}g^{-6}a^{-5}(x,y,z)-\mathrm{A}\mu k^2x^2\mathrm{C}^{-2}g^{-4}a^{-3}(1,0,0)$$

Multiplying the components of $\mathbf{d}_i+\mathbf{d}_i'$ by (l, m, n) or to the first order by $(xg^{-2}, y, z)pa^{-2}$ respectively and adding, we see that the normal component of the electric force vanishes just inside the surface, so that the surface density is given correctly by the expression for $\sigma+\sigma'$

I think that the analysis given in Notes 3 and 4 proves definitely that the differences between the results obtained by Walker on the one hand, and by Lorentz and myself on the other, cannot be attributed to the use of the quasi-stationary assumption or of the energy method by Lorentz, neither of which is used by myself, but are due to the differences in the fundamental assumptions made by us, both as to the constitution of the electron and as to the character of its motion In view of these results, it seems to me that the two electron models ought to be regarded as essentially distinct, and that the relative merits of the two sets of mass formulæ should be assessed in the light of their agreement with the results of experiment and of their adaptability to the purposes of theory

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On an Expansion of the Point-Potential.

By Arthur W Conway, F R.S

(Received February 11, 1918)

Introduction

By point-potential in this paper is meant the potential scalar or vector of a point-charge which is moving in any manner without any restriction except that the speed is less than that of light Its general form is $f(\tau)/r(1-C^{-1}\partial r/\partial \tau)$, where r is the distance of the point-charge at a time τ from the field-point, C the speed of light, and $f(\tau)$ an arbitrary function of τ, τ being given by the equation $C(t-\tau) = r$ The solution for uniform rectilinear motion of the point-charge was given by Sir J J. Thomson[*] and Heaviside[†] The solution for a general motion of the point-charge was given by Liénard[‡] and Wiechert[§] A method of deducing the result by complex integration was given by the writer,[||] and this method is the basis of the present paper The subsequent literature is extensive and falls, for the most part, under two heads First, calculations of electromagnetic inertia include, for uniform motion, many papers by Heaviside, Searle, Morton, Abraham and others For variable motion a typical paper is Sommerfeld.[¶] A Lagrangian expansion appears to have been given first by Herglotz,[**] by Schott,[††] and in two papers by the author[*] The second head contains calculations of the radiation and of the forces at a considerable distance from the origin Many results were given by Heaviside, principally in letters to 'Nature' His work appears to have escaped notice by most writers on the subject Radiation from circular and elliptic orbits was given by him in 'Nature,' October 30 and November 16, 1908 Similar subjects were treated by Schott[†] Reference should be made for fuller treatment to Vol III of Heaviside's 'Electromagnetic Papers,' and to Schott's 'Electromagnetic Radiation' The latter work contains many references to this part of the subject

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[*] 'Phil Mag,' April 1881

[†] 'Phil Mag,' April, 1889.

[‡] 'L'Éclairage Électrique,' 1898.

[§] 'Archives Néerlandaises,' 1900, p 549

[||] 'Proc London Math. Soc.,' Series 2, vol 1.

[¶] 'Gött Nach,' 1904, pp 107–110

[**] 'Gött. Nach.,' 1903, p 356

[††] 'Ann d. Physik,' vol 25, p. 63 (1908).

In this paper a new type of expansion is studied Let O be any origin of axes having fixed directions in space, and let E be the position of the point-charge, P being the field-point Then the point-potential is expanded in a series $\Sigma Y_n U_n$, where Y_n is a spherical harmonic of the angles made by OP with the axes and U_n is a function depending on the positions of O and E and the distance OP It is thus analogous to the well-known harmonic expansions of r^{-1} or of the more general potential $e^{-kr}r^{-1}$ It is, however, of a more general type, for the origin is unrestricted as to its motion, it may be at rest or it may move with the point-charge or in any other manner whatever As, in addition, the point-charge is unrestricted in motion except that its speed is less than that of light, it is clear that the investigation of the functions U_n is a very wide one In this paper the convergence of the series is investigated and methods given for calculating them As an example a formal solution of the problem of the motion of a point-charge outside a fixed spherical conductor is given In the last section the connection with the Lagrangian expansion is given

The Point-Potentials

The electric force (X, Y, Z) and the magnetic force (α, β, γ) are given in terms of a scalar potential ψ and a vector potential (F, G, H) by means of the equations

$$(X, Y, Z) = -(\partial/\partial x, \partial/\partial y, \partial/\partial z)\psi - C^{-1}(\dot{F}, \dot{G}, \dot{H}),$$

$$(\alpha, \beta, \gamma) = \text{curl}\,(F, G, H)$$

Let $x_e(t)$, $y_e(t)$, $z_e(t)$, be the co-ordinates of the point-charge at the time t The equation in u

$$C^2(t-u)^2 = \{x-x_e(u)\}^2+\{y-y_e(u)\}^2+\{z-z_e(u)\}^2$$

has one and only one real root lying between O and t, provided that the velocity of the point-charge is less than C, the velocity of radiation, and that $C^2t^2 > \{x-x_e(0)\}^2+\{y-y_e(0)\}^2+\{z-z_e(0)\}^2$ We shall call this zero τ.

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[*] 'Proc. Roy Irish Academy,' vols. 27 and 28, the former paper being read November 30, 1907

[†] 'Ann. d Physik,' vol 24, p 637 (1907).

The complex integral

$$\Psi = \int f(u)\,du/[C^2(t-u)^2-\{x-x_e(u)\}^2-\{y-y_e(u)\}^2-(z-z_e(u)\}^2]$$

taken round a closed contour which contains only one zero, τ, of the denominator and no pole of the numerator is a solution of the equation $\nabla^2-C^{-2}\partial^2/\partial t^2=0$ The functions ψ, F, G, H are then obtained by putting, respectively, $ieC\pi^{-1}$, $iex_e'(u)\pi^{-1}$, $iey_e'(u)\pi^{-1}$, $iez_e'(u)\pi^{-1}$ instead of $f(u)$ where e is the charge associated with the point-charge. The proofs of these statements will be found in the paper by the author.*

The Harmonic Expansion

If the position of the origin instead of being fixed has for co-ordinates $x_0(t), y_0(t), z_0(t)$, the typical potential Ψ now becomes

$$\int\frac{f(u)\,du}{[C^2(t-u)^2-\{x+x_0(t)-x_e(u)\}^2-\{y+y_0(t)-y_e(u)\}^2-\{z-z_0(t)-z_e(u)\}^2]}$$

Writing

$$\begin{aligned}\xi &= \rho\sin\epsilon\cos\omega = x_e(u)-x_0(t),\\ \eta &= \rho\sin\epsilon\sin\omega = y_e(u)-y_0(t),\\ \zeta &= \rho\cos\epsilon \qquad\quad = z_e(u)-z_0(t),\\ \mu &= \cos\chi = \cos\theta\cos\epsilon+\sin\theta\sin\epsilon\cos(\phi-\omega),\end{aligned} \tag{1}$$

where (r, θ, ϕ), are the polar co-ordinates of the field-point P referred to axes through the new origin having fixed directions,

$$\nu = \{C^2(t-u)^2-\rho^2-r^2\}/2r\rho,$$

we get

$$\Psi = \int f(u)\,du/2r\rho(\nu+\mu) \tag{2}$$

Whenever we wish to put in evidence the dependence of ξ, η, ζ, ρ, etc, on the variable u we can write them $\xi(u)$, $\eta(u)$, $\zeta(u)$, $\rho(u)$, etc

We shall now consider the real zeros of the four functions $C(t-u)\pm r\pm\rho(u)$. We first distinguish between two cases. If we conceive a sphere drawn with its centre at the origin at time t and radius equal to the distance from this point to the position of the point-charge at time t, the field-point P may be *outside* or *inside* this sphere The former case will be called the *external* case and the latter the *internal* case We shall always suppose *that the sphere of radius* r, *the centre being the origin at time* t, *is completely inside the sphere of radius* Ct *having its centre at the initial position of the point-charge* This condition makes $Ct\pm r\pm\rho(0)$ positive.

The differential coefficients, $-C\pm\rho'(u)$, are negative, for $\rho'(u)$, the velocity of the point-charge resolved along ρ, is less than C The four functions thus are all positive when $u = 0$, and go on decreasing as u increases. They each therefore possess a unique zero for the range of positive values of u Taking them in turn we find that

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* 'Proc. London Math Soc.,' Series 2, vol. 1.

$C(t-u)-r-\rho$ is negative when $u = t$ and hence possesses a real zero τ_1 between 0 and t

$C(t-u)-r+\rho$ is negative when $u=t$ provided that $r > \rho(t)$, i.e., the external case, and hence possesses in this case a real zero τ_2 between 0 and t In the internal case its zero $\tau_2 > t$

$C(t-u)+r-\rho$ is negative when $u = t$ provided that $r < \rho(t)$, and hence has a zero τ_3 in this case (the internal) between 0 and t In the external case the zero τ_3 is greater than t

$C(t-u)+r+\rho$ has no zero lying between 0 and t

The zero τ is the zero of

$$\begin{aligned} 2r\rho\,(\nu+\mu) &\equiv C^2(t-u)^2-r^2-\rho^2+2r\rho\cos\chi \\ &\equiv C^2(t-u)^2-(r+\rho)^2+4r\rho\cos^2\tfrac{1}{2}\chi \\ &\equiv C^2(t-u)^2-(r-\rho)^2-4r\rho\sin^2\tfrac{1}{2}\chi \end{aligned}$$

From these identities we see that τ_2 (or τ_3) $> \tau > \tau_1$, and that $\nu-1$ is zero when $u = \tau_1$ and is negative between τ_1 and τ_2 (or τ_3), and also that $\nu+1$ is positive in the same range and zero where $u = \tau_2$ (or τ_3)

The series $S_1 = \sum\limits_0^\infty(-)^n(2n+1)P_n(\mu)Q_n(\nu)$ is known to be absolutely and uniformly convergent on the following conditions If we represent μ and ν as points in a plane of complex variables then the point ν must lie outside the ellipse having the points $+1$ and -1 as foci, and passing through the point μ $Q_n(\nu) = \frac{1}{2}P_n(\nu)\log(\nu+1)/(\nu-1) + Z_n(\nu)$, where $Z_n(\nu)$ is a series of positive integral powers of ν $Q_n(\nu)$ has thus two logarithmic poles at $\nu=-1$, or $u=\tau_2$ (or τ_3), and $\nu = 1$ or $u = \tau_1$ If $1 \geqq \mu \geqq -1$ the series S_1 is convergent provided that ν is complex This will be the case if u is complex Thus we have

$$\Sigma(-)^n(2n+1)\int P_n(\mu)Q_n(\nu)f(u)\,du/2r\rho$$

convergent under these circumstance, the integral being taken around the poles τ_1 and τ_2 (or τ_3), and the contour containing no other poles of the integrand. If in particular we put $\mu = 1/\sqrt{2}$ we have the series

$$S_2 = \Sigma(-)^n(2n+1)\int P_n(1/\sqrt{2})\,Q_n(\nu)f(u)\,du/2r\rho$$

convergent. Converting this to a real integral taken from $u = \tau_1$ to $u = \tau_2$, for the external case (the reasoning for the internal case will be exactly similar) we get

$$S_2 = -\pi i \sum_0^\infty (-)^n (2n+1) \int_{\tau_1}^{\tau_2} P_n(1\sqrt{2})\, P_n(\nu) f(u)\, du/2r\rho$$

Since S_2 is absolutely convergent we have

$$\pi \sum_0^\infty (2n+1) \left| \int_{\tau_1}^{\tau_2} P_n(1/\sqrt{2})\, P_n(\nu) f(u)\, du/2r\rho \right|,$$

the sum of the moduli, convergent

We now consider the case in which μ is, as is generally the case, a function of u and let us consider the series

$$S_3 = -\pi i \sum_0^\infty (-)^n (2n+1) \int_{\tau_1}^{\tau_2} P_n(\mu)\, P_n(\nu) f(u)\, du/2r\rho$$

$$= -\pi i \sum_0^\infty (-)^n (2n+1) \int_{\tau_1}^{\tau_2} \frac{P_n(\mu)}{P_n(1/\sqrt{2})}\, P_n(1/\sqrt{2}) f(u)\, du/2r\rho,$$

$$|S_3| \leq \pi \sum_0^\infty (2n+1) \int_{\tau_1}^{\tau_2} \left| \frac{P_n(\mu)}{P_n(1/\sqrt{2})} \right| \quad |P_n(1/\sqrt{2}) f(u)\, du/2r\rho|$$

We have the known result that, for large values of n, $P_n(\mu)$ tends to the value $\sqrt{(2/n\pi \sin\chi)} \quad \cos[(n+\frac{1}{2})\theta - \frac{1}{4}\pi]$ and thus

$$\left| \frac{P_n(\mu)}{P_n(1/\sqrt{2})} \right| \leq \frac{1}{\sqrt{(2 \sin \frac{1}{4}\pi \sin \chi_1)}}$$

$\sin\chi_1$ is the least value of $\sin\chi$ inside the range, it being supposed that χ is not near 0 or π Comparing with S_2, we thus see that S_3 is absolutely convergent From this fact, together with the identity

$$\sum_0^n (-)^n (2m+1) P_m(\mu) P_m(\nu)$$
$$= \frac{1}{\nu+\mu} + \frac{(-)^n (n+1)}{\nu+\mu} \{P_n(\mu) Q_{n+1}(\nu) + P_{n+1}(\mu) Q_n(\nu)\}$$

We get finally

$$\Psi \equiv \int f(u)\, du/2r\rho(\nu+\mu) = \sum_0^\infty (-)^n (2n+1) \int P_n(\mu) P_n(\nu) f(u)\, du/2r\rho$$

Since $\mu = \cos\chi = \cos\theta\cos\epsilon + \sin\theta\sin\epsilon\cos(\phi-\omega)$ we obtain the expansion of Ψ in spherical harmonics as follows

$$\int f(u)\, du/2r\rho(\nu+\mu) = \sum_0^\infty (-)^n (2n+1) P_n(\cos\theta) A_n$$
$$+ \sum_0^\infty \sum_{m=1}^{m=n} (-)^n (2n+1)\, 2 \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) \cos m\phi\, B_n^m$$
$$+ \sum_0^\infty \sum_{m=1}^{m=n} (-)_n (2n+1)\, 2 \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta) \sin m\phi\, C_n^m, \quad (3)$$

on making use of the identity

$$P_n(\mu) = P_n(\cos\theta)\,P_n(\cos\epsilon)$$
$$+2\sum_{m=1}^{m=n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)\,P_n^m(\cos\epsilon)\cos m(\phi-\omega),$$

the coefficients being given by the expressions

$$A_n = \int P_n(\cos\epsilon)\,Q_n(\nu)f(u)\,dn/2r\rho,$$
$$B_n^m = \int P_n^m(\cos\epsilon)\cos m\omega\,Q_n(\nu)f(u)\,du/2r\rho,$$
$$C_n^m = \int P_n^m(\cos\epsilon)\sin m\omega\,Q_n(\nu)f(u)\,du/2r\rho \tag{4}$$

The proof of convergence breaks down if at any point of the range of integration $\sin\chi = 0$ or is small Let O be the origin at the time t, E_1 and E_2 the positions of the point-charge for $u = \tau_1$ and $u = \tau_2$ respectively, and E any position of the point-charge for a value of u lying between τ_1 and τ_2, P being the field-point The angle χ is then the angle between OP and OE It is zero if O, P, and E are collinear The cone formed by the lines OE lying between OE_1 and OE_2 will meet the sphere of radius r and centre O in two curves, and the expansion (3) has been proved convergent for all points of this sphere except near these curves But the function Ψ, provided that $f(u)$ is finite between τ_1 and τ_2, is finite and continuous over this sphere Hence it possesses a harmonic expansion convergent everywhere on the sphere, which must, from the unique property of such expansions, be coincident with (3) Hence (3) is everywhere convergent, the conditions being that the velocity of the point-charge is less than C, and that the sphere of radius Ct having its centre at the initial position of the point-charge encloses completely the sphere having O as centre and OP as radius All the above results hold also for the internal expansion

It is possible to differentiate the expression Ψ any number of times with respect to x, y, z and t, and still leave the expansion in the same harmonics. This property will be of use to us. This is effected as follows The result of putting $x+dx$ for x in Ψ is the same as $\xi-dx$ for ξ Thus

$$\frac{\partial\Psi}{\partial x} = -\Sigma(-)^n(2n+1)\,P_n(\cos\theta)\frac{\delta A_n}{\delta\xi}+\text{etc}, \tag{5}$$

where $\dfrac{\delta A_n}{\delta\xi}$ means $\displaystyle\int\frac{\partial}{\partial\xi}P_n(\cos\epsilon)\,Q_n(\nu)f(u)/2r\rho$, etc

Again if t becomes $t+dt$ the co-ordinates $x, y, z, \xi, \eta, \zeta$ become $x-\dot{x}_0dt$, $y-\dot{y}_0dt$, $z-\dot{z}_0dt$, $\xi-\dot{x}_0dt$, $\eta-y_0\,dt$, $\zeta-\dot{z}_0dt$, and thus these co-ordinates may be treated as constants and

$$\frac{\partial\Psi}{\partial t} = \Sigma(-)^n(2n+1)\,P_n(\cos\theta)\frac{\delta A_n}{\delta t}+\text{etc} \tag{5}$$

where $\frac{\delta A_n}{\delta t}$ means $\int \frac{\partial}{\partial t} P_n \cos \epsilon\, Q_n(\nu) f(u)/2r\rho$, etc., the differentiation referring only to where t occurs explicitly, i.e. in $\nu = \{C^2(t-u)^2 - r^2 - \rho^2\}/2r\rho$

Methods of Calculating the Coefficients

It will be noticed that the functions A_n, B_n, C_n are all of the one type

$$I = \int f(u)\, Q_n(\nu)\, du/2r\rho$$

for the quantities ϵ and ω are functions of u. Three methods have been found for calculating the coefficients.

First Method.—If we put $\lambda = C(t-u) - r$ we have

$$Q_n(\nu) = \tfrac{1}{2} P_n(\nu) \log \frac{\lambda+\rho}{\lambda-\rho} + Z_n(\nu)$$

where τ_2 is a root of $\lambda+\rho = 0$ and τ_1 is a root of $\lambda-\rho = 0$. a, b, and c being any quantities, we have the identity

$$\frac{a^n}{2^n n!\, b^n}\left(\frac{\partial}{a\partial a}\right)^n \frac{1}{a}\{b^2-(c\pm a)^2\}^n = \frac{1}{a} P_n\left(\frac{c^2-a^2-b^2}{2ab}\right) \tag{9}$$

therefore
$$\frac{r^n}{2^n n!}\left(\frac{\partial}{r\partial r}\right)^n \cdot \frac{(\rho^2-\lambda^2)^n}{r\rho^n} = \frac{1}{r} P_n(\nu)$$

and
$$\frac{r^n}{2^n n!}\left(\frac{\partial}{r\partial r}\right)^n \frac{(\rho^2-\lambda^2)^n}{2r\rho^n} \log \frac{\lambda+\rho}{\lambda-\rho} - \frac{1}{r} Q_n(\nu)$$

has no pole, logarithmic or otherwise, inside the contour of integration. Hence we have

$$I = \frac{r^n}{2^n n!}\left(\frac{\partial}{r\partial r}\right)^n \int \frac{(\rho^2-\lambda^2)^n}{4r\rho^{n+1}} \log \frac{\lambda+\rho}{\lambda-\rho} f(u)\, du, \tag{10}$$

$$= -\pi i \frac{r^n}{2^n n!}\left(\frac{\partial}{r\partial r}\right)^n \int_{\tau_1}^{\tau_2} \frac{(\rho^2-\lambda^2)^n}{2r\rho^{n+1}} f(u)\, du \tag{11}$$

For the internal case, making use of the identity (9) and putting $\lambda_1 = C(t-u) + r$ we get

$$I = -\pi i \frac{r^n}{2^n n!}\left(\frac{\partial}{r\partial r}\right)^n \left\{ \int_{t_0}^{\tau_2} \frac{(\rho^2-\lambda_1^2)^n}{2r\rho^{n+1}} f(u)\, du - \int_{t_0}^{\tau_1} \frac{(\rho^2-\lambda^2)^n}{2r\rho^{n+1}} f(u)\, du \right\}, \tag{12}$$

where t_0 is any convenient lower limit which does not involve r.

Second Method.—Under the conditions laid down in § 3, the zeros τ_1 and τ_2 (in the external case) are unique, and are thus given by Lagrange's expansion. In other words, a contour enclosing the points τ_1 and τ_2, and no other zero of the functions $\lambda \pm \rho$, can be drawn in the u-plane, such that on it $|\rho| < |\lambda|$. We have, therefore,

$$\frac{(\rho^2-\lambda^2)^n}{4r\rho^n} \log \frac{\lambda+\rho}{\lambda-\rho} = \frac{(\rho^2-\lambda^2)^n}{2r\rho^n} \sum_0^\infty \frac{1}{(2s+1)}\left(\frac{\rho}{\lambda}\right)^{2s+1}$$

In this expansion it is only negative powers of λ that will contribute to the integral and these terms are

$$\frac{2^n n!}{r}\sum_0^\infty \frac{1}{(2s+1)(2s+3)\ldots(2s+2n+1)}\frac{\rho^{2s+n}}{\lambda^{2s+1}}$$

Performing the integration with respect to u we get

$$\mathrm{I} = -\frac{\pi\iota}{\mathrm{C}}r^n\left(\frac{\partial}{r\partial r}\right)^n\frac{1}{r}\Sigma\,\frac{1}{1\;3\;5\quad(2n+1)}\left[\rho^n f(u)+\frac{\mathrm{C}^{-2}}{2(2n+3)}\frac{\partial^2}{\partial u^2}\rho^{n+2}f(u)+\quad.\right]$$

where, in the square bracket, $u = t-r/\mathrm{C}$

Defining the functions of Bessel type $\mathrm{E}_n(z)$ and $\mathrm{S}_n\,(z)$ in the following manner

$$\mathrm{E}_n(z) = z_n\left(\frac{\partial}{z\partial z}\right)^n\frac{e^z}{z} = \left\{1-\frac{n(n+1)}{2}\frac{1}{z}+\quad(-)^n\frac{1\;3.\;\;(2n-1)}{z_n}\right\}\frac{e^z}{z}$$

and

$$\mathrm{S}_n(z) = z^n\left(\frac{\partial}{z\partial z}\right)^n\frac{\sinh z}{z} = \frac{1}{1.3\;\;.(2n+1)}\left\{z^n+\frac{z^{n+2}}{2(2n+3)}+\quad\right\},$$

and denoting by p the operation $\partial/\partial u$ followed by putting $u = t$, then we have the following symbolical form

$$\mathrm{I} = (-)^n\pi\iota\mathrm{C}^{-2}p\mathrm{E}_n(-pr/\mathrm{C})\,\mathrm{S}_n(p\rho/\mathrm{C})f(u) \tag{13}$$

In like manner for the internal case we get

$$\mathrm{I} = (-)^n\pi\iota\mathrm{C}^{-2}p\mathrm{E}_n(-p\rho/\mathrm{C})\,\mathrm{S}_n(pr/\mathrm{C})f(u) \tag{14}$$

Third Method—Taking a fixed origin so that $x_0(u) = y_0(u) = z_0(u) = 0$, we may now put in the preceding expression (13) $u = t$ and $p = \partial/\partial t$ In the case of periodic motion we may require to separate the coefficients into their different periodic parts. This can be effected as follows

We have $$\mathrm{S}_n(p\rho/\mathrm{C}) = \tfrac{1}{2}\int_{-1}^{+1}e^{-pl\rho/\mathrm{C}}\,\mathrm{P}_n(l)\,dl$$

If $$e^{-pl\rho/\mathrm{C}}f(u) = \Sigma\,\mathrm{A}e^{\iota kt}\,(1 > l > -1)$$

we get $$\mathrm{I} = (-)^n\tfrac{1}{2}\pi\iota\mathrm{C}^{-2}\Sigma(\iota k)\,\mathrm{E}_n(-\iota kr/\mathrm{C})\,e^{\iota kt}\int_{-1}^{+1}\mathrm{A}_k\mathrm{P}_n(l)\,dl, \tag{15}$$

where the summation extends to the coefficients A

For the internal case we can in like manner make use of the theorem

$$\mathrm{E}_n(-p\rho/\mathrm{C}) = \int_1^\infty e^{-pl\rho/\mathrm{C}}\,\mathrm{P}_n(l)\,dl,$$

provided that $e^{-pl\rho/\mathrm{C}} = \Sigma\,\mathrm{A}_k e^{\iota kt}, (\infty > l > 1)$ This is only possible if throughout the motion $\rho > r$

In the contrary case we can write

$$\mathrm{E}_n(-p\rho/\mathrm{C}) = \mathrm{R}_n(-p\rho/\mathrm{C})+\int_0^1 e^{-pl\rho/\mathrm{C}}\,\mathrm{P}_n(l)\,dl \tag{16}$$

The second term can be treated as in (15) The term $R_n(-p\rho/C)$ will contain only negative powers of ρ and must be treated differently The expansion can be made from the formula $P_n(\partial/\partial z)r^{-1} = R_n(r)$

Examples of Expansion of the Point-Potential

We shall here, for brevity, give examples of the potentials only Examples of calculating the forces will be given later

1 Let the point-charge have a uniform velocity along the z-axis and let the origin be taken at the point-charge. There is thus in this case no internal expansion Then

$$x_0(t) = x_e(t) = y_0(t) = y_e(t) = 0, \qquad z_0(t) = z_e(t) = vt,$$

$$\rho = \zeta = z_e(u) - z_0(t) = v(u-t), \qquad \xi = \eta = 0$$

We have

$$\psi = Ce\Sigma(-)^n \frac{(2n+1)}{2^n n!} r^n \left(\frac{d}{rdr}\right)^n \int_{r_1}^{r_2} \frac{[\rho^2 - \{C(t-u)-r\}^2]^n}{2r\rho^{n+1}} du$$

from (3), (4), and (11)

The integral becomes on introducing a variable

$$w = \{C(t-u)-r\}/v(t-u),$$

$$\tfrac{1}{2}(-)^n r^{n-1} v^{-1} \int_{-1}^{+1} (1-w^2)^n (C/v - w)^{-n-1} dw = (-)^n 2^n r^{n-1} v^{-1} Q_n(C/v),$$

and thus

$$\psi = \frac{Ce}{vr} \sum_0^\infty (-)^n (4n+1) \frac{1 \cdot 3 \ldots (2n+1)}{2 \cdot 4 \ldots 2n} P_{2n}(\cos\theta) Q_{2n}\left(\frac{C}{v}\right), \tag{17}$$

a result which can be verified by expanding the known explicit value of ψ for this case, $e\{(1-v^2/C^2)(x^2+y^2)+z^2\}^{-\frac{1}{2}}$

2 In the same case let the origin be fixed Then

$$x_0(t) = y_0(t) = z_0(t) = x_e(t) = y_e(t) = 0, \qquad \rho = vu$$

On introducing a variable $w = \{r - C(t-u)\}/vu$ we get the integral

$$\int_{r_1}^{r_2} (\rho^2 - \lambda^2)^n du/2r\rho^{n+1} = 2^n v^{-1} r^{-1} (Ct-r)^n Q_n(C/v)$$

Using the formula

$$r^n \left(\frac{\partial}{r\partial r}\right)^n \frac{(Ct-r)^n}{r} = (-)^n \frac{n!}{r} P_n\left(\frac{Ct}{r}\right)$$

we get $$\psi = Cev^{-1}r^{-1} \sum_0^\infty (2n+1) P_n(\cos\theta) P_n(Ct/r) Q_n(C/v). \tag{18}$$

For the internal case it is most inconvenient to use the symbolical formula (15) We have to evaluate

$$\psi = -eC^{-1} \sum_0^\infty (2n+1) pE_n(-p\rho/C) S_n(pr/C) P_n(\cos\theta). \tag{19}$$

We get, on expansion,

$$-C^{-1}p\,E_n(-p\rho/C)\,S_n(pr/C)$$

$$=\frac{r^n}{1\cdot3\cdot5\ldots2n+1}\left[1+\frac{1}{2(2n+3)}\frac{p^2r^2}{C^2}+\frac{1}{2\cdot4\,(2n+3)(2n+5)}\frac{p^4r^4}{C^4}+\ldots\right]$$

$$\times\left[1\cdot3\ldots(2n+1)\left\{\frac{1}{\rho^{n+1}}-\frac{1}{2!(2n-1)}\frac{p^2}{C^2\rho^{n-1}}+\frac{1}{4!(2n-1)(2n-3)}\frac{p^4}{C^4\rho^{n-3}}-\ldots\right\}\right.$$

$$\left.-\frac{(-)^n}{1\cdot3\cdot5\ldots(2n+1)}\left\{\frac{p^{2n}\rho^n}{C^{2n}}+\frac{1}{2\,(2n+3)}\cdot\frac{p^{2n+2}\rho^{n+2}}{C^{2n+2}}+\ldots\right\}\right]$$

Since $\rho = vt$ we have $p^m\rho^n = 0$ if $m > n$ and

$$p^2\rho^{-(n-1)} = n\,(n-1)\,\rho^{-(n+1)}v^2,$$

$$p^4\rho^{-(n-1)} = (n+1)\,n\,(n-1)(n-2)\,\rho^{-(n+1)}v^4$$

Thus the second square bracket yields

$$\rho^{-(n+1)}\left[1-\frac{n\,(n-1)}{2!(2n-1)}\frac{v^2}{C^2}+\text{etc.}\right]=\rho^{-(n+1)}\frac{n!}{1\cdot3\cdot5\ldots(2n-1)}\frac{v^n}{C^n}P_n\!\left(\frac{C}{v}\right)$$

and the final result becomes

$$\frac{n!}{1\cdot3\cdot5\ldots(2n-1)}\frac{r^nv^n}{C^n}P_n\!\left(\frac{C}{v}\right)\left\{1+\frac{1}{2\,(2n+3)}\frac{p^2\rho^2}{C^2}+\ldots\right\}\rho^{-(n+1)}$$

$$=\frac{C}{rv}\frac{n!}{1\cdot3\cdot5\ldots(2n-1)}P_n\!\left(\frac{C}{v}\right)\left\{\left(\frac{r}{Ct}\right)^{n+1}+\frac{(n+1)(n+2)}{2(2n+3)}\left(\frac{r}{Ct}\right)^{n+3}+\ldots\right\}$$

$$=(2n+1)\,\frac{C}{rv}P_n\!\left(\frac{C}{v}\right)Q_n\!\left(\frac{Ct}{r}\right),$$

so that

$$\psi = Cer^{-1}v^{-1}\sum_0^\infty(2n+1)\,P_n(\cos\theta)\,Q_n(Ct/r)\,P_n(C/v) \tag{20}$$

Comparing with (18) we notice that when $r = vt$ the two expressions are, as they ought to be, equal.

3. In the case of simple harmonic rectilinear motion let us take

$$x_0(t) = y_0(t) = z_0(t) = x_e(t) = y_e(t) = 0, \qquad z_e(t) = \rho\,(t) = a\sin kt$$

Making use of the equations

$$\tfrac{1}{2}\int_{-1}^{+1}P_n(\mu)e^{x\mu}\,d\mu = S_n(x), \qquad e^{ix\sin\phi} = \sum_{-\infty}^{\infty}e^{is\phi}J_s(x),$$

we get

$$S_n(pa\sin ku\,C^{-1}) = \sum_{s=-\infty}^{s=\infty}e^{iskt}\,\tfrac{1}{2}\int_{-1}^{+1}J_s(ska\mu C^{-1})\,P_n(\mu)\,d\mu$$

and

$$-pC^{-1}E_n(-pr\,C^{-1})\,S_n(pa\sin ku\,C^{-1}), \tag{21}$$

$$=-iskC^{-1}\sum_{s=\infty}^{s=-\infty}e^{iskt}E_n(-iskrC^{-1})\,\tfrac{1}{2}\int_{-1}^{+1}J_s(ska\mu C^{-1})\,P_n(\mu)\,d\mu,$$

and the scalar potential ψ

$$= -iske\mathrm{C}^{-1}\sum_{n=0}^{n=\infty}\sum_{s=\infty}^{s=\infty}(2n+1)e^{iskt}\mathrm{E}_n(-iskr\mathrm{C}^{-1})\mathrm{P}_n(\cos\theta)$$
$$\times\tfrac{1}{2}\int_{-1}^{+1}\mathrm{J}_s(ska\mu\mathrm{C}^{-1})\mathrm{P}_n(\mu)\,d\mu, \qquad (22)$$

which can thus easily be written down in a series consisting of sin *skt* and cos *skt*

The vector-potential can, in like manner, be calculated by operating with (21) on $\rho'(u)/\mathrm{C}$, *i e*, $-ka\cos ku/\mathrm{C}$ or what is more convenient $-ka\mathrm{C}^{-1}e^{iku}$, and taking the real part. We get that $\mathbf{H}$ is the real part of

$$iseak^2\mathrm{C}^{-2}\sum_{s=-\infty}^{s=\infty}(2n+1)e^{iskt}\mathrm{E}_n(-iskr\mathrm{C}^{-1})\mathrm{P}_n(\cos\theta)$$
$$\times\tfrac{1}{2}\int_{-1}^{+1}\mathrm{J}_{s-1}(ska\mu\mathrm{C}^{-1})\mathrm{P}_n(\mu)\,du.$$

The field at a great distance from the origin has been calculated by Schott[*] When r is great $-iskr\mathrm{C}^{-1}\mathrm{E}_n(-iskr\mathrm{C}^{-1})$ becomes $e^{-iskr\mathrm{C}^{-1}}$ and thus we get

$$\psi = \frac{e}{r}+\frac{2e}{r}\sum_{n=0}^{n=\infty}\sum_{s=0}^{s=\infty}\cos sk(t-r\mathrm{C}^{-1})(2n+1)\mathrm{P}_n(\cos\theta)$$
$$\times\tfrac{1}{2}\int\mathrm{J}_s(ska\mu\mathrm{C}^{-1})\mathrm{P}_n(\mu)\,d\mu$$
$$= \frac{e}{r}+\frac{2e}{r}\sum_{s=1}^{s=\infty}\cos sk(t-r\mathrm{C}^{-1})\mathrm{J}_s(ska\mu\mathrm{C}^{-1}), \qquad (23)$$

and
$$\mathbf{H} = -\frac{eak}{r\mathrm{C}}\sum_{s=-\infty}^{s=\infty}\cos sk(t-r\mathrm{C}^{-1})\mathrm{J}_s(ska\mu\mathrm{C}^{-1}) \qquad (24)$$

These expressions agree with Schott's results

For the internal case the motion is not periodic and we cannot expand it in the same form Thus we cannot expand $\mathrm{E}_n(-pa\sin ku\mathrm{C}^{-1})$ in a series of periodic time terms In a slightly different case $\zeta(u) = \rho(u) = a+b\sin ku$ we can get a periodic time-expansion if $r<a-b$ We easily get, in fact, for this case

$$-p\mathrm{C}^{-1}\mathrm{E}_n\{p(a+b\sin ku)\mathrm{C}^{-1}\}$$
$$= isk\mathrm{C}^{-1}e^{iskt}\int_1^{\infty}\mathrm{P}_n(\mu)e^{-\mu iska\mathrm{C}^{-1}}\sum_{-\infty}^{\infty}\mathrm{J}_s(-\mu bsk\mathrm{C}^{-1})\,d\mu.$$

4 For the case of uniform circular motion in which for the sake of generality we take the origin on the axis of the circular orbit, put $\xi = a\sin\epsilon\cos\Omega u$, $\eta = a\sin\epsilon\sin\Omega t$, $\zeta = a\cos\epsilon$, so that $\rho = a$ and $\cos\chi = \cos\theta\cos\epsilon+\sin\theta\sin\epsilon\cos(\phi-\Omega t)$.

Writing

$$\psi = e\Sigma(2n+1)\mathrm{P}_n(\cos\theta)\mathrm{A}_n+e\Sigma\Sigma(2n+1)\mathrm{P}_n^m(\cos\theta)$$
$$\times(\mathrm{B}_n^m\cos m\phi+\mathrm{C}_n^m\sin m\phi),$$

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[*] 'Electromagnetic Radiation,' p. 120

then

$$A_n = -pC^{-1}E_n(-prC^{-1})S_n(paC^{-1})P_n(\cos\epsilon) = a^n r^{-(n+1)}P_n(\cos\epsilon)$$

and

$$B_n^m + iC_n^m = \frac{2(n-m)!}{2(n+m)!}(-pC^{-1})E_n(-prC^{-1})S_n(paC^{-1})P_n^m(\cos\epsilon)e^{mi\Omega},$$

$$= \frac{2(n-m)!}{(n+m)!}(-mi\Omega C^{-1})E_n(-mri\Omega C^{-1})S_n(mai\Omega C^{-1})P_n^m(\cos\epsilon)e^{mi\Omega t} \qquad (25)$$

For the internal case we interchange the letters a and r in the above equations

5 We can employ our formulæ to verify the remarkable expression obtained by Schott* for the potential of a polyperiodic motion at a distant point At a great distance $-pC^{-1}E_n(-prC^{-1})$ becomes $r^{-1}e^{-prC^{-1}}$ and thus

$$\psi = er^{-1}e^{-prC^{-1}}\Sigma(2n+1)S_n(p\rho C^{-1})P_n(\cos\chi) = er^{-1}e^{-prC^{-1}}e^{pC^{-1}\rho\cos\chi}$$

If $$\rho\cos\chi = \Sigma a_s\sin(\Omega_s + \alpha_s),$$

then $$e^{pC^{-1}\rho\cos\chi} = \Sigma J_s(-ia_s pC^{-1})\ \ \Sigma J_{s'}(-ia_{s'}pC^{-1})\ \ e^{i(\Omega t+\alpha)}$$

where $$\Omega = s\Omega_s + s'\Omega_{s'} + \qquad \text{and} \qquad \alpha = s\alpha_s + s'\alpha_{s'} +$$

and so the result follows at once

6 As an example of aperiodic motion consider the rectilinear case $\xi = 0$, $\eta = 0$, $\zeta = kt^s$ We have

$$\psi = e\Sigma(2n+1)(-)^n r^n\left(\frac{\partial}{r\partial r}\right)^n e^{-prC^{-1}}r^{-1}S_n(p\zeta C^{-1})P_n(\cos\theta)$$

The coefficient of $(2n+1)P_n(\cos\theta)$

$$= (-)^n r^n\left(\frac{\partial}{r\partial r}\right)^n \frac{e^{-prC^{-1}}r^{-1}}{1\,3.\ (2n+1)}\sum_{m=0}^{m=\infty}\frac{p^{2m}C^{-2m}k^{n+2m}t^{(n+2m)s}}{2\ 4\ \ 2m\ (2n+3)\ .(2n+2m+1)}$$

$$= (-)^n r^n\left(\frac{\partial}{r\partial r}\right)^n \frac{e^{-prC^{-1}}r^{-1}}{1\ 3\ \ (2n+1)}\sum_{m=0}^{m=\infty} l_n^m k^{n+2m}(Ct)^{ns+2m(s-1)},$$

where $$l_n^m = \frac{(n+2m)s!\,C^{-(n+2m)s}}{2\ 4\ \ 2m\ \ (2n+3)\ \ (2n+2m+1)\ \ (n+2m)!}$$

We have $$r^n\left(\frac{d}{rdr}\right)^n\frac{(Ct-r)^s}{r} = (-)^n\frac{s!(C^2t^2-r^2)^{-\frac{1}{2}(n-s)}}{r}P_n^{n-s}(Ct/r),$$

in which if $n-s$ is negative, the Legendre function is defined as follows (Hobson),

$$P_n^{-m}(z) = \frac{1}{\Gamma(1+m)}\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}m}F(-n, n+1, 1+m, -\tfrac{1}{2}(x-1)),$$

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* 'Electromagnetic Radiation,' p. 114

and so the coefficient becomes,

$$\frac{(-)^n}{1\ 3\ \ (2n+1)}\sum_{m=0}^{m=\infty}\frac{(n+2m)s!}{2\ 4\ \ 2m\ \ 2n+3\ \ (2n+m+1)}\ .$$
$$\times\left\{\frac{k(C^2t^2-r^2)^{\frac{1}{2}(s-1)}}{C^{s}}\right\}^{n+2m}P_n^{-(n+2m)(s-1)}(Ct/r)$$

A similar expansion can be found for the internal case

7. Let an electrified sphere move without rotation in any manner, and let us take the origin at the centre. Put $\xi = x_e(u)-x_e(t)$ $\eta = y_e(u)-y_e(t)$, $\zeta = z_e(u)-z_e(t)$, where x_e, y_e, and z_e are the co-ordinates of the centre of the sphere Let the co-ordinates of any point on the sphere (of radius a) be ξ_0, η_0, ζ_0, and let the surface density be given by the solid harmonic $Y_s(\xi_0, \eta_0, \zeta_0)$ The potential of the sphere at an external point is given by

$$\Psi = \int dS_0'\, Y_s(\xi_0, \eta_0, \zeta_0)\, e^{\xi_0\partial/\partial\xi+\eta_0\partial/\partial\eta+\zeta_0\partial/\partial\zeta}\psi,$$

where ψ is the potential of unit charge at the centre of the sphere Performing the surface integration, and noticing that by (5) we have $\partial^2/\partial\xi^2+\partial^2/\partial\eta^2+\partial^2/\partial\zeta^2 = \nabla^2 = p^2C^{-2}$, we get

$$\Psi = 4\pi a^2 S_s(paC^{-1})\, Y_s(\partial/\partial\xi, \partial/\partial\eta, \partial/\partial\eta)\,\psi$$

We have by a well known theorem of Hobson that, if $f(\rho)$ is a function of ρ, $\quad Y_s(\partial/\partial\xi, \partial/\partial\eta, \partial/\partial\zeta) f(\rho) = Y_s(\xi, \eta, \zeta)\,\rho^s(\partial/\rho\partial\rho)^s f(\rho)$,

and therefore $\quad \Psi = 4\pi a^2 S_s(paC^{-1})(\rho^s(\partial/\rho\partial\rho)^s\psi \quad Y_s(\xi, \eta, \zeta)$

For instance the coefficient of $(2n+1)P_n(\cos\theta)$ is

$$4\pi a^2 S_s(paC^{-1})\{-pC^{-1}E_n(-prC^{-1})\}\rho^s\left(\frac{\partial}{\rho\partial\rho}\right)^s S_n(p\rho C^{-1})\, Y_s(\xi, \eta, \zeta)$$

Inside the sphere the scalar potential is

$$4\pi a^2 E_s(-paC^{-1})\, Y_s(\partial/\partial\xi, \partial/\partial\eta, \partial/\partial\zeta)\,\psi_i$$

where $\quad \psi_i = -pC^{-1}\Sigma(2n+1)S_n(pr/C)S_n(p\rho/C)P_n(\cos\chi)$

In like manner the scalar potential of a solid sphere of uniform density and unit charge is, for outside points $3S_1(pa/C)\psi$, and for inside points

$$3(r^3/a^3)S_1(pr/C)\psi+3(r^3/a^3)E_1(-pr/C)\psi_i-3E_1(-pa/C)\psi_i.$$

The Polar Components of the Forces

Taking a fixed origin so that $u = t$, $p = \partial/\partial t$, and (ξ, η, ζ) and (ρ, ϵ, ω) are now respectively the Cartesian and polar co-ordinates of the point-charge at the time t We have then for the external case

$$\psi = -eC^{-1}\sum_0^\infty(2n+1)pE_n(-pr/C)S_n(p\rho/C)P_n(\cos\chi),$$

$$F = -eC^{-2}\sum_0^\infty(2n+1)pE_n(-pr/C)S_n(p\rho/C)P_n(\cos\chi)\,\xi, \text{ etc}$$

For the internal case we interchange r and ρ.

In the solution of spherical boundary problems the forces resulting from the above potentials must be put into the well known forms of two types analogous to those used in discussing the vibrations of a sphere and allied problems.[*] In our notation the forces are the real parts of the following, the directions being dr, $rd\theta$, $r\sin\theta d\phi$.

Type I

Electric Force

$$\frac{n(n+1)}{r}\mathrm{Y}_n{}^m\mathrm{E}_n\mathrm{G}_n{}^m,\quad -\sin\theta\frac{\partial\mathrm{Y}_n{}^m}{\partial\mu}\frac{d(r\mathrm{E}_n)}{rdr}\mathrm{G}_n{}^m,\quad \frac{\partial\mathrm{Y}_n{}^m}{\sin\theta\partial\phi}\frac{d(r\mathrm{E}_n)}{rdr}\mathrm{G}_n{}^m \tag{27}$$

Magnetic Force

$$0,\quad \frac{p}{\mathrm{C}}\frac{1}{r\sin\theta}\frac{\partial\mathrm{Y}_n{}^m}{\partial\phi}\mathrm{E}_n\mathrm{G}_n{}^m,\quad \frac{p}{\mathrm{C}}\frac{\sin\theta}{r}\frac{\partial\mathrm{Y}_n{}^m}{\partial\mu}\mathrm{E}_n\mathrm{G}_n{}^m \tag{28}$$

Type II

Electric Force

$$0,\quad -\frac{p}{\mathrm{C}}\frac{1}{r\sin\theta}\frac{\partial\mathrm{Y}_n{}^m}{\partial\phi}\mathrm{E}_n\Gamma_n{}^m,\quad -\frac{p}{\mathrm{C}}\frac{\sin\theta}{r}\frac{\partial\mathrm{Y}_n{}^m}{\partial\mu}\mathrm{E}_n\Gamma_n{}^m \tag{29}$$

Magnetic Force

$$\frac{n(n+1)}{r}\mathrm{Y}_n{}^m\mathrm{E}_n\Gamma_n{}^m,\quad -\sin\theta\frac{\partial\mathrm{Y}_n{}^m}{\partial\mu}\frac{d(r\mathrm{E}_n)}{rdr}\Gamma_n{}^m,\quad \frac{\partial\mathrm{Y}_n{}^m}{\sin\theta\partial\phi}\frac{d(r\mathrm{E}_n)}{rdr}\Gamma_n{}^m, \tag{30}$$

where $\mathrm{Y}_n{}^m$ denotes $2(2n+1)\dfrac{(n-m)!}{(n+m)!}\mathrm{P}_n{}^m(\mu)e^{im\phi}$, when m is not zero and $\mathrm{Y}_n{}^0$ or Y_n denotes $(2n+1)\mathrm{P}_n{}^m(\mu)$, E_n is $\mathrm{E}_n(-pr/\mathrm{C})$. The functions $\mathrm{G}_n{}^m$, $\Gamma_n{}^m$ are arbitrary functions of t, and might be termed the characteristic functions of these solutions. For the internal case we interchange r and ρ.

After some reduction we find for the external case of motion of a point-charge, the normal electric force

$$-ep\mathrm{C}^{-1}\Sigma\frac{(n+1)(2n+1)}{r}\mathrm{P}_n(\cos\chi)\mathrm{E}_n(-pr/\mathrm{C})\{\mathrm{S}_n(p\rho/\mathrm{C})+\rho\mathrm{C}^{-1}\mathrm{S}_{n+1}(p\rho/\mathrm{C})\}$$
$$+ep\mathrm{C}^{-2}\Sigma\frac{(2n+1)\rho}{r}\mathrm{E}_n(-pr/\mathrm{C})\mathrm{S}_n(p\rho/\mathrm{C})\frac{d}{dt}\mathrm{P}_n(\cos\chi) \tag{31}$$

and for the normal magnetic force

$$-ep\mathrm{C}^{-2}\Sigma\frac{(2n+1)\rho}{r}\mathrm{E}_n(-pr/\mathrm{C})\mathrm{S}_n(p\rho/\mathrm{C})$$
$$\left\{-\omega\sin\epsilon\frac{\partial}{\partial\epsilon}+\dot{\epsilon}\frac{\partial}{\sin\epsilon\partial\omega}\right\}\mathrm{P}_n(\cos\chi), \tag{32}$$

where, since $\cos\chi=\cos\theta\cos\epsilon+\sin\theta\sin\epsilon\cos(\phi-\omega)$, $(2n+1)\mathrm{P}_n(\cos\chi)$ is the real part of $\mathrm{Y}_n\mathrm{P}_n(\cos\epsilon)+\Sigma\,\mathrm{Y}_n{}^m\mathrm{P}_n{}^m(\cos\epsilon)e^{-im\omega}$.

By comparison with the expressions (27) to (30) we can at once infer that the moving point-charge is equivalent to a charge e at the origin, together with two trains of vibrators, the characteristic functions of which G_n^m, Γ_n^m, are given by the equations

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[*] *Cf* Bateman, 'Electrical and Optical Wave-Motion,' Chapter III.

$$G_n^m = -\frac{ep\mathrm{C}^{-1}}{n}\{S_n(p\rho/\mathrm{C})+\dot{\rho}\mathrm{C}^{-1}S_{n+1}p\rho/\mathrm{C}\}\,P'^m_n(\cos\epsilon)\,e^{-m\iota\omega}$$

$$+\frac{ep\mathrm{C}^{-2}}{n(n+1)}\rho S_n(p\rho/\mathrm{C})\frac{d}{dt}(P_n^m\cos\epsilon^{-m\iota\omega}), \quad (33)$$

$$\Gamma_n^m = -\frac{ep\mathrm{C}^{-2}}{n(n+1)}\rho\,S_n(p\rho/\mathrm{C})\left\{-\omega\sin\epsilon\frac{\partial}{\partial\epsilon}+\epsilon\frac{\partial}{\sin\epsilon\partial\omega}\right\}P_n^m(\cos\epsilon)\,e^{-m\iota\omega} \quad (34)$$

For the internal case we find

$$G_n^m = \frac{ep\mathrm{C}^{-1}}{n+1}\{E_n(-p\rho/\mathrm{C})+\rho\mathrm{C}^{-1}E_{n-1}(-p\rho/\mathrm{C})\}\,P'^m_n(\cos\epsilon)\,e^{-m\iota\omega},$$

$$+\frac{ep\mathrm{C}^{-2}}{n(n+1)}\rho E_n(-p\rho/\mathrm{C})\frac{d}{dt}P'^m_n(\cos\epsilon)\,e^{-m\iota\omega}), \quad (35)$$

$$\Gamma_n^m = -\frac{ep\mathrm{C}^{-2}}{n(n+1)}\rho\,E_n(-p\rho/\mathrm{C})\left\{-\omega\sin\epsilon\frac{\partial}{\partial\epsilon}+\dot{\epsilon}\frac{\partial}{\sin\epsilon\partial\omega}\right\}P_n^m(\cos\epsilon)\,e^{-m\iota\omega} \quad (36)$$

As an example, let us take a point-charge moving in a general manner outside a perfect spherical conductor, and let G_n^m, Γ_n^m, denote the functions (35) and (36), and let the effect of the sphere be represented by the characteristic functions L_n^m, Λ_n^m of first and second types respectively as given by (27) and (28) The boundary condition is that the resultant tangential electric force vanishes Hence it follows that when $r = a$

$$\frac{d}{rdr}\{rE_n(-pr/\mathrm{C})\}L_n^m+\frac{d}{rdr}\{rS_n(pr/\mathrm{C})\}\,G_n^m = 0$$

and
$$E_n(-pr/\mathrm{C})\,\Lambda_n^m+S_n(pr/\mathrm{C})\,\Gamma_n^m = 0,$$

and thus symbolically

$$L_n^m = -\frac{\dfrac{d}{da}\{aS_n(pa/\mathrm{C})\}}{\dfrac{d}{da}\{aE_n(-pa/\mathrm{C}))\}}\,G_n^m \quad (37)$$

and
$$\Lambda_n^m = -\frac{S_n(pa/\mathrm{C})}{E_n(-pa/\mathrm{C})}\,\Gamma_n^m \quad (38)$$

If we take the case of uniform motion in a straight line $\xi = 0$, $\eta = 0$, $\rho = \zeta = vt$. We have for this case $G_n^m = 0$, and $\Gamma_n^m = 0$ unless when m is zero

$$G_n = e\,\frac{pC^{-1}}{n+1}\{E_n(-p\rho/C)+\rho C^{-1}E_{n-1}(-p\rho/C)\}$$

$$= -\iota\,\frac{C^n}{p^n}\left(\frac{C^2}{v^2}-1\right)P_n'\left(\frac{C}{v}\right)n\,!/C^{n+1}t^{n+1}$$

To calculate the numerator of (37) we find that

$$n\,!\,\frac{C^n}{p^n}S_n(pa/C)\,C^{-(n+1)}t^{(n+1)} = \frac{1}{a}Q_n(Ct/a)$$

and

$$L_n = e\left(\frac{C^2}{v^2}-1\right)P_n'\left(\frac{C}{v}\right)\frac{\frac{d}{da}Q_n(Ct/a)}{\frac{d}{da}\,aE_n(-pa/C)}$$

We can put this symbolical expression into the form of a definite integral by the aid of the equation

$$Q_n(x) = \int_0^\infty \epsilon^{-kx}S_n(k)\,dk$$

and we get

$$L_n = e\left(\frac{C^2}{v^2}-1\right)P_n'\left(\frac{C}{v}\right)\frac{d}{da}\int_0^\infty \frac{S_n(k)}{\frac{d}{dk}\{kE_v(k)\}}\,\epsilon^{-kCt/a}\,dk$$

In the case of periodic motion the symbolical expressions (37) and (38) can be easily evaluated. Take the case of a circular orbit, ρ and ϵ being constants, and $\omega = \Omega t$. We get in this case

$$L_n^m = -e\,\frac{\frac{d}{da}\{a\,S_n(pa/C)\}}{\frac{d}{da}\{a\,E_n(-pa/C)\}}\,\frac{pC^{-1}(n+p\rho/C)}{n(n+1)}E_n(-p\rho/C)\,P_n^m(\cos\epsilon)\,e^{-m\iota\omega},$$

$$\Lambda_n^m = -e\,\frac{S_n(pa/C)}{E_n(-pa/C)}\,\frac{p\rho C^{-2}\Omega}{n(n+1)}E_n(-p\rho/C)\sin\epsilon\frac{\partial}{\partial\epsilon}P_n^m(\cos\epsilon)\,e^{-m\iota\omega},$$

where $p = -m\iota\Omega$

Connection with the Lagrangian Expansion

If $R^2(u) = \{x-x_s(u)\}^2+\{y-y_s(u)\}^2+\{z-z_s(u)^2\}$, then the zeros of $R(u)$, regarded as a function of u, are even in number, and all imaginary. We can then construct a two-sheeted Riemann surface, having connections along the lines joining pairs of these zeros such that no connection meets the real axis between 0 and t. The equation $C^2(t-u)^2 = R^2(u)$ has one real root τ between 0 and t. It is a zero of $C(t-u)-R(u)$, but not of $C(t-u)+R(u)$. Let τ be situated in the upper sheet of the Riemann surface. Taking a closed contour in this sheet, enclosing τ and no other zero, we have, integrating about this contour,

$$\begin{aligned}
\Psi &= \int f(u)\,du/\{C^2(t-u)^2-R^2(u)\} \\
&= \int f(u)\,du/2R(u)\{C(t-u)-R(u)\}-\int f(u)\,du/2R(u)\{C(t-u)+R(u)\} \\
&= \int f(u)\,du/2R(u)\{C(t-u)-R(u)\} \\
&= \tfrac{1}{2}\int f(u)\,du\sum_0^\infty\frac{(Ru)^{n-1}}{C^{n+1}(t-u)^{n+1}} \\
&= -\pi\iota\sum_0^\infty\frac{(-)^n}{n!\,C^{n+1}}\left(\frac{\partial}{\partial t}\right)^n f(t)\{R(t)\}^{n-1}
\end{aligned}$$

which is the Lagrangian Expansion

This expression may be written symbolically $-\frac{\pi\iota}{C}\frac{e^{-pR(t)/C}}{R(t)}$ where $p=\partial/\partial t$

This symbolical expression has been termed the Pan-potential by Heaviside *

We can pass from this expression to the Harmonic Expansion as follows We have

$$R^2(t)=\Sigma[x-x_0(u)-\{x_e(t)-x_0(u)\}]=r^2+\rho^2-2r\rho\cos\chi,$$

where
$$r^2=\Sigma\{x-x_0(u)\}^2$$
$$\rho^2=\Sigma\{x_e(t)-x_0(u)\}^2$$

and
$$\cos\chi=\Sigma\{x-x_0(u)\}\{x_e(t)-x_0(u)\}/r\rho$$

and u is any variable We have then

$$\begin{aligned}
\frac{e^{-pR(t)/C}}{R(t)}&=\frac{e^{-p(r^2+\rho^2-2r\rho\cos\chi)^{\frac12}/C}}{(r^2+\rho^2-2r\rho\cos\chi)^{\frac12}} \\
&= -pC^{-1}\Sigma(2n+1)\,E_n(-pr/C)\,S_n(p\rho/C)\,P_n(\cos\chi)
\end{aligned}$$

As this series is identically independent of u, we may treat u as a constant, and, after the differentiations, put $u=t$

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* 'Nature,' January 1, 1902.

Scattering of Light by Dust-free Air, with Artificial Reproduction of the Blue Sky.—Preliminary Note

By the Hon R. J STRUTT, F.R.S., Professor of Physics, Imperial College, South Kensington

(Received February 12, 1918.)

[PLATE 4.]

It is now well established* that the luminosity and blue colour of the sky on very clear days and at considerable altitudes above the sea-level can almost be accounted for by the scattering of light by the molecules of air, without postulating suspended particles of foreign matter, such as were thought necessary by the earlier writers.

This conclusion depends on the measured opacity of the atmosphere, deduced from observations such as those of Abbot and Fowle† of the sun's radiation at various zenith distances. The opacities measured at Mount Wilson for different wave-lengths are found to be nearly in agreement with what would be expected if scattering by the molecules were alone operative, leaving little room for the action of larger particles.

I have not been able to find any reference in the later literature to the possibility of observing in the laboratory the scattering of light by pure air, filtered free from dust. Tyndall, who devoted more attention to such questions than any other experimenter of his day, states that nothing of the kind can be observed. He says "I, at that time (1872), found that London air, which is always thick with motes, and also with matter too fine to be described as motes, after it had been filtered by passing through densely packed cotton wool ... showed, when examined by a concentrated luminous beam, no trace of mechanically suspended matter. The particular portion of space occupied by such a beam was not to be distinguished from adjacent space." "The purely gaseous portion of our atmosphere was thus shown to be incompetent to scatter light,"‡ and as far as I have been able to learn it has generally been supposed, perhaps in view of Tyndall's statement just quoted, that very great thicknesses of air would have to be illuminated before the scattered light would be appreciable. Certainly such was my own notion.

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* See Lord Rayleigh, 'Phil. Mag.,' vol. 47, p. 375 (1899); Schuster, 'Nature,' vol. 81, p. 97 (1909), or Schuster's 'Optics,' 2nd ed., p. 326 (1909).

† 'Annals of the Astrophysical Observatory of the Smithsonian Institution,' vol. 2, 1908.

‡ 'Phil. Trans.,' vol. 166, p. 27 (1876).

I was first led to question this by noticing that the searchlights used for anti-aircraft purposes, both in town and country, always exhibit a distinct scattering along their track, even on the clearest nights, and high above the ground level When the air is very clear, the track of the beam is comparatively faint, and of a bluish colour But it is always perfectly easy to see This, though suggestive, does not, of course, definitely prove that the observed scattering is independent of dust

The following simple considerations show, however, that the scattering of light by pure air should be within the range of laboratory experiment

The brightness of the clear sky in sunlight may be estimated at 550,000 times its value in full moonlight, since that is the comparative photometric value of the sun and the moon If, instead of 5 miles "of homogeneous atmosphere" illuminated by the sun, we substitute a layer 1/550000th part of this, or 0 58 inch thick, the intensity should be equal to that of the moonlit sky It is easy to arrange conditions at least as good as this in the laboratory, whether sunlight or arc-light is used , and, therefore, the scattered light, like the light of the moonlit sky, should be quite bright enough for observation

In arranging an experiment for this purpose, the chief essential was to avoid as far as possible stray light diffused from the wall of the vessel used, and to observe the beam transversely against the blackest possible background

The source of light was a hand-regulated carbon arc, taking about 12 ampères A quartz condenser (used on account of its transparency to blue and violet light) was adjusted to give a convergent beam This passed through a quartz water-cell to take out heat rays, then into the experimental vessel The latter was in the form of a cross, constituted of brass tubing 1½ inch diameter, as shown in the figure (drawn to half size) The interior was dead black. The light entered by the quartz window A and passed on through the rectangular diaphragm B to the far end, where it was stopped by the closed end of the tube The beam was observed through the glass window D The side piece E constituted a black cave, and the mouth of it afforded the necessary background against which a feeble transverse luminosity of the beam would stand out well.* In spite of all precautions, enough light was diffused by the walls of the tube to enclose this black background in a luminous ring, which, however, was not bright enough to cause inconvenience The beam passed diametrically across this luminous ring, which is seen in all the photographs

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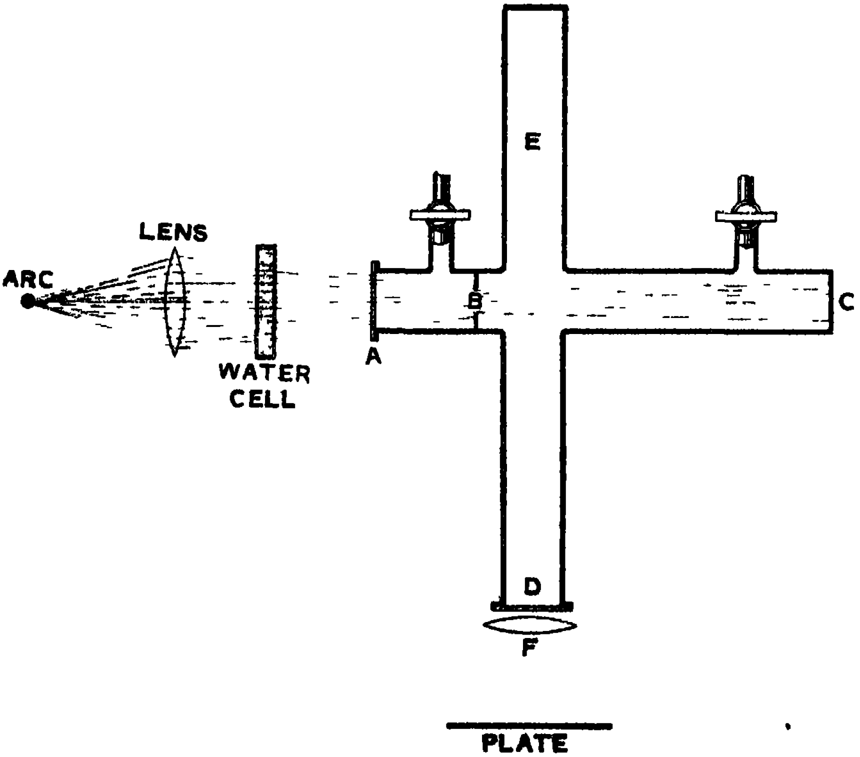
* The vessel used by Tyndall—a cylindrical glass tube of 3 inches diameter with the beam passing axially along it—would not, I feel sure, give a black enough background to make it possible to detect the scattering of dust-free air In saying this it is not intended to depreciate his work, which at the time marked an important advance

For photography, the lens F, of 6 cm. focal length, was arranged to form an image on a sensitive plate held in an ordinary camera. For visual observation, the plate holder was removed, and the image viewed with an eyepiece of 3 cm focal length This moderate magnification was found to be of considerable advantage, though, of course, it does not increase the intrinsic brightness of the scattered light



With ordinary untreated air in the apparatus, a very bright track due to scattering by the dust particles was observed Passing a current of air dried by phosphorus pentoxide, and filtered by cotton wool, the dust particles became fewer It took a little time to get rid of them, because the shape of the vessel was unfavourable for a wash through of air After a few minutes, however, the dust particles became so scarce that they could only be seen occasionally crossing the track, and they soon vanished entirely

At this stage there was a blue track along the beam, which, though much fainter than the original dusty track, was visible without difficulty when the eyes had been rested in the dark. It was much bluer than the original dust track, which latter was about the same colour as the arc, and also much bluer than the ring of light scattered by the walls of the vessel, which has been mentioned above This faint blue track could be well photographed in 1 minute on a rapid plate with aperture about $f/2$ * Photograph No. 1 (Plate 4) was taken on an ordinary plate, using a filter transparent only to the ultra-violet (nitrosodimethylaniline with cobalt glass) No 2 was taken with an isochromatic plate and yellow screen It will be observed how much stronger the image is relative to the walls of the vessel in the former case This pair of photographs shows objectively what was seen visually in the blue colour of the beam contrasting with the yellow colour of the surrounding ring of light due to the vessel

At this point it is important to concentrate attention on the question of whether dust had really been got rid of, and whether the effects described were not really due to very fine dust Special attention was paid to this question

In the first place, no loss of brightness in the scattered light could be observed when the air was more elaborately filtered A tube 2 inches wide and 4 feet long, tightly packed with cotton-wool, was used But this gave exactly the same result as a much smaller and shorter filtering tube. It made no difference whether the air current was slow or fast, nor did passage of the current of filtered air for several hours induce any further change The air current was driven by pressure, not by suction

If there was any slight leak in the apparatus, filtered air may have passed out, but dusty air could not pass in It did not matter whether the air was dried or not As an additional precaution to remove dust, Aitken's method was resorted to The experimental vessel above described containing filtered air, with some water on the bottom, was connected with a gasholder, also containing filtered air under a pressure of 15 inches of water The connection was closed, and the experimental vessel opened for a moment to the atmosphere At first, a somewhat sparse cloud was observed when this was done, showing that some dust particles had escaped the filter, or had lurked in one of the blind alleys of the apparatus After repeated expansion the water drops visible in the beam were reduced to a very small number, five or ten, and, finally, they were got rid of altogether But this additional purification was not observed to reduce the scattered light

When the vessel was exhausted, the blue track disappeared, nothing

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* The aberrations of a single lens working at that aperture are great, but the images were good enough for the present purpose

remaining except the above-mentioned ring of light diffused by the vessel Re-admitting filtered air, the blue track re-appeared exactly as before

These various tests and precautions seem to prove beyond doubt that the observed scattering was not due to dust surviving imperfect filtration, or to any false light, but that it is a property of pure air Its intensity is of the order of magnitude to be expected[*]

The bare possibility remained, however, that what had been observed was a fluorescence of the air rather than the anticipated scattering The most direct test of this would be a spectroscopic examination of the scattered light, and a two hours' exposure brought out faintly the great cyanogen band at λ 388, which is photographically the most conspicuous feature of the carbon arc spectrum

It was not convenient to give longer exposures with the carbon arc, which requires constant attention, and a special arrangement was set up, using a powerful quartz-mercury lamp made by the Westinghouse-Cooper-Hewitt Company The beam from the lamp broadside on was concentrated to a quartz condenser, which latter was set air-tight in the entrance to a special chamber made of cardboard, and in essential design similar to the much smaller brass apparatus of the figure The cardboard vessel was made as dust-proof as possible, and Tyndall's device of covering the bottom with glycerine served in the course of a day or two to catch and remove the dust The spectrograph was connected air-tight to the vessel, without any intervening window, and it was pointed to the focus with its collimator tube parallel to the length of the lamp, and therefore looking through the greatest depth of luminous air In three days' exposure the photograph No 3, Plate 4, was obtained The spectrum below, No 4, is that of the lamp itself, with the exposure adjusted to give about the same intensity in the middle of the spectrum It will be noticed that the upper spectrum shows no signs of any constituent except the known mercury lines Thus the lateral emission is scattered light, not fluorescent light Secondly, it will be seen how far the intensity of the spectrum is shifted to the ultra-violet The visual lines of the lamp do not appear in the upper photograph at all, and the far ultra-violet lines are absent from the lower one[†] This effect is of the same type as the blue colour of the sky, and is fully accounted for by the accepted theory

Returning to the experiments with the carbon arc, the scattered beam in oxygen was indistinguishable in colour and intensity from that in air This circumstance alone is almost enough to show that the light is not fluorescent. The same colour was obtained from carbon dioxide, but the intensity was greater, perhaps double In hydrogen the scattered light was much fainter and difficult to see distinctly, unless the eye was very well rested in the dark beforehand Photographically, it came out plainly with five minutes' exposure, and the intensity was not apparently inconsistent with a value ¼ that for air

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[*] I have further evidence that dust has nothing to do with the observed effects, but this will best be given with the quantitative measurement to be published later

[†] They would of course come out with longer exposure

Finally, the test of polarisation has been applied, both visually and photographically Visual examination with a Nicol showed that the beam was invisible when the prism was held with its short diagonal horizontal Thus the vibrations were transverse to the direction of propagation, as in all cases of scattering by particles small compared with the wave-length For photography, a double image prism was mounted over the photographic lens The result is shown in photograph No 5, Plate 4 The approximately complete extinction of the beam in the lower image will be noticed This is the image in which the vibrations (if any) would be parallel to the length of the beam

The intensity of the beam should not be compared with that of the ring of light scattered by the walls of the vessel, which is itself partly polarised and suffers something of the same changes as the beam when the prism is rotated. The proper comparison is between the two images of the beam, or rather between the image and the absence of an image

No. 6, Plate 4, corresponds to No 5 just described, but was taken with the ordinary (dusty) air of the room, and the exposure was shorter. In this case the scattered beam is much more intense, and is far indeed from being perfectly polarised

Whether the polarisation in pure air and other gases is absolutely complete is an important question, which will be examined in a future paper

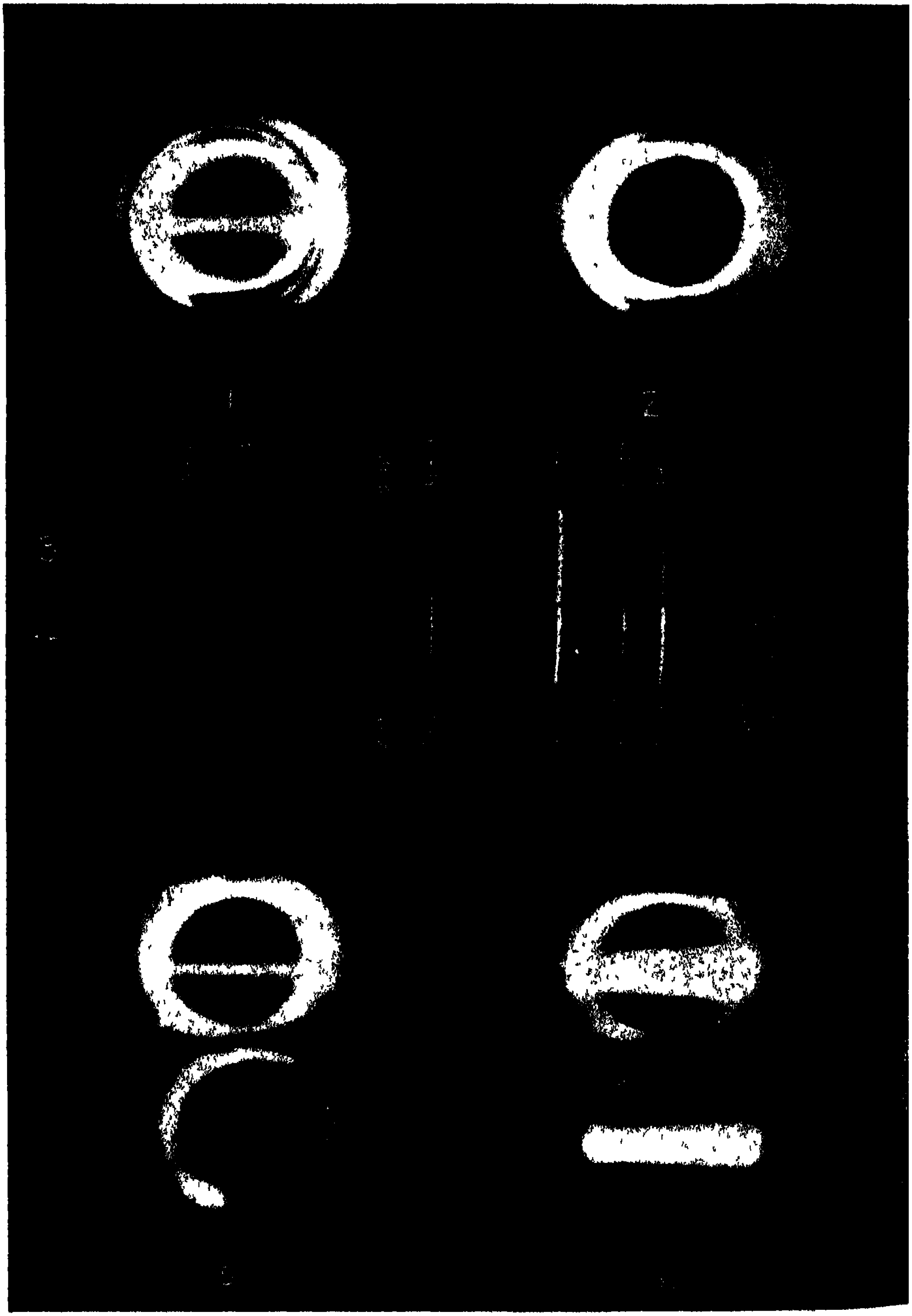
All the results here described are in qualitative agreement with the received theory of scattering Quantitative measurements are in progress.

Summary.

(1) By proper arrangement of the experimental conditions it is possible to observe the scattering of light by pure air, free of dust, in a small-scale laboratory experiment

(2) Similar results can be obtained with other gases. Hydrogen gives much less scattering than air, oxygen about the same, carbon dioxide decidedly more.

(3) The scattered light in air and in all the other gases is blue—the blue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2

of the sky, illustrating very directly the theory that attributes the blue of the sky to scattering by the molecules of air.*

(4) The scattered light is almost completely polarised.

[*Note added April* 19, 1918.—In examining the literature, I regret that I overlooked the importance of a paper by Cabannes ('Comptes Rendus,' vol. 160, p. 62, January 11, 1915), though I had consulted a somewhat meagre abstract of it. He detected the scattering by pure air, and made approximate measurements of its amount. This paper has, however, been allowed to stand unaltered, as the method of treatment is independent, and several important points, such as the polarisation of the scattered light and the effect of different gases, are not mentioned by Cabannes.

DESCRIPTION OF PLATE.

1. Beam in dust-free air viewed transversely. Ultra-violet filter. The oval outline is light diffused by walls of vessel. The beam is seen passing across this oval.
2. Similar conditions, except that a yellow filter is substituted. Beam very much fainter relative to light diffused by walls of vessel.
3. Spectrum of light from mercury lamp scattered by dust-free air and showing ultra-violet lines 2536 and 2654, but not the yellow and green lines.
4. Spectrum of mercury lamp direct, showing yellow line 5890 and green line 5460, but not the far ultra-violet lines.
5. Beam in pure air through a double image prism. Vibrations in upper image vertical, in lower horizontal. The beam is invisible in the lower image, showing that polarisation of the scattered light is nearly complete.
6. Similar photograph with dusty air. The beams are very strong, and comparable in intensity.

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* Tyndall obtained the blue by means of fine-grained fogs, precipitated from organic vapours. This was a valuable contribution, but his fogs were of course both chemically and physically very different from dust-free air.

The Formation of Nitrites from Nitrates in Aqueous Solution by the Action of Sunlight, and the Assimilation of the Nitrites by Green Leaves in Sunlight.

By BENJAMIN MOORE, D.Sc., F.R.S.

Action of Light Rays on Organic Compounds, and the Photosynthesis of Organic from Inorganic Compounds in Presence of Inorganic Colloids.

By BENJAMIN MOORE, D.Sc., F.R.S., and T. A. WEBSTER.

[These papers are published in Series B, No. 627, June 1, 1918.]

Investigations on Textile Fibres.

By W. Harrison, M Sc.

(Communicated by Prof W H Bragg, F.R S. Received December 13, 1917)

[Plates 5 and 6]

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Although fibres form the most important raw material in the textile industries, few or no attempts have been made to investigate their fundamental properties On this account most of the phenomena met with in the treatment of textiles have not been explained The most important of these phenomena are those which occur during the application of stress, heat, and moisture

An opinion commonly accepted by workers in the wool industries is that heat and moisture combined soften wool and render it plastic. This opinion appears to be right in principle, but is only qualitative, since wool exhibits plasticity under ordinary conditions. With regard to cotton, numerous investigations have been made on its chemical derivatives and their products of hydrolysis, but little attention has been paid to its physical properties

It is known that the treatment of cotton, while under tension, with concentrated solutions of sodium hydrate, produces changes in its physical state resulting in an increase in lustre This process is known as mercerisation. An explanation of the increased lustre was first put forward by Lange,* who stated that the surface of the fibres was rendered much smoother by the treatment Hübner and Pope† were unable to confirm Lange's observations, and suggested that the increased lustre was due to reflection from spiral grooves on the surface of the fibre The author‡ proved by means of photomicrographs that Lange's theory was substantially correct, but that the shape of the fibre in cross-section had an important bearing on its lustre In the the same papers it was shown by photographs taken in the Cardioid ultra-microscope that a change in the state of aggregation of the cellulose in cotton fibres was produced by mercerisation Further, by the use of polarised light, an explanation was obtained for the fact that cotton fibres shrink in length when treated with sodium hydrate solution while in a loose condition

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* 'Farber-Zeitung,' p 197 (1898).

† 'Journ Soc Chem Ind.,' vol 23, p. 410 (1904)

‡ 'Journ Soc. Dyers and Colourists,' vol. 31, p 1981 (1915), also 'Trans. Nat Assoc. Cotton Manufacturers,' Boston, U.S.A, vol. 101, p 201 (1916)

Polarised light has been used by several authors for the examination of nitrated cotton · Chardonnet* and Liebschutz† Ambronn‡ stated that an idea of the percentage of combined nitrogen could be obtained in this way De Mosenthal§ found that nitrated cotton of the same degree of nitration, prepared by different methods as to temperature, acid mixture, and time of immersion, showed different colours in polarised light, and that the appearance differed with the raw material used No explanation of these results was offered

The investigations described in the present paper were undertaken with a view to determining some of the fundamental properties of textile fibres which govern their behaviour in manufacturing processes, and thereby to establish a scientific basis for research into such processes

Effect of Stress, Moisture, and Heat on Textile Fibres

It is a well known fact, first established by Sir David Brewster in 1814, that substances of a colloidal nature such as glass, gelatine, and indiarubber, exhibit double refraction when subjected to stress, but lose this property when the stress is removed Permanent double refraction can be produced in these substances in different ways in the glass by heating it nearly to the melting point and cooling rapidly, in the gelatine by applying considerable pressure, and in unvulcanised indiarubber by subjecting it to distortion while warm and cooling it in position The permanent double refraction produced in these different ways is due to internal stresses resulting from the permanent strains left in the materials These stresses can be removed from the glass by annealing, from the gelatine by soaking in water, and from the indiarubber by heating or soaking in a solvent

It is known that all textile fibres are doubly refracting, and it will be shown in the course of this paper that this property is due to the presence of internal stresses.

By the application of external stresses in certain directions the double refraction of all fibres is increased The effect of pressure was investigated by compressing the fibres between a thick glass plate and a grooved celluloid film placed on a second glass plate, the whole being placed on the stage of a microscope and examined in polarised light between crossed nicols By this method, the difference between the compressed and uncompressed portions could be clearly seen

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* Compare Carl Suvern, 'Die Künstliche Seide,' p. 23 (Berlin, 1900)

† 'Moniteur Scientifique,' p 119 (1891).

‡ 'Koll. Zeit.,' vol. 13, p. 200 (1913).

§ 'Journ. Soc. Chem. Ind,' vol. 26, p. 444 (1907)

The effect of gradually increasing the pressure on a fibre of English wool, until compressed level with the top of the grooves in the celluloid film, is illustrated in Plate 5, fig 1 Part of the illumination is due to the effect of the pressure on the celluloid, but this is relatively small and does not interfere with the results, which can also be obtained by compressing crossed fibres between parallel glass plates The interference figures shown in fig 1, B and C, indicate spreading of the fibre substance in all directions away from the centre of pressure, *i e*, the point of contact between the cylindrical fibre and the cylindrical surfaces of the grooved celluloid film When the portions in the hollows of the groove became compressed, they spread towards the original centre of pressure and the interference figures disappeared, fig 1, D The maximum pressure applied was approximately 5 tons to the square inch of the surface of contact between fibre and celluloid, and in all cases the latter retained an impression of the fibre, showing that celluloid is not relatively very hard compared with wool and cotton.

No interference figures were observed during the gradual application of pressure on cotton fibres, fig 2 Spreading of the fibre substance only occurred where directly under pressure even when compressed level with the top of the grooves in the celluloid

The cotton fibre thus offers greater resistance to deformation than the wool fibre

With both wool and cotton, after removal of the pressure, the fibres did not return to their original shape and the increased double refraction in the compressed portions remained permanent This is illustrated with wool in fig 3 and with cotton in fig 4 The bright portions in fig 3, C, and fig 4, C, are evidence of internal stresses resulting from the permanent strains left in the fibres after compression The tendency to return to its original shape is small in the case of wool even when only slight pressure has been used, but is somewhat greater with cotton For the photomicrographs a cotton fibre mercerised under tension was chosen on account of its greater regularity in structure, ordinary cotton behaves in exactly the same way

Similar experiments were made with dry fibres of silk, linen and ramie, and these also were permanently deformed by pressure and retained internal stresses, but the effect was not so pronounced as with wool.

Fibres deformed in the manner described above were found to return to their original shape when placed in cold water. This is shown with wool in fig. 3, E, and with cotton in fig 4, E This return took place almost immediately with wool, but more slowly with cotton, particularly when high pressure had been applied The length of time under compression appeared to have an influence on the result obtained

Similar results were given by fibres which had been bent or twisted, when placed in cold water they gradually returned to their original shape Fibres doubled and twisted as in figs 6 and 7, placed loosely in cold water, gradually untwisted and straightened themselves out This occurred quite rapidly when the fibres were doubled and the free ends held tightly while twisting On the other hand, when the fibres were first twisted on their axes and then doubled, untwisting took place very slowly and only from the free ends onwards This retardation is due to the fact that when twisted in this latter manner, the torsional stresses in the one portion of the doubled fibre tend to balance those in the other, and it is only by the untwisting of the free ends of the fibre that these torsional stresses are released

Fibres subjected to a limited amount of extension when dry do not return to their original length when kept loosely in a dry atmosphere, but do so rapidly when placed in water.

The effect of a humid atmosphere is similar to that of cold water, but the release of internal stresses produced by deformation takes place much more slowly with the former

The results obtained by compressing wet fibres at the ordinary temperature agreed with what was expected from the above, the fibres behaved like other elastic bodies The wool fibre appeared to be more elastic than any other.

Experiments were made at higher temperatures by the use of a grooved aluminium sheet in place of the celluloid, the fibres being compressed between this grooved plate and a plain one, and in all cases an impression of the fibre remained on the metal

With perfectly dry fibres, compression in a dry atmosphere at 100° C produced effects similar to those obtained at lower temperatures. All the fibres experimented with were permanently deformed when compressed in contact with water at 100° C The result obtained with wool is shown in fig. 5, A. The difference in luminosity, as seen in polarised light between crossed nicols, of the compressed and uncompressed portions, fig 5, B, is mainly due to variations in thickness. Wool fibres treated in this manner do not return to their original shape when boiled in water while loose, because none of the internal stresses produced by the compression remain.

This was proved by subjecting a wool fibre to extension, until the interference colour shown in polarised light changed, and then boiling the fibre in water, when the colour returned to that shown before extension, and the fibre itself showed no tendency to return to its original length when placed loosely in boiling water

Experiments made at different temperatures showed that no permanent deformation was produced by compressing wool fibres in contact with water below 80° C for five minutes, longer times were not used

Cotton fibres exhibit similar properties in boiling water, but the amount of permanent deformation is not so great as that produced on wool by similar treatment, and some of the internal stresses remain

The results of the experiments already described show that in the dry state textile fibres exhibit a kind of plasticity, in which strains produced by externally applied stresses are accompanied by corresponding internal stresses, both strains and internal stresses disappearing when the fibres are placed in cold water The effect of heat on dry fibres, up to the temperature 100° C, is not to alter the nature of this plasticity, it may, however, alter its magnitude, but this has not been investigated. Water at the ordinary temperature removes this plasticity, and causes the fibres to become elastic, they are deformed by stress but return to their original shape on removal of that stress, unless the elastic limit has been passed. The elastic limit appears to be higher for wool than for other fibres At high temperatures water renders fibres truly plastic, externally applied stress produces deformation, but little or no internal stresses Cotton retains some of the internal stresses, but wool appears not to do so.

The quantitative measurement of these properties presents difficulties, on account of the small dimensions of many of the fibres The author hopes to be able to overcome these difficulties later

The experiments already mentioned throw some light on the differences in the impression given to the sense of touch by fabrics composed of different fibres In the act of feeling fabrics, the individual fibres are bent considerably, and it is in the resistance to bending, and the tendency to unbend when released, that the real difference lies

It is extremely difficult to imitate the act of feeling fabrics with such small quantities as are required for microscopic examination, but an idea of the manner of bending of fibres was obtained by twisting them so as to form loops. An ordinary fibre of English wool bends like an indiarubber cord, forming wrinkles on the inside of the loop (fig 6, A). Hollow fibres, which are by no means uncommon in Scotch wool, bend like an indiarubber tube, with flattening of the central canal at the bend

As already mentioned, bent wool fibres return to their original shape in water, or more slowly in a moist atmosphere, when free to move In practice, it is known that the impression given to the sense of touch by fabrics composed of wool is to a large extent dependent on the amount of moisture present in the fabrics Thus the deductions which may be drawn from the experiments described are in accordance with practical facts

Ordinary cotton fibres, on account of their hollow ribbon-like structure, bend at very sharp angles, see fig 7, A Fibres mercerised under tension, being more cylindrical, bend in a more circular curve, see fig 7, B Fibres mercerised while in a loose condition form wider loops than either of the other two This is shown in fig 7, C, a wider loop was formed in spite of the fact that more twists were present in the fibre This is due partly to the increased diameter resulting from the transverse swelling, and partly to a difference in the colloidal state of the fibre substance produced by treatment It is known that cotton fibres treated in this way feel more like wool than ordinary cotton fibres or those mercerised under tension

The Cause of the Double Refraction of Textile Fibres

The double refraction exhibited by cotton fibres has already been shown by the author* to be due to strains such as are produced on any elastic body by the application of stress Such strains are permanent and are accompanied by internal stresses The evidence brought forward in support of this view was the following —

When cotton fibres are treated with an ammoniacal copper solution swelling takes place, and often barrel-shaped formations such as those shown in fig. 8, A, are produced It is only in those portions which have not swelled or done so to a slight extent, as in the rings of the barrel-shaped formations, that double refraction persists, fig 8, B

A film prepared from a solution of cotton in ammoniacal copper solution shows double refraction only when subjected to stress, but retains this property when the stress is removed.

The direction of strain is parallel with the axis of the fibre. This was proved by turning the fibre about the plane of polarisation, it appeared darkest when parallel to or at right angles to the plane of polarisation and brightest when at 45° to that plane The double refraction of the fibre is increased by stretching, hence the natural condition of cotton fibres corresponds to that produced by tension on an elastic body. The internal stresses are represented by a negative pressure acting along the axis, or a positive pressure acting in all other directions.

* *Loc cit* (Refs. 3 and 4).

All the natural fibres examined appear to have internal stresses of the same kind as those present in cotton

It is well known that many crystals exhibit double refraction, and many chemists attribute this to a particular type of molecular configuration or of crystalline structure Since colloids such as indiarubber and cellulose exhibit double refraction only when subjected to stress, the effect of stress should be either to change the molecular configuration or impart some temporary crystalline structure to the colloid It is difficult to see how the relatively small pressure required to impart double refraction to indiarubber could alter the configuration of its molecules, and there is no known method of proof applicable in this case

In order to test whether stress imparts crystalline structure, X-ray photographs of pieces of stretched and unstretched indiarubber were taken by the Laue method, but no evidence of crystalline structure of any kind was obtained

The most probable effect of stress on substances of colloidal nature is to alter the distances between the molecules or atoms composing those molecules, unequally in different directions

Effect of Chemical Treatment on the Double Refraction of Fibres

When cotton fibres are treated with concentrated solutions of sodium, potassium, or tetramethylammonium hydroxides,* they shrink in length and swell transversely, with almost complete removal of the central canal Fibres treated in this way when turned about the plane of polarisation show almost equal illumination whatever their position with respect to that plane From this observation, the author† concluded that the shrinkage in length was due to the balancing of the internal stresses

Naegeli‡ observed that nitrated cotton containing less than 11 7 per cent of nitrogen had positive double refraction, but that containing more than 12 5 per cent of nitrogen showed negative double refraction This latter corresponds to the condition produced on an elastic body by compression in the direction of the axis The internal stresses would be represented by a positive pressure acting along the axis of the fibre This is confirmed by the fact that stretching a highly nitrated cotton fibre reduces the negative double refraction, and eventually makes it positive

Ordinary cotton fibres show numerous irregularities when examined in polarised light, which are not seen in ordinary light (fig 9, A), and in transverse sections there is no evidence of a regular distribution of stresses (fig 9, B) When treated with concentrated alkalies while under tension, the fibres show a somewhat regular distribution of the internal stresses, both longitudinally (fig 10, A) and transversely (fig 10, B)

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* Knecht and the Author, 'Journ Soc Dyers and Cols,' vol 28, p 224, (9), (1912).

† *Loc cit* (Ref 3)

‡ 'Ber der Bayer Akad,' vol. 1, p 307, (10), (1862).

Most of the comparatively thick fibres of English wool, when examined in polarised light, show different colours in various parts of the fibres This is partly due to differences in thickness, as shown by the fact that portions change in colour when fibres are rotated on their axes, and partly to variations in the internal stresses

A fibre of wool showing variations of the latter kind was held straight between two clamps and boiled for five minutes in water, when the variations disappeared, and the fibre showed even colouring (Figs. 11, A and B, illustrate this experiment)

The same fibre, after treatment, was stretched 25 per cent, when the colour changed except in that portion which before treatment differed from the rest of the fibre photographed Further boiling in water while stretched again removed the inequalities in colour, fig 11, D

The results of this experiment point to inequalities in the physical state of the fibres in different portions

The modifications in the condition of internal stress produced on wool fibres by boiling in water are similar to those produced on cotton fibres by treatment with concentrated solutions of alkalies Cold water, although it removes internal stresses formed by the action of externally applied stress, has no effect on the naturally occurring internal stresses, which must therefore differ from the former kind

The same difference has been observed with artificial fibres, and is undoubtedly due to the fact that the stresses present in the formed fibres are due to changes in volume, whereas externally applied stress produces only changes in shape.

The action of concentrated sulphuric acid on wool in modifying the internal stresses is illustrated by fig 12 Sulphuric acid 1·80 sp gr caused swelling of the wool fibre (fig. 12, A and F), which was attacked first on the outside and gradually through to the inside This is proved by the manner of removal of double refraction as shown by Photos A, C, D, and E Treatment with cold water after 40 minutes in contact with the acid caused return of the double refraction and transverse shrinkage of the fibre (Photos G and H).

As would be expected from this experiment, the physical properties of the wool were changed considerably by the treatment Such wool was found to be unshrinkable, and when stretched extended much further before breaking than ordinary wool The maximum effect was given in the time taken to completely change the whole of the fibres

The Probable Causes of the Internal Stresses Existing in Natural Fibres

In order to ascertain in what manner internal stresses could arise in fibres, experiments were made with artificial fibres A concentrated solution of gelatine was run through a fine capillary tube into alcohol, and the resulting threads rapidly dried in a current of air On examination in polarised light these fibres showed double refraction similar to the natural fibres This property is due to internal stresses arising from the drying of the jelly Although shrinkage would occur in all directions, the stresses due to shrinkage lengthways would act on the whole cross-section of the fibre, whereas those due to shrinkage towards the diameter would be concentrated on it, consequently, the pressure acting transversely would be greater than that acting longitudinally

Fibres prepared from gelatine in the above manner, when placed in water swelled rapidly and the double refraction disappeared but no shrinkage lengthways took place, the swelling transversely, however, was greater than that longitudinally Treatment of the gelatine fibres with formaldehyde vapour prevented the removal of double refraction by cold water

Fibres were prepared by forcing viscous solutions of cellulose into a coagulating liquid, the resulting fibres, although wet, were doubly refracting The internal stresses present in such fibres are formed during the forcing of the viscous solution through the fine capillary, and made permanent by solidification The same cellulose solution, forced through the capillary into the air, swelled out as it left the capillary, presumably owing to the effect of the internal stresses It is obvious that, the more viscous the material, the longer the time required for the balancing of internal stresses, and, with a very viscous material, even a slow process of coagulation would be quite sufficient to make the stresses permanent

Artificial silk is made by forcing cellulose solutions into a coagulating liquid, the resulting fibres being eventually dried under slight tension Such fibres show double refraction quite strongly, and, under the action of stress, heat, and moisture, behave in a similar manner to ordinary cotton fibres It is thus possible to produce, artificially, fibres having internal stresses similar to those present in the natural fibres

There are several possible causes for the internal stresses present in the natural fibres. If fibres are composed of spindle cells, which have themselves originated from spherical cells, the forces which produced the change are obviously those of extension, and in a viscous or plastic body would produce

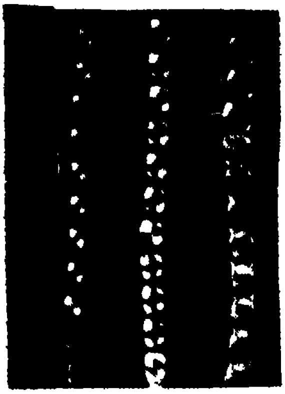


Fig. 1

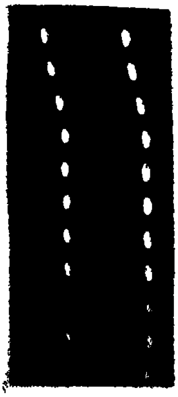


Fig. 2

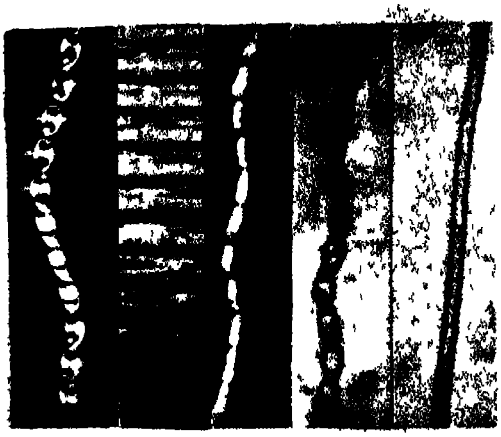


Fig. 3

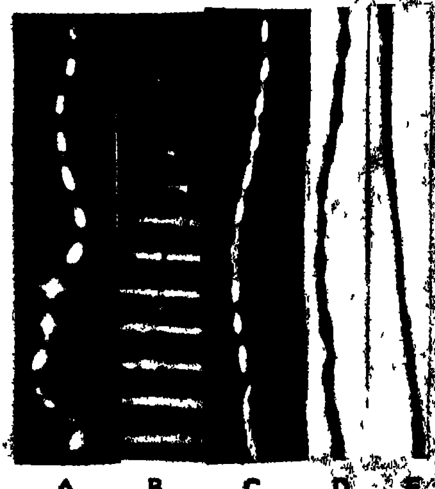


Fig. 4

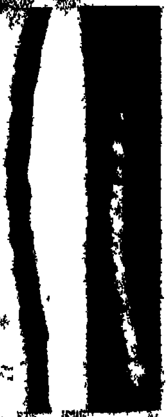


Fig. 5

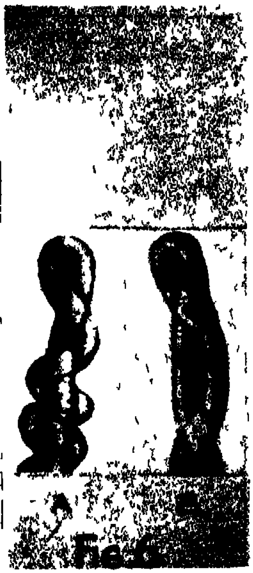


Fig. 6

Fig. 7

FIG 10

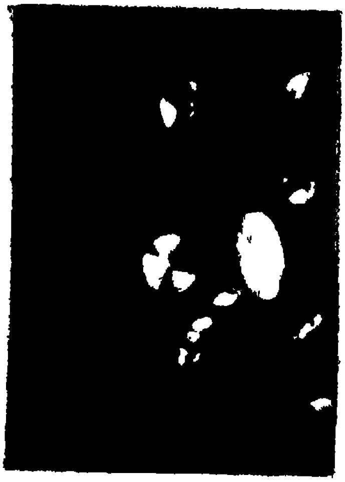


FIG 11

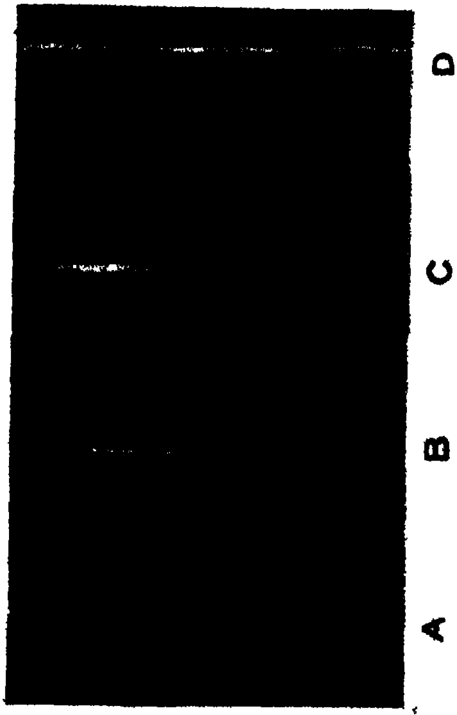
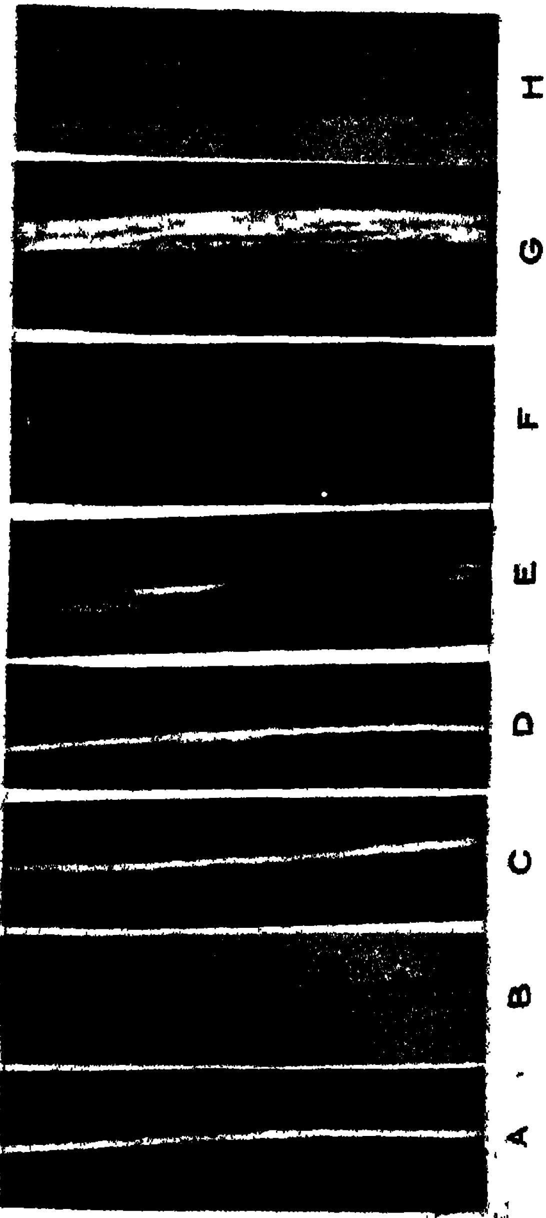


FIG 12



internal stresses similar to those in natural fibres, such stresses would be made permanent by the coagulation of the viscous material

A phenomenon of this kind occurs when a drop of oleic acid is placed in a 10-per-cent. solution of ammonia, cylindrical formations arise which show double refraction. This is one of the well known cases of the so-called liquid crystals.

On the other hand, if fibres are formed by the forcing, probably by osmotic pressure, of a viscous or plastic material through a nozzle or pore in the skin of a seed or an animal, conditions will arise similar to those discussed with reference to cellulose solution

It is likely that the air plays some part in the hardening of the materials of which the natural fibres are composed

DESCRIPTION OF PLATES

PLATE 5

Fig 1—A fibre of English wool as seen in polarised light between crossed nicols in various stages of compression on a grooved celluloid plate.

Fig 2.—A cotton fibre in two stages of compression

Fig 3.—A, a wool fibre compressed on a grooved celluloid plate, polarised light; B, the impression produced on the celluloid, ordinary light, C, the wool fibre after removal of pressure, polarised light, D, the same seen in ordinary light; E, after immersion in cold water

Fig. 4.—A similar set of experiments to the above, with a fibre of mercerised cotton.

Fig 5—A wool fibre compressed under boiling water on a grooved aluminium plate A, ordinary light B, polarised light.

Fig. 6—Illustrating the bending of wool fibres A, an ordinary fibre of English wool, B, a hollow fibre from Scotch wool.

Fig 7—Illustrating the bending of cotton fibres. A, a fibre of Egyptian cotton, B, mercerised under tension, C, mercerised without tension

Fig 8—A fibre of Egyptian cotton swelled by treatment with a saturated solution of copper carbonate in ammonia of sp gr, 0 880, showing barrel-shaped formations. A, ordinary light, B, polarised light, crossed nicols, showing internal stresses in narrow portions.

Fig 9.—Fibres of ordinary Egyptian cotton as seen in polarised light between crossed nicols. A, whole fibre, B, transverse sections

PLATE 6

Fig. 10.—Fibres of mercerised Egyptian cotton A, whole fibre, B, transverse sections.

Fig 11—A fibre of English wool as seen in polarised light, crossed nicols. A, original fibre, B, after boiling in water five minutes, C, the same fibre stretched 25 per cent.; D, the stretched fibre boiled in water for five minutes.

Fig. 12—Illustrating the action of sulphuric acid of sp gr, 1·80 on a fibre of English wool. A, original fibre, polarised light; B, the same, ordinary light, C, the fibre after one minute treatment with the acid, D, after 5 minutes; E, after 30 minutes; C, D, and E in polarised light; F, after 35 minutes, seen in ordinary light, G, the same fibre treated with cold water after 40 minutes in contact with the acid, seen in polarised light; H, the same in ordinary light.

The Ultra-Violet Band of Ammonia, and its Occurrence in the Solar Spectrum.

By A Fowler, F R S, and C C L Gregory, B A

(Received December 15, 1917)

(Abstract)

A question of great interest in connection with the solar spectrum is that of the origin of the thousands of unidentified faint lines which were photographed and catalogued by Rowland Some of these lines may possibly be identical with faint lines in metallic spectra which have not yet been completely tabulated, but in view of the presence of bands of cyanogen, carbon, and hydrocarbon, the possibility of the correspondence of most of them with band spectra of other substances should not be overlooked

As a contribution to this inquiry, the present investigation was undertaken primarily in order to determine whether Group P in the ultra-violet region of the solar spectrum might not be mainly due to the presence of ammonia in the absorbing atmosphere of the sun Ammonia was already known to give a remarkable band in this region, having its greatest intensity near λ 3360, but existing records of the component lines were inadequate for comparison with the solar tables Photographs were accordingly taken with instruments of various dispersions, ranging up to that of the third order of a 10 feet concave grating, a copper arc in an atmosphere of ammonia being employed as the source in the latter case In view of the unusual appearance of the band, an attempt has also been made to elucidate the chief features of its structure

The chief ammonia band consists of a bright central maximum about λ 3360, a secondary maximum about λ 3371, and a number of lines, which are arranged in groups of three, extending to a considerable distance in both directions The lines composing the two maxima are very closely crowded together and have been found to be arranged in series of ordinary type The components of the groups of three are widely separated near the central maxima, but the intervals rapidly diminish and there is final coalescence at λ 3450 towards the red, and at λ 3287 towards the violet, where the lines fade out The groups of three, however, are not symmetrical with respect to the central maxima, and they show marked peculiarities, so that they are very imperfectly represented by the formulæ usually employed for band spectra

The comparison with the solar spectrum is considerably complicated by

the presence of lines of metallic origin, but there is abundant evidence that many of the fainter solar lines are due to ammonia The most convincing proof of the presence of ammonia in the sun is perhaps afforded by the strongest parts of the central maximum, as indicated by the following extract from a table which is given in the paper —

Ammonia		Sun			Remarks
λ (Rowland scale)	Intensity	λ Rowland	Intensity	Origin	
3360 82	3	3360 828	0		Continuous background in both spectra
60 72	4	60 741	0		
60 62	5	60 ·631	0		
		60 485	1	Cr	
60 45	8	60 444	2	Ni	Continuous background in both spectra
60 33	3	60 345	0		
60 24	5	60 258	0		
60 18	10	60 181	2		
3360 ·08	4	3360 066	0		

It will be seen that there is a complete correspondence of the solar and laboratory spectra as regards these two clusters, except that the line 3360 44 in the sun may be slightly reinforced by a line which Rowland attributes to nickel, though it is not given as such by Exner and Haschek The agreement is emphasised by the occurrence of patches of continuous background (dark in the sun) which are identical in the two spectra The coincidences in the case of the central maximum, as a whole, are scarcely less striking, and it is clear that the solar Group P is largely due to ammonia

There is also a sufficiently consistent representation of the groups of three, and of the lines composing the secondary maximum of the ammonia band. Of the 260 band-lines of ammonia in the region λ 3450 to λ 3286, there are 140 which correspond with previously unidentified faint lines of the solar spectrum About 100 of the remaining lines are obscured by lines for which metallic origins have been found, or fall upon lines which are too strong in the sun to be assigned solely to ammonia, and the few which fail to appear in the sun are all of low intensity

The Presence in the Solar Spectrum of the Water-vapour Band λ 3064

By A FOWLER, F R S

(Received December 15, 1917)

The recent discovery of the ammonia band λ 3360 in the solar spectrum* suggested that additional solar lines might be identified by a search for other bands which are produced under generally similar conditions in laboratory experiments In the first instance, comparison has been made with the water-vapour band λ 3064, which accompanies the ammonia band in the ammonia flame and in the arc in ammonia

This band was discovered by Liveing and Dewar† in 1880, and independently, about the same time, by Huggins ‡ It was found by Liveing and Dewar that the band appeared in the flame of hydrogen or of hydrocarbons, and less strongly in the flames of non-hydrogenous gases, such as carbonic oxide and cyanogen, if burnt in moist air, it occurred also in the uncondensed discharge through moist hydrogen or other gases, but disappeared when the gas and apparatus were thoroughly dried, or when a large coil, with condenser, was the source of discharge It was accordingly concluded by Liveing and Dewar that the spectrum in question was that of water-vapour, but they were careful to explain that in writing of this and other spectra which had been traced to compounds, they abstained from speculating upon " the particular molecular condition or stage of combination or decomposition which may give rise to such spectra " Subsequent work has likewise indicated that both hydrogen and oxygen are essential to the development of the spectrum under consideration, but the precise nature of the combination involved cannot yet be deduced from the spectroscopic evidence

The band λ 3064 was afterwards found by Liveing and Dewar to be associated with others of lower intensity, extending as far as λ 4100 towards the red, and to λ 2268 in the ultra-violet. Extensive measures of the lines composing these bands were made by the same observers §

The chief band was measured more completely by Meyerheim|| in 1904,

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* Communicated to the Royal Society on December 15, 1917, by A Fowler and C C L. Gregory

† 'Proc Roy Soc.,' vol 30, pp. 494, 580 (1880)

‡ 'Proc Roy Soc,' vol 30, p. 576 (1880)

§ 'Phil. Trans.,' vol 179, p 27 (1888)

|| 'Zeitschr f Wiss. Phot,' vol 2, p. 131.

and by Grebe and Holtz* in 1912. The latter series of wave-lengths have been found to be sufficiently accurate for effective comparison with the solar tables of Rowland .The wave-lengths are given on the International scale, and the lines are classified in three orders of intensity, 3 indicating the brightest, and 1 the weakest lines The head of the band is marked by a pair of lines at 3063 547 (2) and 3063 713 (3), from which point the measured lines extend on the red side to 3291 687 (1) Strong lines persist to a considerable distance from the head, but in the less refrangible part of the band, the lines become generally weak and relatively far apart

The result of the comparison is to prove that the band 3064 is quite strongly represented in the solar spectrum, and accounts for a large number of lines which were previously unidentified It will suffice for the present purpose to give details relating to the brighter lines of the band, as in the appended Table To facilitate comparison with the solar lines, the wave-lengths given by Grebe and Holtz have been reduced to Rowland's scale by the addition of the following corrections, derived from the curve given by Kayser† —

Range of wave lengths	Correction from I A to Rowland
3057—3072	+0 121
3072—3087	122
3087—3102	123
3102—3117	124
3117—3132	125
3132—3147	126
3147—3162	127
3162—3177	128
3177—3192	+0 129

The corrected wave-lengths are given in the first column of the Table In Grebe and Holtz' list there are 53 lines which are classed as of intensity 3, but three of them are comparatively faint in my own photographs, and have therefore been omitted As the scale of intensities adopted by Grebe and Holtz appears to be unduly restricted, my own estimates, on the basis of 6 for the strongest lines, are given in the second column of the Table The following three columns show the wave-lengths, intensities, and origins of solar lines, as tabulated by Rowland The sixth column indicates the difference "sun *minus* water-vapour," except in the case of near coincidences with metallic lines The last column indicates the distances of the solar lines which are nearest to those which correspond to water-vapour

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* 'Ann d Physik,' vol. 39, p 1243.

† 'Handb d. Spect,' vol. 6, p 390

Water vapour		Sun (Rowland)			Sun *minus* W V	Next nearest solar line
λ (G and H)	Intensity (F)	λ	Intensity	Origin		
3063 834	5	3063 836	2		+002	+076
64 307	5	64 323	2		+016	+152
67 354	4	67 369	3	Fe		
68 039	5	68 046	2		+007	−151
74 477	3	74 492	1		+015	+050
78 552	5	78 552	3	Fe		
80 084	5	80 086	4	Fe ?		
81 648	5	81 ·657	1		+009	−090
83 384	5	83 389	1		+005	+100
85 304	5	85 313	2		+009	+125
87 449	5	87 451	1		+002	+106
89 843	6	89 851	2		+008	+123
89 968	6	89 ·974	2		+006	−123
91 301	5	91 ·319	1N		+018	−142
91 473	5	91 477	1		+004	−158
92 506	5	92 509	1		+003	+070
92 898	5	92 818 92 ·957	4 1	Al Al		
94 729	5	94 ·732	2		+003	+096
95 452	5	95 453	3*		+001	+098
96 242	4	96 244	2		+002	−099
96 938	5	97 ·008	5	—, Mg		
98 698	5	98 ·693	2		−005	+132
99 524	5	99 523	1		−001	+157
3099 689	3	3099 680	0N		−009	+100
3101 343	4	3101 347	1		+004	+175
02 256	5	02 253	1		−003	+151
05 775	5	05 ·782	1		+007	−114
06 133	5	06 137	2		+004	−136
06 658	5	06 664	2	Zr		
12 187	5	12 183	2	Fe, Ti ?		
13 476	5	13 490	1		+014	+070
17 311	4	17 ·309	1		−002	+048
22 661	6	22 680	2		+019	+094
24 ·051	5	24 ·070	1		+019	+138
27 790	3	27 ·784	2d ?*		−006	+175
28 389	4	28 402	1		+013	+097
30 366	5	30 380	3	V		
34 451	5	34 451	1		000	+059
36 996	5	37 005	1		+009	+135
40 843	5	40 872	3	Co, -		
43 901	5	43 879	4	Ti		
45 633	3	45 641	0		+008	−157
47 562	5	47 562	1		000	+153
51 110	5	51 120	1		+010	+092
54 619	5	54 608	1	—, Fe		
58 628	4	58 635	0		+007	+112
62 004	4	62 ·014	0		+010	+051
66 445	4	66 447	0		+002	−080
69 718	3	69 727	0		+009	+187
3183 082	3	3183 101	3	Fe, Ni ?		

* Some other substance probably also involved.

Of the 50 lines included in the Table, it will be seen that, while 12 are obscured in the sun by closely adjacent metallic lines, the remaining lines show a very close agreement in wave-length with solar lines, and a sufficiently consistent correspondence of intensities The average deviation of the 38 water-vapour lines from the corresponding solar lines, taken without regard to sign, is, in fact, only 0 007 Å, while the range is from +0 019 to −0 009 The differences are not too great to be considered as arising from errors in the two sets of measurements and uncertainty as to the corrections from the international to the Rowland scale Moreover, the deviations are clearly to some extent systematic, as indicated by the preponderance of positive signs From the last column of the Table, it results that the average distance of the next nearest solar lines is 0 111 Å, and there is consequently no reason to suppose that the coincidences are accidental

The total number of lines in Grebe and Holtz' list is 257, 63 of them are approximately coincident with, and therefore obscured by, lines in the sun which are of metallic, or probable metallic, origins, according to Rowland Five, which cannot be traced in the sun, although not masked by metallic lines, are too faint to appear in my own photographs or in Meyerheim's list, so that their apparent absence is of no significance With very few exceptions, which require further investigation, all the remaining lines are closely coincident with solar lines, but it is possible that several of them are slightly too strong in the sun to be attributed solely to water-vapour, notwithstanding the close agreement in wave-lengths. At least 150 of the solar lines in the region of the λ 3064 band, however, may confidently be assigned exclusively to water-vapour It is further probable that many additional solar lines will be traced to water-vapour when sufficiently accurate measures of the other bands become available for comparison

Besides accounting for a large number of previously unidentified solar lines, the identification of the water-vapour band in the solar spectrum is of interest as furnishing further evidence of the existence of oxygen in the sun

The Lunar and Solar Diurnal Variations of Water-Level in a Well at Kew Observatory, Richmond.

By E G BILHAM, A R C Sc, D I C, B Sc, F R Met Soc

(Communicated by Sir Napier Shaw, F R S Received December 11, 1917)

(Abstract)

In a paper communicated to the Royal Meteorological Society,* it was shown that the experimental well at Kew Observatory responded to the lunar fortnightly oscillation of mean level in the River Thames, which is 300 yards from the Observatory at its nearest point The sensitiveness of the water-level to barometric pressure has also been investigated, and the results have been given in a paper recently read before the Royal Society † The present paper deals with the effects of the short-period tides in the solar and lunar series, S_1, S_2, S_3, S_4, and M_1, M_2, M_3, M_4

Two-hourly measurements, both in lunar and solar time, were made on the traces obtained during the first two years, August, 1914–August, 1916, omitting days of very irregular movement. Monthly mean inequalities were then computed Well marked solar and lunar diurnal variations were found in each month, taking the form of double oscillations with two maxima and two minima during the 24 hours The range of movement was in each case found to be highly associated with the mean height of the water in the well, the correlation coefficients being 0 89 (lunar) and 0 90 (solar). A similar relation had been previously found to exist in the case of barometric pressure ‡

Harmonic coefficients were computed for means of groups of months representing high, intermediate, and low levels, the classification being the same as that described in the paper on the effects of barometric pressure §

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* In the press.

† 'Roy Soc Proc,' A, vol 94, pp 165–181 (1917)

‡ *Loc cit*, p 172.

§ *Loc cit.*, p 169

Lunar Diurnal Variation

(i) Mean of all results

$$0{\cdot}120 \sin (t_L + 1^\circ) + 0{\cdot}398 \sin (2t_L + 163^\circ) + 0{\cdot}028 \sin (3t_L + 322^\circ) + 0{\cdot}013 \sin (4t_L + 349^\circ),$$

(ii) High level

$$0{\cdot}169 \sin (t_L + 336^\circ) + 1{\cdot}357 \sin (2t_L + 166_0) + 0{\cdot}084 \sin (3t_L + 229^\circ) + 0{\cdot}035 \sin (4t_L + 355^\circ),$$

(iii) Intermediate level

$$0{\cdot}289 \sin (t_L + 14^\circ) + 0{\cdot}199 \sin (2t_L + 83^\circ) + 0{\cdot}068 \sin (3t_L + 95^\circ) + 0{\cdot}034 \sin (4t_L + 212^\circ),$$

(iv) Low level

$$0{\cdot}067 \sin (t_L + 48^\circ) + 0{\cdot}132 \sin (2t_L + 32^\circ) + 0{\cdot}012 \sin (3t_L + 14^\circ) + 0{\cdot}016 \sin (4t_L + 351^\circ)$$

Amplitudes are in millimetres, and t_L is reckoned from the time of the moon's northerly meridian transit at the rate of 15° per lunar hour

The chief points of interest in the above expressions are—

(1) The decrease in amplitude of all terms except the first with diminishing water-level

(2) The retardation in the phase of M_2, which represents a change of 4·4 hours between high and low levels

(3) The relative insignificance of M_4 In the Thames, near Richmond, the ratio of M_4 to M_2 appears to be about 0·45

The high value of the first order amplitude in (iii) is considered to be anomalous

Solar Diurnal Variation

(v) Mean of all results

$$0{\cdot}122 \sin (t + 317^\circ) + 0{\cdot}456 \sin (2t + 342^\circ) + 0{\cdot}145 \sin (3t + 169^\circ) + 0{\cdot}035 \sin (4t + 311^\circ),$$

(vi) High level

$$0{\cdot}394 \sin (t + 300^\circ) + 0{\cdot}861 \sin (2t + 351^\circ) + 0{\cdot}379 \sin (3t + 170^\circ) + 0{\cdot}085 \sin (4t + 297^\circ),$$

(vii) Intermediate level

$$0{\cdot}233 \sin (t + 157^\circ) + 0{\cdot}352 \sin (2t + 345^\circ) + 0{\cdot}015 \sin (3t + 298^\circ) + 0{\cdot}062 \sin (4t + 283^\circ),$$

(viii) Low level

$$0{\cdot}170 \sin (t + 33^\circ) + 0{\cdot}155 \sin (2t + 310^\circ) + 0{\cdot}049 \sin (3t + 162^\circ) + 0{\cdot}025 \sin (4t + 358^\circ),$$

t is reckoned from midnight G M T

Except in the case of high levels, where S_3 has a significant amplitude, the first and second order terms are the only important ones. The variations in amplitude and phase of the semi-diurnal oscillation are similar in character to the lunar results but the actual change of phase in the solar term amounts to only 1 2 hours between high and low levels

The ratio of S_2 to M_2 in the river is approximately 0 25 The fact that the corresponding ratio in the well was in all cases much greater suggested that the solar semi-diurnal oscillation was not due solely to the solar tide Using results given in the previous communication we may eliminate the diurnal variation of barometric pressure, when we are left with the following expressions for the effects of solar tides

Diurnal Variation of Water-level due to Solar Tides

(ix) Mean of all results

$$0\ 278 \sin(t+356^\circ)+0\ 069 \sin(2t+43^\circ),$$

(x) High level

$$0\ 270 \sin(t+343^\circ)+0\ 361 \sin(2t+41^\circ),$$

(xi) Intermediate level

$$0\ 178 \sin(t+96^\circ)+0\ 170 \sin(2t+14^\circ),$$

(xii) Low level

$$0\ 250 \sin(t+17^\circ)+0\ 065 \sin(2t+219^\circ)$$

The process of elimination is subject to considerable error on account of the smallness of the quantities dealt with The amplitudes of the semi-diurnal oscillation are, however, of the expected order of magnitude, and the ratio of S_2 to M_2 for high level (0 27) is in good agreement with the result from the River Thames The change of phase between high and low levels represents a lag of 5 1 days, a result similar to that given by the lunar analysis

Combining the lunar and solar semi diurnal tidal oscillations it is computed that "spring-tides" occur in the well 3 3 days after they occur in the river, so that this represents the actual time taken by solar and lunar semi-diurnal waves to travel through the subsoil from the river to the well The corresponding time in the case of the lunar-fortnightly tide is 5 1 days

For valuable assistance in this enquiry my grateful acknowledgments are due to Dr C Chree, Superintendent of Kew Observatory, and to Mr. H. W. T. Roberts, who kindly supplied tidal data for a number of stations on the River Thames.

Internal Ballistics

By Lieut.-Colonel A. G. Hadcock

(Communicated by Sir George Greenhill, F.R.S. Received October 27, 1917.)

1. *Introduction.*

My object is to explain and illustrate a method which I have devised for obtaining the pressure-volume relation of the gases in the bore of a gun, from the instant of ignition of the charge to the instant when the shot leaves the gun. By the aid of the expressions given below, the pressure-volume curve, or the indicator diagram, of the charge can be plotted when the nature and weight of charge is given.

The peculiarity of this problem, in comparison with similar problems of all kinds of internal combustion engines other than the gun, is that energy is produced in the gun by the burning of solid fuel within the gun itself during a considerable fraction of the expansion stroke. In other internal combustion engines, energy is added to the charge by combustion of the charge almost at constant volume, and no account need be taken of the rate of combustion of the fuel. In the gun problem, the rate at which the charge is burnt, that is the rate at which energy is produced, is fundamental to any investigation of the subject. There is the additional complexity that a discontinuity exists in the expansion of the gases at the point where the charge is just burnt. Up to this point expansion proceeds with the continuous addition of energy from the burning charge. After this point expansion proceeds nearly adiabatically. The influence of the resistance of the driving band on the subsequent expansion of the gases, and on the maximum pressure produced by the charge, is another vital element in the problem.

Whichever of the possible methods of thermodynamics be chosen to solve the problem of internal ballistics, many assumptions have to be made to meet the conditions which arise in practice. I have chosen to use the pressure-volume equation for the expansion of the gas of the general form

$$p\left(\frac{v-a}{z}\right)^{\epsilon} = k, \text{ a constant,}$$

because its constant can be more easily adjusted to meet the conditions found from experiments than with other methods.

p is the pressure in tons per square inch.

v the gravimetric volume of the charge (inches^3/lb.) (This artillery term will be explained later.)

z is the fraction of the charge burnt.

a is a constant called the co-volume (inches^3/lb.)

2. *Rate of Burning of the Charge*

The historical aspect of the subject up to 1875 has been dealt with by Noble and Abel.[*] Since that date much has been done, and our knowledge of the subject has been greatly increased by Noble,[†] and also by other experimenters. The later enquiries have followed, almost without exception, Noble's lead in making use of strong steel vessels or cylinders, closed by screw plugs. In this vessel the explosives have been burnt, and from the results the rate of burning of the explosive under different pressures has been determined, but the conclusions of the various investigators are by no means in agreement. Although they all assume that combustion takes place by parallel layers from the outer surface of the explosive, and they concur that the rate of burning, or the thickness σ inches burnt per second, varies as the pressure p raised to some power s, they do not concur on the value to be assigned to the power. Thus, in Sarrau's determination, the power $s = \frac{1}{2}$ for the old French black gunpowder, while, for similar powder, Sebert and Hugoniot, as a result of trials carried out in a 10-cm gun, came to the conclusion that $s = 1$.[‡] Vieille[§] adopts for colloidal powders $s = \frac{2}{3}$, while Centerval,[||] for the Swedish type of tubular ballistite, found that $s = 0.9$.

Major Mansel (1903),[¶] from experiments carried out in closed vessels with cylindrical modified cordite, formulates the expression $2\sigma = bp + a$ inches per second for the double thickness burnt off the diameter. Captain Grave, of the Russian Artillery (1904), for pyrocollodion powder, and Lees and Petavel,[**] with Mark I cylindrical cordite, follow with similar results. Petavel[††] points out that the shape of the explosion vessel has a considerable effect on the cooling curve of the explosive.

Charbonnier[‡‡] selects $s = 1$ as the value of the power for French B.M. powder, and Bianchi,[§§] of the Italian Artillery, also accepts a similar relation for Italian ballistite. The choice seems, therefore, to rest between the expressions

$2\sigma = bp + a$, inches per second for the double thickness of the film burnt and
$2\sigma = bp$, inches per second (1)

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[*] "Researches on Explosives, Fired Gunpowder," 'Phil. Trans.,' vol. 165, Part I.
[†] "Researches on Explosives," 'Phil. Trans.,' A, vols. 205 and 206.
[‡] 'Mémorial de l'Artillerie de la Marine,' series 2, vol. 1 (1882).
[§] 'Mémorial des Poudres et Salpêtres,' 1893.
[||] 'Calcul des Éléments Ballistiques Intérieurs,' Stockholm, 1902.
[¶] See also 'Phil. Trans.,' A, vol. 207.
[**] 'Roy. Soc. Proc.,' A, vol. 79.
[††] 'Phil. Trans.,' A, vol. 205.
[‡‡] 'Mémorial de l'Artillerie de la Marine,' 33, 1905.
[§§] 'Nozioni Fondamentali di Balistica Interna,' Turin, 1914.

In any case the constant a is small, and the different values given to it, and to b, by the several investigators, may be partly due to the differences in the composition of the explosive, but the presence of the first-named constant and its variation seems to point conclusively to the cooling effect of the explosion vessel. This effect would manifest itself principally when charges of low gravimetric density were fired, and the observed pressures would consequently be lower than the amount of powder burnt seemed to warrant. In a gun the cooling surface of the bore is continually increasing, so that a inches per second, a constant for the closed vessel, would on this hypothesis become a variable, and thus agree with Petavel's observation. It is usual to assume that, in a gun, the action of the explosion is so rapid that the cooling effect of the bore may be ignored. We shall therefore adopt the condition (1), which becomes for infinitely short lengths of time $\frac{d\sigma}{dt} = \frac{1}{2}bp$, suitably selecting the value of b, and accounting for the loss of energy by cooling and other losses—such as the momentum of the recoiling gun, friction in the bore, and the rotation of the shot by the rifling, by introducing a factor of effect, which is equivalent to assuming an increase in the weight of the projectile. This factor will also take into account the wear of the gun.

Cordite, and other similar explosives, made by different makers, varies considerably in the rate of burning. This is due to the diversity of methods adopted in the manufacture, besides also the natural variations of the raw materials. In order, therefore, to obtain the ballistics stipulated in the specification governing supply, the dry diameter of such cordite frequently differs from that of normal cordite. The rate of burning considered in this paper is that found by experiment for normal cordite.

For cylindrical modified cordite, heated to 80° F., before firing, the rate of radial combustion is 1 3361 p inches per second, on the diameter.

With tubular cordite there is reason for believing that the rate of combustion inside the tube is higher than that outside, owing to the generated gases inside the tube having to escape. This would induce a slightly greater pressure inside the tube than that outside. There are two methods of dealing with this feature: (*a*) by assuming a slight increase in the rate of combustion for both inside and outside of the tube when compared with the combustion of cylindrical cordite; (*b*) by assuming a different rate of combustion for the inside surface of the tube.

In the present case we shall adopt (*a*), and assume that the rate of combustion per second is 1 44 p instead of 1·3361 p. This value was obtained independently, from actual ballistic results in a gun (see Example 2), but it corresponds with the pressure coefficient of the formula deduced from Table D of "Investigation of the Law of Burning of Modified Cordite."[*] This Table D is the result of experiments with modified tubular cordite in closed vessels, and the formula I deduced from it is

Reduction in annulus at 80° F in 0.001 second = $0.001407p + 0.000473$.

If z denotes the fraction of the charge ϖ burnt up to any time t, then for sticks of explosive, of which the length is great in comparison to the other dimensions, it can be easily proved that

$$\frac{dz}{dt} = \mathrm{M}\phi(z)p, \tag{2}$$

where M denotes some function of the original sectional dimensions and the rate of burning per second, and $\phi(z)$ is a function of z

Thus for solid cordite, of diameter D inches, the sectional area of the burnt portion when a layer σ thickness has been consumed is

$$\frac{\pi}{4}\{\mathrm{D}^2-(\mathrm{D}-2\sigma)^2\}$$

Therefore the burnt fraction of the whole charge

$$z = \frac{\frac{\pi}{4}\{\mathrm{D}^2-(\mathrm{D}-2\sigma)^2\}}{\frac{\pi}{4}\mathrm{D}^2} = 1-\left(1-\frac{2\sigma}{\mathrm{D}}\right)^2$$

and $$\frac{dz}{dt} = \frac{4}{\mathrm{D}}\left(1-\frac{2\sigma}{\mathrm{D}}\right)\frac{d\sigma}{dt} = \frac{2}{\mathrm{D}}(1-z)^{\frac{1}{2}}bp.$$

Again for tubular cordite of external diameter D and internal diameter D′, supposing the rate of burning of the interior to be the same as for the exterior,

$$z = \frac{(\mathrm{D}-2\sigma)^2-(\mathrm{D}'+2\sigma)^2}{\mathrm{D}^2-\mathrm{D}'^2}$$

$$\frac{dz}{dt} = \frac{2}{\mathrm{D}-\mathrm{D}'}bp$$

Similarly for strip powder of width e and thickness h

$$\frac{dz}{dt} = \frac{e+h}{eh}\left\{1-\frac{4eh}{(e+h)^2}z\right\}^{\frac{1}{2}}bp$$

It will be seen that of the general form (2)

$$\mathrm{M} = \frac{2b}{\mathrm{D}},\quad \phi(z) = \sqrt{1-z}, \text{ for solid cylindrical cordite.}$$

$$\mathrm{M} = \frac{2b}{\mathrm{D}-\mathrm{D}'},\quad \phi(z) = 1, \text{ for tubular cordite}$$

$$\mathrm{M} = \frac{e+h}{eh}b,\quad \phi(z) = \sqrt{1-\frac{4eh}{(e+h)^2}z} \text{ for strip cordite}$$

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[*] Mansell, 'Phil. Trans.,' A, vol. 207, No. 418.

3 *The Equation of Expansion*

Again gravimetric density is an artillery term which really means the specific gravity of a charge of powder if considered to uniformly occupy some definite space This space is that between the base of the projectile and the breech of the gun, it is continually changing as the shot moves along the bore, and, consequently, the specific gravity of the charge also changes at the same time Gravimetric density is therefore the ratio of the weight of the charge of powder to the weight of water which would fill the space C in a gun behind the projectile The gravimetric volume is the reciprocal of the gravimetric density

In the metric system, since the weight of a cubic decimetre of water, at standard temperature, is 1 kgrm, the gravimetric density (or as it is called in France the "densité de chargement") is

$$\Delta = \frac{\varpi}{C}$$

and the gravimetric volume

$$v = \frac{C}{\varpi}$$

where ϖ is in kilogrammes and C in decimetres

With English units at the standard temperature of 62° F the volume of the Imperial gallon of 10 lb of water, of the Act of Parliament, is 277·3 inches^3, or a cubic foot of water weighs 62·35 lb, thus 27·73 inches^3 of water weighs 1 lb

If C is in inches^3 and ϖ in lb, the gravimetric density

$$\Delta = \frac{27{\cdot}73\varpi}{C}$$

and the gravimetric volume

$$v = \frac{C}{27{\cdot}73\varpi}$$

In the United States the gallon of water is taken to occupy 276·8 cubic inches, which corresponds to a density of water of about 62·4 lb. per cubic foot at a temperature of 54° F

Thus, if x represents the travel, in feet, of the projectile *plus* the equivalent length of the chamber (*i.e.*, the length of a chamber of cross-sectional area equal to A, the area in square inches of the bore, and of chamber capacity C in cubic inches, thus the equivalent length $= \frac{C}{12A}$), then for any point in the bore the gravimetric volume of the charge is

$$v = \frac{12xA}{27{\cdot}73\varpi} = \frac{1}{2{\cdot}31}\frac{xA}{\varpi} \text{ inches}^3/\text{lb}$$

or differentiating

$$dx = \frac{G}{A}\,dv, \qquad (3)$$

by putting $G = 2{\cdot}31\,\varpi$

Note — $\frac{G}{A}$ is the equivalent of length in feet of the bore for one gravimetric volume.

When a fraction z of the charge has been burnt, the space occupied by the solid unburnt portion of the charge is, if δ is the density of the explosive compared with water,

$$27{\cdot}73\,(1-z)\,\varpi\,\frac{1}{\delta}\ \text{inches}^3,$$

so that the available gas space behind the projectile is

$$12xA - 27{\cdot}73\,(1-z)\,\varpi\,\frac{1}{\delta}\ \text{inches}^3$$

The actual gravimetric volume of the gas for the fraction z of the charge burnt is therefore

$$v' = \frac{12xA - 27{\cdot}73\,(1-z)\,\varpi\,\frac{1}{\delta}}{27{\cdot}73\,z\varpi},$$

or by putting $\frac{1}{\delta} = \alpha$ (inches^3/lb.)

$$v' - \alpha = \frac{v-\alpha}{z}, \qquad (4)$$

where v is the gravimetric volume supposing the whole charge had been converted into gas.

The constant α is commonly called the co-volume, in cubic inches per pound, and equals 0·633 for M.D. cordite of which the density δ, compared with water, is 1·58, *i.e.*, 27·73 cubic inches of the explosive will weigh 1·58 lb.

If the gas, produced from the proportion of the charge burnt, expands doing work on the shell, heat is lost, supposing no more cordite is consumed during this process. But actually the cordite continues to burn and imparts additional heat to the already generated gas. There would be thus, during expansion, a loss of heat and at the same time a continual fresh supply of heat until the charge had been completely burnt.

The expansion is, therefore, neither purely adiabatic nor is it isothermal but something between the two. Probably no single condition satisfies the whole process, but for our purpose the mean exponent is sufficiently explicit. The co-efficient γ of adiabatic expansion of the gases of modern gunpowders is usually taken as 1·2, while that for isothermal expansion is 1, the coefficient of expansion up to the time of the total consumption of the cordite is therefore taken as the mean between these two values, viz., 1·1

In the following investigation it is assumed that the gas generated at volume v' from any fraction z of the charge, expands in accordance with the law:—

$$p(v'-\alpha)^{\epsilon} = k, \text{ a constant,}$$

the index ϵ being chosen between the value 1·2 for adiabatic expansion and 1 for isothermal expansion, namely 1·1

Or more conveniently by eq. (4)

$$p\left(\frac{v-\alpha}{z}\right)^{\epsilon} = k \tag{5}$$

The value of k depends largely on the frictional resistance to motion of the shot. It is evident from the form of this equation that for different values of z the pressures and volumes follow clearly defined curves

4 *Expansion Curves for Different Values of z.*

A group of expansion curves for different values of z is shown in fig. 1, they are hyperbolic in character, and the asymptotes are the axis of v and the vertical line α. In this figure the trace of these curves, which may be termed isopyric lines, has been drawn for $z = 0·1$, $0·2$, etc., so that if two of the three variables are known the third can be determined. Thus, knowing z and v, the corresponding p can be found at once

5 *The Pressure Equation and the Variation of the Constant k*

The initial resistance, during engraving, of the copper driving band, which is fitted to the projectile to give rotation, plays a most important part in the development of both the gas pressure and the velocity of the shot. The greater this initial resistance is, the earlier the maximum chamber pressure will be attained, and the higher will be this pressure and the muzzle velocity

The discussion of this part of the subject is deferred to the end of this paper, but mention is made of it here so that the reader may understand that p_0, v_0, u_0, and z_0 are the pressure, gravimetric volume, velocity of the shot and fraction of the charge burnt, when the engraving of the copper band is completed.

The higher resistance while the band is being engraved will alter the value of the constant k to k', so that at the end of the engraving process the gases generated from the proportion z_0 of the charge would expand according to the equation

$$p_0\left(\frac{v-\alpha}{z_0}\right)^{\epsilon} = k', \text{ a constant.}$$

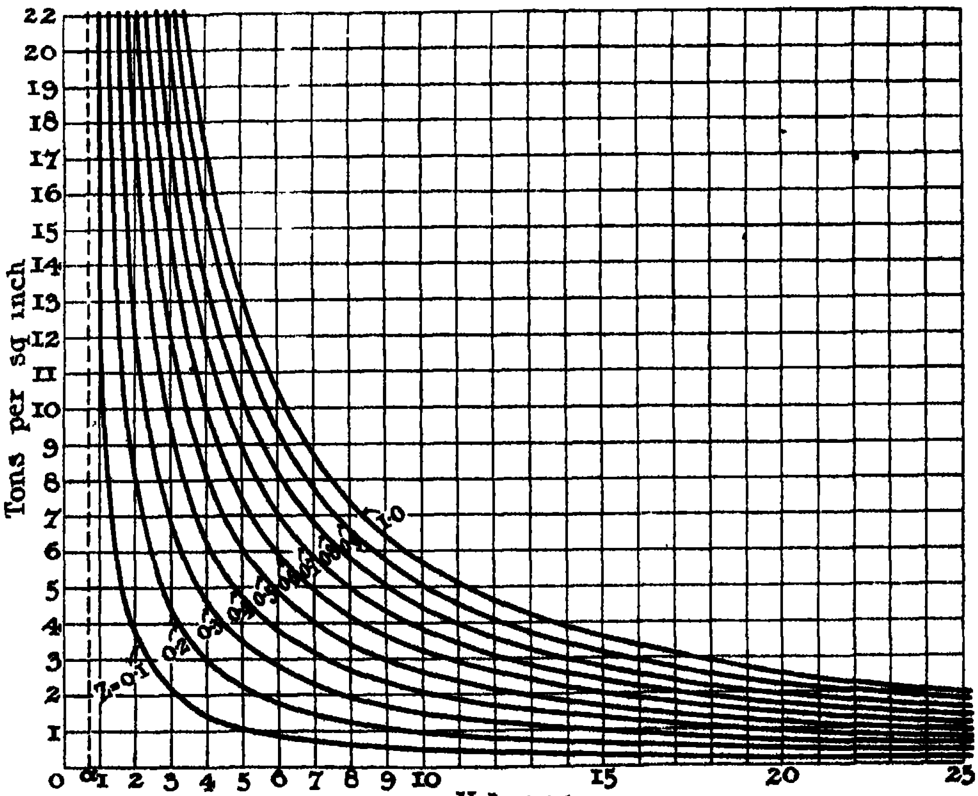

Fig 1.

If p_0' is the pressure, fig 2, which the generated gases at v_0 would have if expanded to v_1, then

$$p_0'\left(\frac{v_1-a}{z_0}\right)^{e}=p_0\left(\frac{v_0-a}{z_0}\right)^{e},$$

or

$$p_0'=p_0\left(\frac{v_0-a}{v_1-a}\right)^{e}$$

The pressure p_1 at volume v_1 is p_0' increased by an increment dp due to the burning of a further proportion dz of cordite Therefore, by algebraical reduction

$$p_1=p_0\left(\frac{v_0-a}{v_1-a}\right)^{e}+dp.$$

Again

$$p_1'=p_1\left(\frac{v_1-a}{v_2-a}\right)^{e},\qquad\text{and}\qquad p_2=p_1\left(\frac{v_1-a}{v_2-a}\right)^{e}+dp$$

$$=p_0\left(\frac{v_0-a}{v_2-a}\right)^{e}+dp\left(\frac{v_1-a}{v_2-a}\right)^{e}+dp,$$

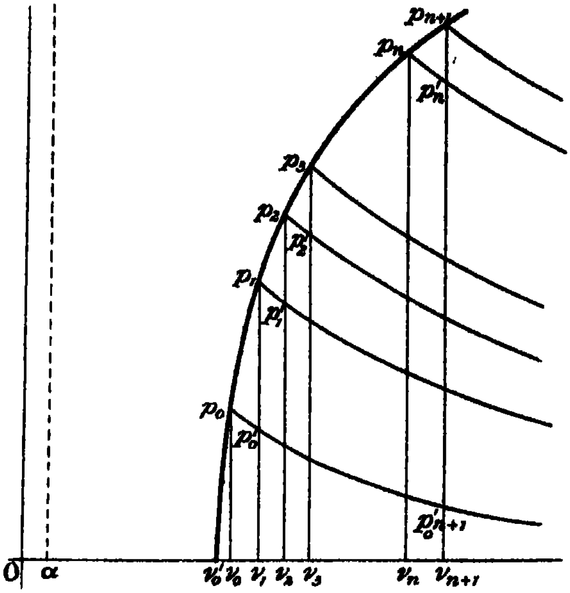

FIG 2

or generally

$$p_n = p_0\left(\frac{v_0-\alpha}{v_n-\alpha}\right)^\epsilon + dp\left(\frac{v_1-\alpha}{v_n-\alpha}\right)^\epsilon + dp\left(\frac{v_2-\alpha}{v_n-\alpha}\right)^\epsilon + \quad + dp,$$

$$= p_0\left(\frac{v_0-\alpha}{v_n-\alpha}\right)^\epsilon + \frac{1}{(v_n-\alpha)^\epsilon}\int_{v_0}^{v_n}(v-\alpha)^\epsilon\, dp \tag{6}$$

At any volume v_{n+1} the pressure

$$p_n' = p_n\left(\frac{z_n}{v_{n+1}-\alpha}\right)^\epsilon \quad \left(\frac{v_n-\alpha}{z_n}\right)^\epsilon = p_n\left(\frac{v_n-\alpha}{v_{n+1}-\alpha}\right)^\epsilon.$$

and from the hypothesis

$$p_n = k\left(\frac{z_n}{v_n-\alpha}\right)^\epsilon + \text{a proportion of the expansion of } z_0$$

$$= \frac{k}{(v_n-\alpha)^\epsilon}(z_n^\epsilon + \kappa z_0^\epsilon) \text{ suppose}$$

So that $$p_n' = \frac{k}{(v_{n+1}-\alpha)^\epsilon}(z_n^\epsilon + \kappa z_0^\epsilon)$$

Also $$p_{n+1} = \frac{k}{(v_{n+1}-\alpha)^\epsilon}(z_{n+1}^\epsilon + \kappa z_0^\epsilon)$$

Therefore $$p_{n+1} - p_n' = \frac{k}{(v_{n+1}-\alpha)^\epsilon}(z_{n+1}^\epsilon - z_n^\epsilon)$$

When p_n approaches very close to p_{n+1}

$$p_{n+1}-p_n' = dp = \epsilon\frac{k}{(v-\alpha)^\epsilon} z^{\epsilon-1}\,dz,$$

where $v-\alpha$ represents generally $v_n-\alpha$

Inserting this value of dp in (6) we get

$$\begin{aligned} p_n &= p_0\left(\frac{v_0-\alpha}{v_n-\alpha}\right)^\epsilon + \epsilon\frac{k}{(v_n-\alpha)^\epsilon}\int_{z_0}^{z_n} z^{\epsilon-1}\,dz \\ &= p_0\left(\frac{v_0-\alpha}{v_n-\alpha}\right)^\epsilon + \frac{k}{(v_n-\alpha)^\epsilon}(z_n^\epsilon - z_0^\epsilon) \\ &= \frac{k}{(v_n-\alpha)^\epsilon}\left\{z_n^\epsilon + \left(\frac{k'}{k}-1\right)z_0^\epsilon\right\} \end{aligned}$$

Therefore $\frac{k'}{k}-1 = \kappa$, and we may write generally for the mean pressure in the bore

$$p = \frac{k}{(v-\alpha)^\epsilon}(z^\epsilon + \kappa z_0^\epsilon) \qquad (7)$$

When all the explosive is burnt $z = 1$, we shall then put $v = v_1$ and $p = p_1$. Thus

$$p_1 = \frac{k}{(v_1-\alpha)^\epsilon}(1 + \kappa z_0^\epsilon) \qquad (8)$$

It has been found with modified cordite and the present form of Service driving bands that the average value of $k' = 80.6$ and of $k = 66$, so that $\kappa = 0.22121$

6. *The Instantaneous Mean Pressure in the Gun*

The pressure in tons per square inch on the face of the breech screw is greater than that acting on the base of the projectile. Experiment proves this to be the case. Crusher gauges inserted in the base of the projectile always give a lower pressure reading than those placed in the chamber at the breech. It is reasonable to suppose that the pressure in the bore falls more or less regularly from the breech to the base of the shot, and that the mean rate of burning of the explosive is that due to the mean pressure in the bore

The unburnt portion of the charge is assumed to be spread out equally over the space between the breech of the gun and the base of the projectile. Let u be the velocity in feet per second of the shot relative to the gun, lu the mean velocity of the charge, P the pressure in tons per square inch on the breech and p_s that on the base of the shot, W the weight in pounds of the projectile and w that of the charge, A the area in square inches of the bore, μ the coefficient of friction, supposed proportional to p_s, of the shot passing along the bore, and μ' the coefficient of friction of the gases in the bore

At the breech the momentum of the charge and projectile is

$$\frac{1}{2240}\left(\varpi\frac{lu}{g}+W\frac{u}{g}\right)\text{sec tons},$$

and the impressed force is

$$(P-\mu p_s-\mu' p)\,A \text{ tons}$$

At the base of the shot the momentum is

$$\frac{W}{2240}\,\frac{u}{g}\text{ sec tons},$$

and the impressed force is

$$p_s(1-\mu)\,A \text{ tons}$$

Consequently

$$\frac{(l\varpi+W)}{2240}\,\frac{du}{g\,dt}=(P-\mu p_s-\mu' p)\,A$$

and

$$\frac{W}{2240}\,\frac{du}{g\,dt}=p_s(1-\mu)\,A$$

The mean frictional resistance μ' of the gas is unknown, it may be included in the general loss μ and accounted for in the factor of effect Therefore eliminating $\frac{du}{dt}$ and putting $(1-\mu)=f$, the factor of effect, we obtain

$$P=\left(l\,\frac{\varpi}{W}f+1\right)p_s\ (\text{tons/inch}^2),$$

It is found that the value of l which suits practical results best is $\frac{2}{5}$, then for the mean pressure

$$p=\tfrac{1}{2}(P+p_s)=\left(\frac{1}{5}\,\frac{\varpi}{W}f+1\right)p_s=Bp_s, \qquad (9)$$

putting

$$\frac{1}{5}\,\frac{\varpi}{W}f+1=B \text{ for convenience,}$$

and then

$$P=\left(2-\frac{1}{B}\right)p\ (\text{tons/inch}^2) \qquad (10)$$

7 *Motion of the Shot*

The equation of motion of the shot is

$$\frac{W}{2240}\cdot\frac{d^2x}{g\,dt^2}=A\,(1-\mu)\,p_s\ (\text{tons})=Afp_s,$$

where μ includes the frictional and other losses, estimated as a fraction

If u is the velocity, in feet per second, of the projectile at any moment then by (9) the above expression can be written

$$m\,\frac{du}{dt}=\frac{A}{B}fp,$$

if for convenience we put m instead of $\frac{W}{2240}\,\frac{1}{g}$

Eliminating the mean pressure p and dt by (2)

$$m\,du = \frac{Af}{MB}\frac{dz}{\phi(z)}$$

Integrating between limits u and u_0, z and z_0

$$u - u_0 = \frac{Af}{mMB}V^z_{z_0} \tag{11}$$

where

$$V^z_{z_0} = \int_{z_0}^{z}\frac{dz}{\phi(z)}$$

This result is identical with Charbonnier's* except that he assumes the velocity $u_0 = 0$, immediately after the driving band has been engraved, whereas it may attain to a considerable magnitude

Since u is the velocity at any moment (ft./sec.), udt gives the distance dx (ft.) which the shot moves through in time dt (sec.), and now making v_0 the datum point or origin

$$dx = \frac{Af}{mMB}V^z_{z_0}\,dt$$

Putting $dx = \frac{G}{A}dv$, from (3), and eliminating dt by (2) we get

$$p\,dv = \frac{A^2f}{GmM^2B}V^z_{z_0}\frac{dz}{\phi(z)},$$

by (7)

$$\frac{dv}{(v-\alpha)^\epsilon} = \frac{A^2f}{GmM^2Bk}\frac{V^z_{z_0}}{z^\epsilon + \kappa z_0^\epsilon}\frac{dz}{\phi(z)} \tag{12}$$

Integrating between limits v and v_0, z and z_0 and putting

$$r = \frac{A^2f}{GmM^2Bk}$$

$$\frac{1}{\epsilon-1}\left\{\frac{1}{(v_0-\alpha)^{\epsilon-1}} - \frac{1}{(v-\alpha)^{\epsilon-1}}\right\} = rX^z_{z_0}, \tag{13}$$

where

$$X^z_{z_0} = \int_{z_0}^{z}\frac{V^z_{z_0}}{z^\epsilon + \kappa z_0^\epsilon}\frac{dz}{\phi(z)}$$

Tables I and II give $X^z_{z_0}$ for different values of z and z_0 for cylindrical and tubular modified cordite

As v_0 is known from the initial conditions, equation (13) gives the volume v and hence the distance x for any proportion of the charge burnt The corresponding pressure p is found from (7) and the velocity from (11).

When $z = 1$ the corresponding volume and pressure will be denoted by v_1 and p_1 and at the muzzle of the gun by v_2 and p_2

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* 'Mémorial de l'Artillerie Navale,' 1907, p. 287

Table I.—Cylindrical Cordite

z_0 / z	$V^z_{z_0}$	$X^z_{z_0}$							$Y^z_{z_0}$					
		0·00	0·05	0·10	0·15	0·20	0·25	0·30	0·05	0·10	0·15	0·20	0·25	0·30
0·00	0·000000	0·000000												
0·01	0·010025	0·017520												
0·02	0·020102	0·033012												
0·03	0·030228	0·047766												
0·04	0·040408	0·062116												
0·05	0·050642	0·076294	0·000000						0·000000					
0·06	0·060928	0·090178												
0·07	0·071270	0·104010												
0·08	0·081668	0·117750												
0·09	0·092122	0·131434												
0·10	0·102634	0·145058	0·017843	0·000000					0·070779	0·000000				
0·15	0·156092	0·213126	0·056105	0·011166	0·000000				0·143880	0·071211	0·000000			
0·20	0·211146	0·281696	0·103110	0·038054	0·006281	0·000000			0·221096	0·146310	0·073128	0·000000		
0·25	0·267950	0·351991	0·156069	0·074573	0·029638	0·006724	0·000000		0·303845	0·226732	0·151322	0·076018	0·000000	
0·30	0·326680	0·424254	0·213931	0·118665	0·060185	0·024708	0·005749	0·000000	0·393535	0·313913	0·235997	0·158229	0·079751	0·000000
0·35	0·387548	0·498987	0·276328	0·168631	0·098508	0·051662	0·021688	0·005144	0·491768	0·409393	0·328715	0·248165	0·166333	0·084368
0·40	0·450806	0·577002	0·343350	0·224762	0·143699	0·086443	0·046159	0·019761	0·600506	0·515096	0·431354	0·347734	0·263334	0·177590
0·45	0·516760		0·414948	0·286375	0·195389	0·128390	0·078536	0·042723	0·722255	0·633462	0·546291	0·450184	0·371268	0·281881
0·50	0·585786		0·491892	0·353803	0·253595	0·177397	0·118462	0·073670	0·860316	0·767702	0·676656	0·585567	0·493596	0·400098
0·55	0·658359		0·574783	0·427658	0·318663	0·233658	0·165992	0·112533	1·019230	0·922231	0·826733	0·731104	0·634443	0·536153
0·60	0·735088		0·664475	0·508724	0·391275	0·297723	0·221554	0·159626	1·205370	1·103278	1·002582	0·901614	0·799475	0·695556
0·65	0·816784		0·762411	0·596233	0·471885	0·369678	0·285996	0·215687	1·426368	1·320094	1·220625	1·105748	0·997136	0·886463
0·70	0·904554		0·870601	0·697926	0·563172	0·451606	0·360454	0·282000	1·702683	1·587156	1·472435	1·357114	1·240521	1·121566
0·75	1·000000		0·991546	0·810480	0·667192	0·547036	0·447332	0·360246	2·053291	1·926161	1·803796	1·678461	1·551565	1·421968
0·80	1·105573		1·120931	0·940065	0·787913	0·658863	0·550204	0·454127	2·585115	2·367247	2·249799	2·111020	1·970962	1·826369
0·85	1·225404		1·293126	1·093854	0·932308	0·793625	0·675333	0·569800	3·212239	3·055865	2·899403	2·741075	2·580175	2·415477
0·90	1·367544		1·495902	1·286058	1·113723	0·964384	0·835232	0·718553	4·358092	4·170972	3·982892	3·792018	3·597571	3·398315
0·95	1·552786		1·776840	1·553840	1·368185	1·205405	1·062694	0·932335	6·930018	6·674004	6·415139	6·151401	5·881771	5·604739
0·96	1·600000		1·851951	1·626667	1·436725	1·270806	1·124522	0·990763	7·963675	7·680185	7·392761	7·099735	6·799948	6·491730
0·97	1·653590		1·939017	1·709072	1·516381	1·346494	1·196609	1·058995	9·476762	9·159673	8·823826	8·466013	8·144099	7·790190
0·98	1·717158		2·044909	1·810624	1·612529	1·430194	1·264817	1·142677	12·012401	11·626568	11·222113	10·814393	10·396652	9·986364
0·99	1·800000		2·187393	1·947473	1·744654	1·564538	1·404223	1·256391	17·729911	17·185475	16·630077	15·904520	15·476340	14·873799
1·00	2·000000		2·560643	2·397504	2·111238	1·917568	1·743775	1·581903	∞	∞	∞	∞	∞	∞

Table II —Tubular Powder

$V^z_{z_0}$	$X^z_{z_0}$							$Y^z_{z_0}$					
z \ z_0	0·00	0·05	0·10	0·15	0·20	0·25	0·30	0·05	0·10	0·15	0·20	0·25	0·30
0·00	0·000000												
0·01	0·017610												
0·02	0·032861												
0·03	0·047333												
0·04	0·061321												
0·05	0·074960	0·000000						0·000000					
0·06	0·088327	0·000985											
0·07	0·101472	0·003871											
0·08	0·114430	0·007341											
0·09	0·127226	0·012025											
0·10	0·139881	0·017421	0·000000					0·064668	0·000000				
0·15	0·201484	0·051134	0·009756	0·000000				0·128003	0·061491	0·000000			
0·20	0·261026	0·091339	0·033405	0·008829	0·000000			0·185327	0·120703	0·059413	0·000000		
0·25	0·319053	0·134527	0·061759	0·033060	0·006297	0·000000		0·243193	0·178481	0·117236	0·057956	0·000000	
0·30	0·375982	0·179585	0·095849	0·046717	0·018545	0·004189	0·000000	0·299945	0·235144	0·173864	0·114650	0·056815	0·000000
0·35	0·431986	0·225578	0·131936	0·074115	0·037514	0·015200	0·003493	0·355765	0·290867	0·229550	0·170365	0·115589	0·055884
0·40	0·487063	0·272323	0·170472	0·104745	0·060650	0·031246	0·012846	0·410817	0·345634	0·284464	0·225258	0·167519	0·110861
0·45		0·319375	0·210284	0·137606	0·086911	0·051185	0·026723	0·465188	0·400122	0·338091	0·279472	0·221744	0·165141
0·50		0·366677	0·251148	0·172254	0·115636	0·074154	0·044207	0·518906	0·453824	0·392346	0·333094	0·275358	0·218776
0·55		0·414127	0·292808	0·208299	0·146256	0·099689	0·064596	0·572216	0·507004	0·445473	0·386188	0·328441	0·271862
0·60		0·461650	0·335080	0·245456	0·178454	0·127006	0·087391	0·624990	0·559713	0·498130	0·438809	0·381044	0·324462
0·65		0·509206	0·377829	0·283512	0·211946	0·156099	0·112177	0·677333	0·611994	0·550361	0·491004	0·433218	0·376625
0·70		0·556754	0·420952	0·322303	0·246513	0·186563	0·138661	0·729277	0·663582	0·602201	0·542607	0·484996	0·428390
0·75		0·604270	0·464370	0·361701	0·281982	0·218223	0·166576	0·780857	0·715408	0·653682	0·594250	0·536415	0·479792
0·80		0·651735	0·508021	0·401604	0·318216	0·250870	0·195739	0·832098	0·766596	0·704827	0·645361	0·587496	0·530657
0·85		0·699135	0·551857	0·441931	0·355102	0·284386	0·226060	0·883022	0·817475	0·755662	0·696160	0·638270	0·581606
0·90		0·746461	0·595839	0·482616	0·392546	0·318657	0·257120	0·933649	0·868057	0·806206	0·746669	0·688751	0·632067
0·95		0·793706	0·639985	0·523604	0·430419	0·353586	0·289105	0·983999	0·918365	0·856474	0·796906	0·738959	0·682264
0·96		0·803145	0·648766	0·531835	0·438119	0·360645	0·295604	0·994038	0·928396	0·866497	0·806921	0·748969	0·692259
0·97		0·812560	0·657600	0·540075	0·445774	0·367726	0·302111	1·004064	0·938414	0·876510	0·816926	0·758969	0·702255
0·98		0·822012	0·666437	0·548325	0·453446	0·374829	0·308656	1·014082	0·948424	0·886512	0·826922	0·768960	0·712241
0·99		0·831440	0·675277	0·556584	0·461133	0·381953	0·315227	1·024089	0·958423	0·896503	0·836907	0·778939	0·722217
1·00		0·840864	0·684120	0·564852	0·468835	0·389098	0·321825	1·034086	0·968412	0·906485	0·846884	0·788910	0·732168

After $z = 1$, when the whole of the charge has been converted into gas, ordinary adiabatic expansion is assumed to take place, so that

$$p(v-\alpha)^{\gamma} = \text{constant} = p_1(v_1-\alpha)^{\gamma},$$

where $\gamma = 1{\cdot}2$, and p or p_1 is the mean gas pressure in the bore Writing the equation of motion of the shot in the form

$$m\,u\,du = \mathrm{G}\,f\,p_s\,dv$$

we obtain from the above general condition and from $p = \mathrm{B}p_s$

$$m\,u\,du = \frac{\mathrm{G}f}{\mathrm{B}}p_1(v_1-\alpha)^{\gamma}\frac{dv}{(v-\alpha)^{\gamma}}$$

Integrating between limits u_1 and U, where U is the muzzle velocity, v_1 and v_2, where v_2 is the gravimetric volume of expansion at the muzzle,

$$\frac{m}{2}(\mathrm{U}^2-u_1^2) = \frac{1}{\gamma-1}\frac{\mathrm{G}f}{\mathrm{B}}p_1(v_1-\alpha)\left\{1-\left(\frac{v_1-\alpha}{v_2-\alpha}\right)^{\gamma-1}\right\}, \tag{14}$$

or giving p_1 its value from (8) and transposing

$$\mathrm{U}^2 = u_1^2+\frac{2}{\gamma-1}\frac{\mathrm{G}fk}{m\mathrm{B}}(1+\kappa z_0^{\epsilon})\frac{1}{(v_1-\alpha)^{\epsilon-1}}\left\{1-\left(\frac{v_1-\alpha}{v_2-\alpha}\right)^{\gamma-1}\right\} \tag{15}$$

where the value of u_1 is obtained from (11) and (20)

$$u_1 = \frac{\mathrm{A}}{m\mathrm{MB}}\{f\mathrm{V}^{z}_{z_0}+(1-n)\mathrm{V}_0^{z_0}\} \tag{16}$$

The mean pressure p_2 of the gas when the projectile is just leaving the muzzle is obtained at once from the general condition

$$p_2 = p_1\left(\frac{v_1-\alpha}{v_2-\alpha}\right)^{\gamma} \tag{17}$$

8 *Maximum Pressure*

It is important to know the maximum pressure and its position in the gun To find this position, differentiate equation (7) in the form

$$p(v-\alpha)^{\epsilon} = k(z^{\epsilon}+\kappa z_0^{\epsilon})$$

Thus since z_0^{ϵ} is a constant for any particular set of conditions

$$(v-\alpha)^{\epsilon}\,dp+\epsilon p(v-\alpha)^{\epsilon-1}\,dv = \epsilon k z^{\epsilon-1}\,dz$$

For a maximum value of p, $\quad \dfrac{dp}{dv} = 0$

and $\quad p(v-\alpha)^{\epsilon-1}\,dv = kz^{\epsilon-1}\,dz,$

or $\quad \dfrac{dv}{v-\alpha} = \dfrac{z^{\epsilon-1}\,dz}{z^{\epsilon}+\kappa z_0^{\epsilon}}$

Eliminating $\dfrac{dz}{z^{\epsilon}+\kappa z_0^{\epsilon}}$ between this equation and (12), and writing χ and ζ

for the values of v and z respectively, corresponding with the maximum value of p, the following expression is obtained

$$\frac{d\chi}{(\chi-\alpha)^{\epsilon}} = r\frac{V^{\zeta}_{z_0}}{\zeta^{\epsilon-1}\phi(\zeta)}\frac{d\chi}{\chi-\alpha},$$

so that

$$\frac{1}{\chi-\alpha} = \left\{r\frac{V^{\zeta}_{z_0}}{\zeta^{\epsilon-1}\phi(\zeta)}\right\}^{\frac{1}{\epsilon-1}}$$

Again from equation (13) by treating v and z as corresponding to the maximum value of p and transposing

$$\frac{1}{\chi-\alpha} = \left\{\frac{1}{(v_0-\alpha)^{\epsilon-1}}-(\epsilon-1)rX^{\zeta}_{z_0}\right\}^{\frac{1}{\epsilon-1}}. \tag{18}$$

We have therefore from these two expressions the relation

$$\frac{1}{r(v_0-\alpha)^{\epsilon-1}} = \frac{V^{\zeta}_{z_0}}{\zeta^{\epsilon-1}\phi(\zeta)}+(\epsilon-1)X^{\zeta}_{z_0}$$

$$= Y^{\zeta}_{z_0} \tag{19}$$

Since r and v_0 are known from the initial conditions ζ can be found from Tables I and II, $\frac{1}{\chi-\alpha}$ can then be found from equation (18).

9 *Resistance of Driving Band*

When a projectile, fitted with a copper or other metal driving band, is forced through a short rifled cylinder by hydraulic pressure, the resistance offered by the band depends on the amount of copper to be forced through the rifling grooves. It depends also on the clearance in rear of the band between the projectile and the gun bore

There are two general forms of driving band in use, viz, (*a*) that with a parallel profile of which the external diameter is a little larger than the diameter over the rifling grooves in the gun, (*b*) the gas-check type with a tapered profile of which the largest diameter, towards the base of the projectile, rests against the front cone of the gun chamber and so seals it.

The flat profile of the first type of band does not as a rule properly seal the chamber. The resistance of this band to being forced into the rifling appears to depend on the total shearing circumference of the rifling, and the experimental results already collected seem to agree with this.

With the gas-check type of band, actual trial shows that in many cases the maximum resistance takes place before the band has entered the rifling, that is to say, the maximum pressure is developed in deforming the copper, or other metal of the band, in forcing it through a diameter smaller than itself

Experiments with standard and other bands are still in progress, but the results are not sufficiently advanced for publication Only one or two forms are consequently dealt with for the purpose of illustrating the present paper

The shape of the band differs in most cases, and it is only in the larger natures that the bands are made proportional to the diameter of the bore No general law can at present be laid down for the resistance For a correct determination, recourse must be had to experiments with each type of band

Owing to the wear of the band in high-velocity guns the amount of copper in the section must be considerably greater than with that intended for low velocity guns It is usually preferable to add to the amount of copper where necessary, by increasing the diameter of the band

The resistance begins with a fairly rapid rise, and the curve usually takes a parabolic form, it then flattens, and when about two-thirds of the band is engraved, the resistance gradually falls until the engraving is completed The curve then runs nearly parallel to the datum line

In a gun, during the brief period of engraving the band, the velocity attained by the projectile is never high (say, 60 or 70 ft./sec.) and it can therefore, be assumed that the resistance is similar to that occurring in the statical cylinders As the shot travels along the bore the band wears, moreover, the pressure on the rifling which produces rotation is—speaking broadly—proportional to the gas pressure. We may, therefore, fairly assume that the resistance of the band after engraving is proportional to the pressure

If the curve of resistance during the engraving process is assumed to be a parabola, we shall not be greatly in error, especially if we take into account the mean resistance in the calculation The gas pressure may also be assumed with even more correctness to rise as a parabola during the engraving of the band. In that case the resistance and pressure are proportional and the formulæ already found again apply, the constants must, however, be changed to suit the engraving conditions

Thus let p_s be the gas pressure at any instant (tons/inch^2),
q the resistance at the same instant (tons/inch^2),

then

$$\frac{W}{2240}\frac{du}{g} = A(p_s - q)\,dt$$

$$= \frac{A}{MB}\left(1 - \frac{Bq}{p}\right)\frac{dz}{\phi(z)} \text{ by (2) and (9)}$$

Put $\frac{Bq}{p} = n$, a fraction, n is therefore also equal to the ratio of the mean values of q and p multiplied by B

Integrating, starting with the shot at rest and $z = 0$, and again, for convenience, putting m for $\frac{W}{2240}\frac{1}{g}$

$$m u = \frac{A}{MB}(1-n)V_0^z \qquad (20)$$

But as
$$u = \frac{dx}{dt} = \frac{G}{A}\frac{dv}{dt}$$

$$p\,dv = \frac{A^2(1-n)}{Gm M^2 B} V_0^z \frac{dz}{\phi(z)}$$

Until the band is fully engraved $p = k'\left(\frac{z}{v-\alpha}\right)^\epsilon$,

therefore
$$\frac{dv}{(v-\alpha)^\epsilon} = \frac{A^2(1-n)}{Gm M^2 B k'}\frac{V_0^z}{z^\epsilon}\frac{dz}{\phi(z)},$$

or, integrating between limits v_0', the gravimetric volume of the chamber before the shot moves, and v_0, 0 and z_0,

$$\frac{1}{\epsilon-1}\left\{\frac{1}{(v_0'-\alpha)^{\epsilon-1}} - \frac{1}{(v_0-\alpha)^{\epsilon-1}}\right\} = \gamma_0 Y_0^{z_0} \qquad (21)$$

where
$$\gamma_0 = \frac{A^2(1-n)}{Gm M^2 B k'}$$

$$n = \frac{\text{B mean resistance } (= q)}{\text{mean pressure } (= {}^2/_3 p_0)} = {}^3/_2\frac{Bq}{p_0}, \qquad (22)$$

p_0 is chosen and z found from the column $z_0 = 0·00$ in the Tables. If these values satisfy the expression

$$p = k'\left(\frac{z}{v_0-\alpha}\right)^\epsilon \qquad (23)$$

they are the correct values. Then u_0 can be found from equation (20).

The correct values of p_0 and z_0 are most easily found graphically by plotting the curve eq. (23), then from (21) plotting values of p and z. The point where these lines intersect gives p_0 and z_0.

In the above investigations we have, for the sake of convenience, taken the important process of engraving the driving band last, but actually immediately the charge in the gun is ignited gas is generated, and begins to force the driving band into the grooves. As the gas pressure during this operation is comparatively low, the actual engraving occupies quite an appreciable time, but after the band is engraved the pressure in the gun rapidly rises to its maximum, subsequently falling until the whole of the charge is burnt. It is not possible for the maximum pressure to take place after the combustion of the charge is completed, but the maximum pressure and the completion of the combustion may occur at the same instant.

By solving formula (13) for $z = 1$ the position in the gun v_1 at which all the charge is burnt is found. Whether the charge is completely burnt before the shot reaches the muzzle is of great importance from an artillery point of view, because it is found that greater regularity as regards range and deflection is obtained when the charge is all consumed at a point in the gun some distance from the muzzle.

10. *Values of the Functions and Examples.*

In order to make use of these formulæ I have calculated, see Table I, the values of $V_{z_0}^z$, $X_{z_0}^z$ and $Y_{z_0}^z$, for modified cylindrical cordite for $z = 0, 0·05, 0·10, 0·15, 0·20, 0·25$, and $0·30$. Experience has shown that it is very rarely that so large a fraction of the charge is burnt as $z_0 = 0·30$ before the driving band is fully engraved.

Similar $X_{z_0}^z$ and $Y_{z_0}^z$ functions (see Table II) for modified tubular cordite have also been calculated. The value of $V_{z_0}^z$ for tubular cordite is immediately obtained, as it is identical with $z - z_0$. It should be noted that the rate of radial burning, $1·44p$ inches per second, may require to be modified if the diameter of the central hole is unduly reduced.

To illustrate the method of plotting the pressure-volume diagram, I have chosen an example from ordinary modified cylindrical cordite and one from modified tubular cordite. The driving band in each case was identical, but the guns differed both as regards length and capacity of chamber. The factor of effect may vary considerably, as, even in the same gun with different batches of cordite of the same nature, the ballistics actually obtained may differ by 15–20 ft./sec. in velocity with a corresponding variation in pressure.

For cylindrical cordite the function

$$V_{z_0}^z = \int_{z_0}^z \sqrt{1-z}\,dz = 2\{\sqrt{1-z_0} - \sqrt{1-z}\}$$

For

$$X_{z_0}^z = \int_{z_0}^z \frac{\sqrt{1-z_0} - \sqrt{1-z}}{z^\epsilon + \kappa z_0^\epsilon} \frac{dz}{\sqrt{1-z}},$$

$$Y_{z_0}^z = 2\frac{\sqrt{1-z_0} - \sqrt{1-z}}{z^{\epsilon-1}\sqrt{1-z}} + (\epsilon - 1) X_{z_0}^z,$$

the evaluation was performed by approximate integration.

For tubular cordite

$$V_{z_0}^z = z - z_0,$$

$$X_{z_0}^z = \int_{z_0}^z \frac{z - z_0}{z^\epsilon + \kappa z_0^\epsilon}\,dz,$$

$$Y_{z_0}^z = \frac{z - z_0}{z^{\epsilon-1}} + (\epsilon - 1) X_{z_0}^z.$$

Here again the X and Y functions were obtained by approximate integration

The following examples will show the general method of calculation —

(1) Gun, 6-inch, 50-calibre total bore capacity, including chamber, 9268 cubic inches, chamber capacity, 1658 cubic inches, area of bore, including grooves, 28·942 square inches, charge, 28·625 lb, modified cordite, size 26 (D = 0·1891 inches)

The actual mean muzzle velocity obtained with a shot of 100 lbs was 2905 ft/sec, and a mean chamber pressure of 17·93 tons per square inch

The driving band used was of cupro-nickel, and its mean resistance, when forced through a rifled cylinder, was found to be 1·61 tons per square inch of bore area for a distance of 0·1416 feet

Here $A = 28{\cdot}942$ sq in $D = 0{\cdot}1891$ ins
$W = 100$ lb $\varpi = 28{\cdot}625$ lb

The initial gravimetric volume of the charge is

$$v_0' = \frac{1658}{28{\cdot}625 \times 27{\cdot}73} = 2{\cdot}0887,$$

$$v_0' - \alpha = 2{\cdot}0887 - 0{\cdot}633 = 1{\cdot}4557$$

When the band has fully entered the rifling, the projectile has advanced a distance of 0·1416 feet, thus increasing the chamber capacity by 49 cubic inches The gravimetric volume at this stage is

$$v_0 = \frac{1658+49}{28{\cdot}625 \times 27{\cdot}73} = 2{\cdot}1505,$$

or

$$v_0 - \alpha = 2{\cdot}1505 - 0{\cdot}633 = 1{\cdot}5175$$

At the muzzle the gravimetric volume is

$$v_2 = \frac{9268}{28{\cdot}625 \times 27{\cdot}73} = 11{\cdot}676,$$

$$v_2 - \alpha = 11{\cdot}676 - 0{\cdot}633 = 11{\cdot}043$$

The constants r_0 and r should now be found

$$r_0 = \frac{A^2(1-n)g \times 2240}{GWM^2Bk'}$$

$$= \frac{A^2(1-n)D^2 \times 32{\cdot}2 \times 2240}{2{\cdot}31 \quad \varpi \,.\, 4 \quad (1{\cdot}3361)^2 \,.\, W\left(\frac{1}{5}\frac{\varpi}{W}f+1\right)80{\cdot}6}$$

$$= 54{\cdot}2524 \frac{A^2 \times D^2}{\varpi\left(\frac{1}{5}\varpi + \frac{W}{f}\right)} \frac{1-n}{f}$$

Assume the factor of effect f to be 0·96, then

$$\frac{1}{5}\varpi + \frac{W}{f} = \frac{28\cdot625}{5} + \frac{100}{0\cdot97}$$

$$= 109\cdot892$$

$$r_0 = 0\cdot538116\,(1-n)$$

and as $\epsilon = 1\cdot1$ $\quad (\epsilon-1)\,r_0 = 0\cdot0538116\,(1-n)$

$$r = \frac{A^2 f g\, 2240}{GWM^2Bk} = 66\cdot2537\,\frac{A^2 \times d^2}{\varpi\left(\frac{1}{5}\varpi + \frac{W}{f}\right)}$$

$$r = 0\cdot630868$$

$$(\epsilon-1)\,r = 0\cdot0630868$$

It is necessary to find the ballistic conditions when the driving band is just fully engraved Equation (21), together with the expression

$$p = k'\left(\frac{z}{v_0-\alpha}\right)^{\epsilon} = 80\cdot6\left(\frac{z}{1\cdot5175}\right)^{\epsilon}$$

enables z_0 to be found.

$$\frac{1}{(v_0'-\alpha)^{\epsilon-1}} = 0\cdot96315, \qquad \frac{1}{(v_0-\alpha)^{\epsilon-1}} = 0\cdot95915$$

Therefore

$$\frac{1}{(v_0'-\alpha)^{\epsilon-1}} - \frac{1}{(v_0-\alpha)^{\epsilon-1}} = 0\cdot96315 - 0\cdot95915,$$

or

$$(\epsilon-1)\,r_0\,X_0^{r_0} = 0\cdot004,$$

$$X_0^{r_0} = \frac{0\cdot004}{0\cdot0538116\,(1-n)} = 0\cdot074333\,\frac{1}{1-n}$$

$$B = \frac{1}{5}\,\frac{\varpi}{W}\,f + 1 = 1\cdot05496,$$

From eq. (22)

$$n = \frac{3}{2}\,B\,\frac{q}{p_0} = \frac{3}{2} \times 1\cdot05496 \times \frac{1\cdot61}{p_0} = \frac{2\cdot54773}{p_0}$$

Suppose $p_0 = 4$ tons/inch², then

$n = 0\cdot63693, \qquad 1-n = 0\cdot36307,$

$X_0^{z_0} = 0\cdot204736$ and from Table I, Column 0·00,

$z_0 = 0\cdot14384$

If $p_0 = 5$ tons,

$n = 0\cdot50955, \qquad 1-n = 0\cdot49045,$

$X_0^{z_0} = 0\cdot151562, \qquad z_0 = 0\cdot10478$

or $p_0 = 6$ tons,

$n = 0\cdot42462, \qquad 1-n = 0\cdot57538,$

$X_0^{z_0} = 0\cdot129190, \qquad z_0 = 0\cdot08836$

Plot on squared paper (see fig 3), from $z = 0$ to $z = 0·15$ the curve

$$p = 80·6\left(\frac{z}{1·5175}\right)^{\epsilon}$$

and draw a curve through the three points p_0, z_0 found above Where these curves intersect, viz, $p_0 = 4·65$ tons, $z_0 = 0·1135$, is the pressure in the gun and the fraction of the charge burnt when the driving band is just fully engraved

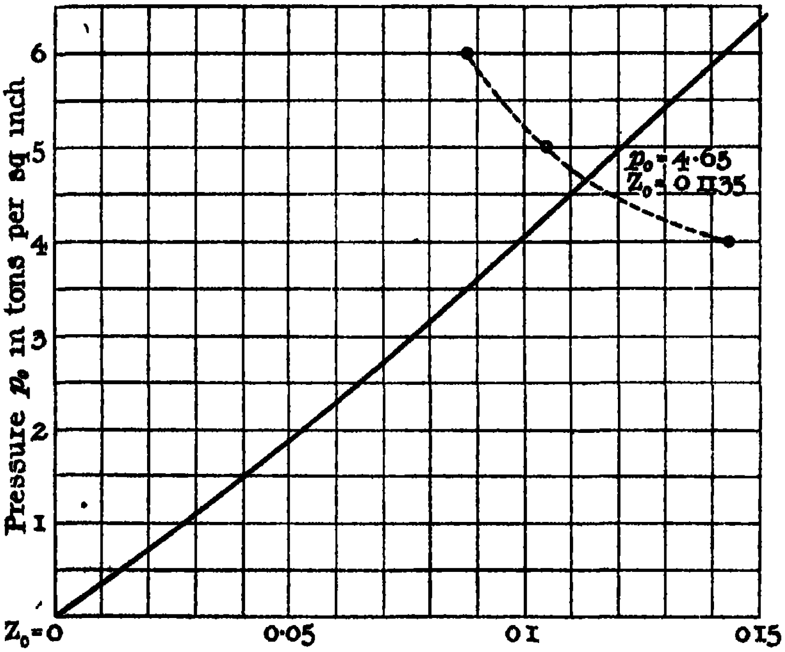

Fig 3

The position in the gun, where the combustion of the charge is completed, is found from equation (13)

$$\frac{1}{(v_1-\alpha)^{\epsilon-1}} = \frac{1}{(v_0-\alpha)^{\epsilon-1}} - (\epsilon-1)\,r\mathrm{X}^1_{z_0}.$$

From Table I by interpolation

$$\mathrm{X}^1_{z_0} = 2·269112,$$

$$(\epsilon-1)\,r\mathrm{X}^1_{z_0} = 0·143151,$$

$$\frac{1}{(v_0-\alpha)^{\epsilon-1}} = 0·95915,$$

$$\frac{1}{(v_1-\alpha)^{\epsilon-1}} = 0·815999,$$

$$(v_1-\alpha) = 7·6402 \qquad v_1 = 8·2732.$$

The pressure can be found from eq (8)

$$1+\kappa z_0^{\epsilon} = 1+0{\cdot}021097,$$

$$p_1 = 7\ 2 \text{ tons/inch}^2 \text{ mean pressure.}$$

The velocity of the shot at the completion of combustion is by eq (16)

$$u_1 = \frac{Ag2240}{WMB} f \left\{ V^1_{z_0} + \frac{1-n}{f} V^{z_0}_0 \right\},$$

$$u_1^2 = 728567500 \left(\frac{AD}{\frac{1}{5}\varpi + \frac{W}{f}} \right)^2 \left(V^1_{z_0} + \frac{1-n}{f} V^{z_0}_0 \right)^2,$$

$$n = \frac{2\ 54773}{4\ 65} \qquad \text{and} \qquad \frac{1-n}{f} = 0\ 47094,$$

$$\frac{1-n}{f} V^{z_0}_0 = 0\ 47094 \times 0\ 117068 \text{ by Table 1,}$$

$$= 0\ 055103,$$

$$V^1_{z_0} = 1{\cdot}882932,$$

$$V^1_{z_0} + \frac{1-n}{f} V^{z_0}_0 = 1\ 938035,$$

$$u_1^2 = 6787363, \qquad u_1 = 2605 \text{ f./s}$$

After the charge is burnt adiabatic expansion takes place according to the expression

$$\frac{2}{\gamma-1} \frac{Gfk32{\cdot}2 \times 2240}{WB} (1+\kappa z_0^{\epsilon}) \frac{1}{(v_1-\alpha)^{\epsilon-1}} \left\{ 1 - \left(\frac{v_1-\alpha}{v_2-\alpha} \right)^{\gamma-1} \right\}$$

$$= 109966350 \frac{\varpi}{\frac{1}{5}\varpi + \frac{W}{f}} (1+\kappa z_0^{\epsilon}) \frac{1}{(v_1-\alpha)^{\epsilon-1}} \left\{ 1 - \left(\frac{v_1-\alpha}{v_2-\alpha} \right)^{\gamma-1} \right\}$$

$$\left(\frac{v_1-\alpha}{v_2-\alpha} \right)^{\gamma-1} = 0\ 928974, \qquad 1 - \left(\frac{v_1-\alpha}{v_2-\alpha} \right)^{\gamma-1} = 0{\cdot}071026,$$

$$\text{const. } \frac{\varpi}{\frac{1}{5}\varpi + \frac{W}{f}} (1+\kappa z_0^{\epsilon}) \frac{1}{(v_1-\alpha)^{\epsilon-1}} \left\{ 1 - \left(\frac{v_1-\alpha}{v_2-\alpha} \right)^{\gamma-1} \right\} = 1695170,$$

$$u_1^2 = 6787363, \qquad U^2 = 8482533, \qquad U = 2913 \text{ f s.}$$

The mean muzzle pressure $\quad p_2 = \left(\frac{v_1-\alpha}{v_2-\alpha} \right)^{1\ 2} p_1 = 4{\cdot}63$ tons/inch^2

For maximum pressure, use eq. (19)

$$\frac{1}{r(v_0-\alpha)^{\epsilon-1}} = 1\ 520367 = Y^{\zeta}_{z_0}.$$

From Table I by interpolation, the fraction ζ of the charge burnt when the maximum pressure is developed is

$$\zeta = 0.69319, \qquad X_{z_0}^{\zeta} = 0.648272, \qquad (\epsilon-1)\,r X_{z_0}^{\zeta} = 0.040897,$$

$$\frac{1}{(v_0-\alpha)^{\epsilon-1}} = 0.959150, \qquad \frac{1}{(\chi-\alpha)^{\epsilon-1}} = 0.918253,$$

$$\chi-\alpha = 2.3463, \qquad \chi = 2.9793 \text{ volumes},$$

$$p_{\max} = \frac{\lambda}{(\chi-\alpha)^{\epsilon}}\left(\zeta^{\epsilon}+\kappa z_0^{\epsilon}\right) = 17.80 \text{ tons/inch}^2.$$

This is the mean pressure in the bore behind the projectile when it has advanced to a distance equivalent to χ volumes.

The pressure on the breech screw face is

$$P = \left(2-\frac{1}{B}\right)p_{\max},$$

$$2-\frac{1}{B} = 2-0.9479 = 1.0521,$$

and

$$P = 1.0521 \times 17.80$$

$$= 18.73 \text{ tons/inch}^2$$

(2) A charge of 30 lb 6 oz of modified tubular cordite gave, in a 6-inch 45-calibre gun, an actual muzzle velocity of 2797 ft./sec to a projectile weighing 100 lb The chamber pressure, as given by crusher gauges, was 15.78 tons per square inch.

The dry dimensions of the cordite tubes were external diameter = 0.463 inch and internal diameter = 0·167 inch.

The capacity of the chamber was 1715 cubic inches and the total capacity of the gun bore 8550 cubic inches The same driving band was used on the projectile as that in the previous example.

Here $A = 28.942 \text{ in}^2$ $\qquad D-D' = 0.463-0.167 = 0.296$ in

$W = 100$ lb $\qquad \varpi = 30.375$ lb.

We have

$$v_0' = \frac{1715}{30.375 \times 27.73} = 2.0361, \qquad v_0'-\alpha = 1.4031$$

$$v_0 = \frac{1715+49}{30.375 \times 27.73} = 2.0943, \qquad v_0-\alpha = 1.4613$$

$$v_2 = \frac{8550}{30.375 \times 27.73} = 10.1508, \qquad v_2-\alpha = 9.5178$$

The factor of effect will be taken as 0·98, the constants r_0 and r are then

$$r_0 = \frac{A^2(1-n)(D-D')^2\times 32{\cdot}2\times 2240}{2{\cdot}31\times\varpi\times 4(1{\cdot}44)^2 W\left(\frac{1}{5}\frac{\varpi}{W}f+1\right)80{\cdot}6},$$

$$= 46{\cdot}7059\,\frac{A^2(D-D')^2}{\varpi\left(\frac{1}{5}\varpi+\frac{W}{f}\right)}\,\frac{1-n}{f},$$

$$= 1{\cdot}06507(1-n),$$

and $$(\epsilon-1)r_0 = 0{\cdot}106507(1-n),$$

$$r = 57{\cdot}0378\,\frac{A^2(D-D')^2}{\varpi\left(\frac{1}{5}\varpi+\frac{W}{f}\right)}, \qquad = 1{\cdot}27467,$$

$$(\epsilon-1)r = 0{\cdot}127467$$

In order to find z_0 we have

$$\frac{1}{(v_0'-\alpha)^{\epsilon-1}} = 0{\cdot}96670, \qquad \frac{1}{(v_0-\alpha)^{\epsilon-1}} = 0{\cdot}96278,$$

and eq. (21) . $$(\epsilon-1)r_0X_0^{z_0} = \frac{1}{(v_0'-\alpha)^{\epsilon-1}}-\frac{1}{(v_0-\alpha)^{\epsilon-1}},$$

$$= 0{\cdot}96670-0{\cdot}96278,$$

$$= 0{\cdot}00392$$

Therefore $$X_0^{z_0} = \frac{0{\cdot}00392}{0{\cdot}106507}\,\frac{1}{1-n} = \frac{0{\cdot}036805}{1-n}$$

$$B = \frac{1}{5}\frac{\varpi}{W}f+1 = 1{\cdot}05954,$$

By plotting, as in the first example

$$p = 80{\cdot}6\left(\frac{z}{1{\cdot}4613}\right)^{\epsilon},$$

and z_0 assuming, in order to obtain the value of $X_0^{z_0}$, that $p = 4$ tons, 3·5 tons, and 3 tons per square inch, it will be found that

$$p_0 = 3{\cdot}64 \text{ tons/in}^2$$

and $$z_0 = 0{\cdot}0875$$

The position in the gun where combustion is completed is given by

$$\frac{1}{(v_1-\alpha)^{\epsilon-1}} = \frac{1}{(v_0-\alpha)^{\epsilon-1}}-(\epsilon-1)rX_{z_0}^{1}.$$

From Table II by interpolation

$$X_{z_0}^{1} = 0{\cdot}723306,$$

then $$(\epsilon-1)rX^1_{z_0} = 0{\cdot}092198,$$

$$\frac{1}{(v_0-\alpha)^{\epsilon-1}} = 0{\cdot}962780,$$

and $$\frac{1}{(v_1-\alpha)^{\epsilon-1}} = 0{\cdot}870582,$$

whence $$v_1-\alpha = 3{\cdot}9986$$

For the pressure when combustion is completed

$$1+\kappa z_0^{\epsilon} = 1+0{\cdot}015171$$

and $$p_1 = \frac{k}{(v_1-\alpha)^{\epsilon}}(1+\kappa z_0^{\epsilon})$$

$$= 14{\cdot}59 \text{ tons/in}^2$$

The velocity of the projectile at the completion of combustion is obtained from

$$u_1^2 = 627224000\,\frac{A^2(D-D')^2}{\left(\frac{1}{5}\varpi+\frac{W}{f}\right)^2}\left(V^1_{z_0}+\frac{1-n}{f}V_0^{z_0}\right)^2,$$

$$\frac{1-n}{f}V_0^{z_0} = \frac{0{\cdot}29675}{0{\cdot}98}\times 0{\cdot}0875 = 0{\cdot}026495,$$

$$V^1_{z_0} = 0{\cdot}912500$$

Therefore

$$V^1_{z_0}+\frac{1-n}{f}V_0^{z_0} = 0{\cdot}938955$$

so that $$u_1^2 = 3472240, \qquad u_1 = 1863 \text{ f/s}$$

After the charge is burnt the same formula as for Example 1 is used, viz.,

$$U^2 = u_1^2+109966350\,\frac{\varpi}{\frac{1}{5}\varpi+\frac{W}{f}}(1+\kappa z_0)\frac{1}{(v_1-\alpha)^{\epsilon-1}}\left\{1-\left(\frac{v_1-\alpha}{v_2-\alpha}\right)^{\gamma-1}\right\},$$

$$\frac{v_1-\alpha}{v_2-\alpha} = \frac{3{\cdot}9986}{9{\cdot}5178}, \qquad \left(\frac{v_1-\alpha}{v_2-\alpha}\right)^{\gamma-1} = 0{\cdot}840763, \qquad 1-\left(\frac{v_1-\alpha}{v_2-\alpha}\right)^{\gamma-1} = 0{\cdot}159237,$$

so that

$$U^2 = 3472240+4347900 = 7820140, \qquad U = 2796 \text{ f/s}$$

For maximum pressure

$$\frac{1}{r(v_0-\alpha)^{\epsilon-1}} = 0{\cdot}755317 = Y^{\zeta}_{z_0}$$

From Table II we find that the fraction ζ of the charge burnt when the maximum pressure is developed is

$$\zeta = 0{\cdot}77300$$

From Table II $$X^{\zeta}_{z_0} = 0{\cdot}519863,$$

and $$(\epsilon-1)\,r X^{\zeta}_{z_0} = 0{\cdot}066265,$$

and since $$\frac{1}{(v_0-\alpha)^{\epsilon-1}} = 0{\cdot}962780,$$

$$\frac{1}{(\chi-\alpha)^{\epsilon-1}} = 0{\cdot}896515, \qquad \chi-\alpha = 2{\cdot}98143,$$

$$p_{\text{max}} = \frac{k}{(\chi-\alpha)^{\epsilon}}\left(\zeta^{\epsilon}+\kappa z_0^{\epsilon}\right) = 15{\cdot}25 \text{ tons/in}^2$$

This is the mean pressure in the bore At the breech the maximum pressure is

$$P_{\text{max}} = \left(2-\frac{1}{B}\right)p_{\text{max}},$$

$$\frac{1}{B} = 0{\cdot}9439, \qquad 2-\frac{1}{B} = 1{\cdot}0561,$$

$$P = 1{\cdot}0561 \times 15{\cdot}25 = 16{\cdot}1 \text{ tons/in}^2$$

The results of the calculation of Example 1 are plotted in fig 4, the isopyric lines begin with that for $z_0 = 0{\cdot}1135$, the fraction of the charge burnt when the driving band is just engraved The trace of this curve follows the equation

$$p = 80{\cdot}6\left(\frac{0{\cdot}1135}{v'-\alpha}\right)^{1{\cdot}1}$$

The remainder of these curves for $z = 0{\cdot}2$, $0{\cdot}3$, etc, follow the equation

$$p = 66\frac{z^{1{\cdot}1}+\kappa z_0^{1{\cdot}1}}{(v-\alpha)^{1{\cdot}1}}$$

where $\kappa z_0^{1{\cdot}1} = 0{\cdot}021097$

The pressure curve begins at $v_0'-\alpha$ at 0, it rises nearly vertically until it reaches a value of 4·65 tons when it cuts the isopyric line $z_0 = 0{\cdot}1135$ As the pressure curve continues to rise to its maximum, and then to fall, it intersects the remainder of the isopyric lines, so that the diagram gives, for any mean gas pressure in the gun, the position $v-\alpha$ of the shot and the fraction z of the charge burnt. When the charge has just been completely burnt the pressure curve joins the isopyric line $z = 1{\cdot}0$. After this the expansion proceeds adiabatically until the shot leaves the gun This figure has been plotted for convenience with $v-\alpha$ equal to 1, 2, 3, etc

The velocity curve is shown in the dash-and-dot line, when the driving band is just engraved the projectile has a velocity of 77 ft/sec

The pressure curve can be more expeditiously drawn by using logarithmic ruled paper, as shown in fig 5 The isopyric curves then become a group of inclined parallel straight lines

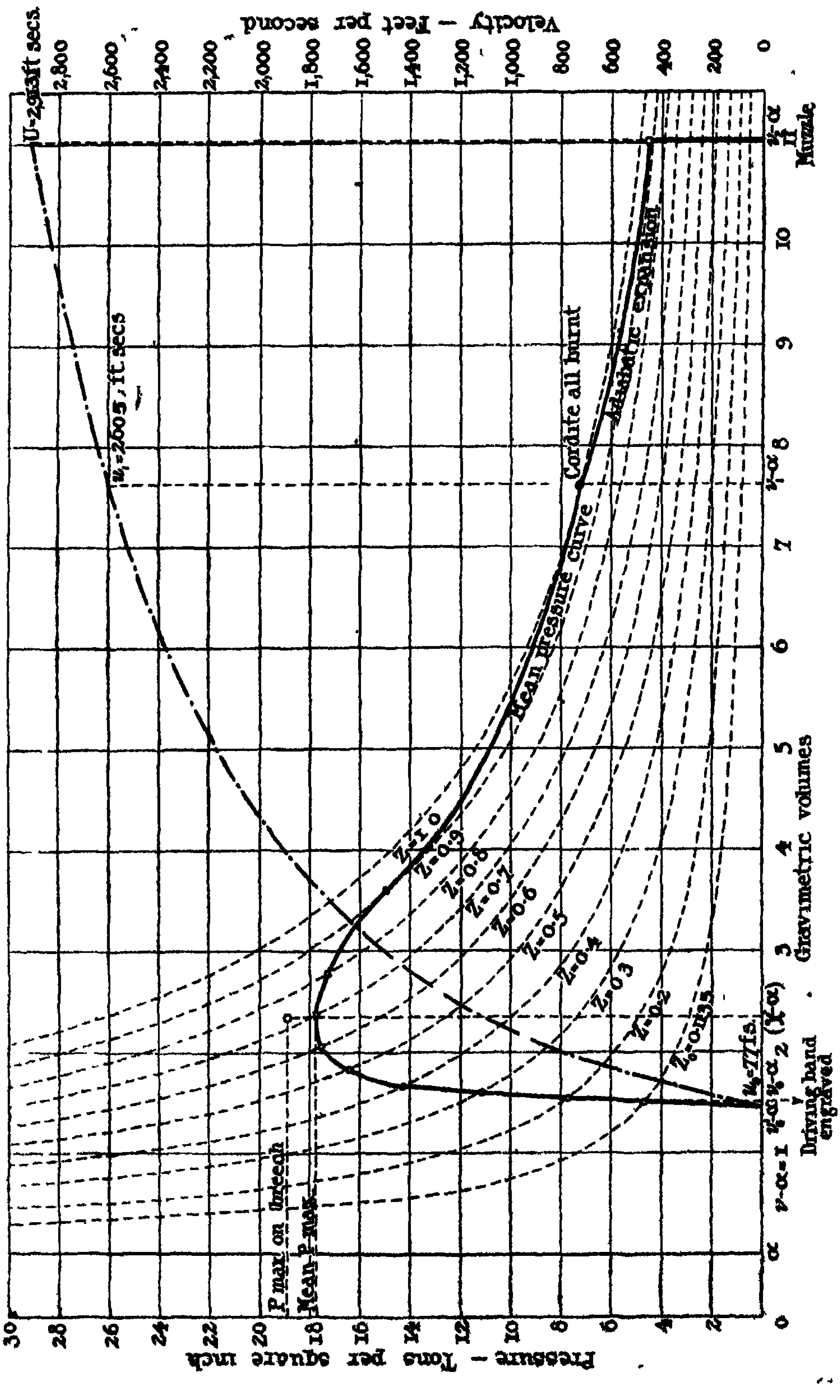
Velocity — Feet per second
2,800
2,600
2,400
2,200
2,000
1,800
1,600
1,400
1,200
1,000
800
600
400
200
0
Pressure — Tons per square inch
30
28
26
24
22
20
18
16
14
12
10
8
6
4
2
0
Gravimetric volumes
α
v-α=1
v_0-α
v_0-α 2
(v_1-α)
3
4
5
6
7
v_1-α
8
9
10
v_2-α
11
Muzzle
Driving band engraved
U=2911 ft secs
u_1=2605 ft secs
u_0=77 f.s.
Cordite all burnt
Adiabatic expansion
Mean pressure curve
Z=1.0
Z=0.9
Z=0.8
Z=0.7
Z=0.6
Z=0.5
Z=0.4
Z=0.3
Z=0.2
Z_0=0.135
P max on breech
Mean P max

Fig 4.

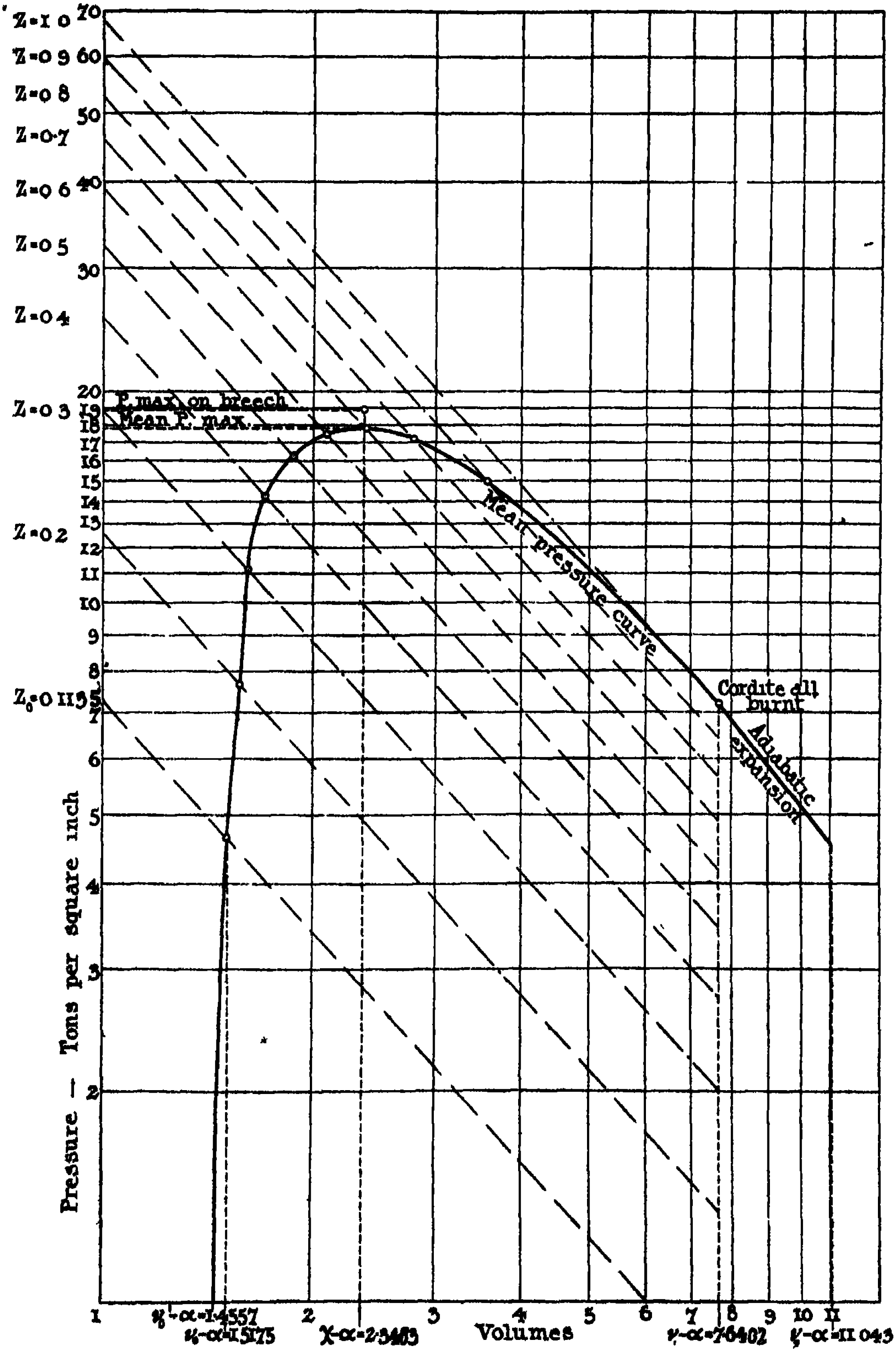
Z = 1·0
Z = 0·9
Z = 0·8
Z = 0·7
Z = 0·6
Z = 0·5
Z = 0·4
Z = 0·3
Z = 0·2
Z_0 = 0·115
70
60
50
40
30
20
19
18
17
16
15
14
13
12
11
10
9
8
7·5
7
6
5
4
3
2
P. max. on breech
Mean P. max.
Mean pressure curve
Cordite all burnt
Adiabatic expansion
Pressure — Tons per square inch
1
$x_0' - \alpha = 1·4557$
$x_0 - \alpha = 1·5175$
2
$X - \alpha = 2·3463$
3
4
Volumes
5
6
7
$x_1 - \alpha = 7·6402$
8
9
10
11
$\xi - \alpha = 11·043$

Fig 5

The maximum pressure found above of 18 73 tons per square inch on the breech screw is nearly a ton too high, as found by the crusher gauges, but

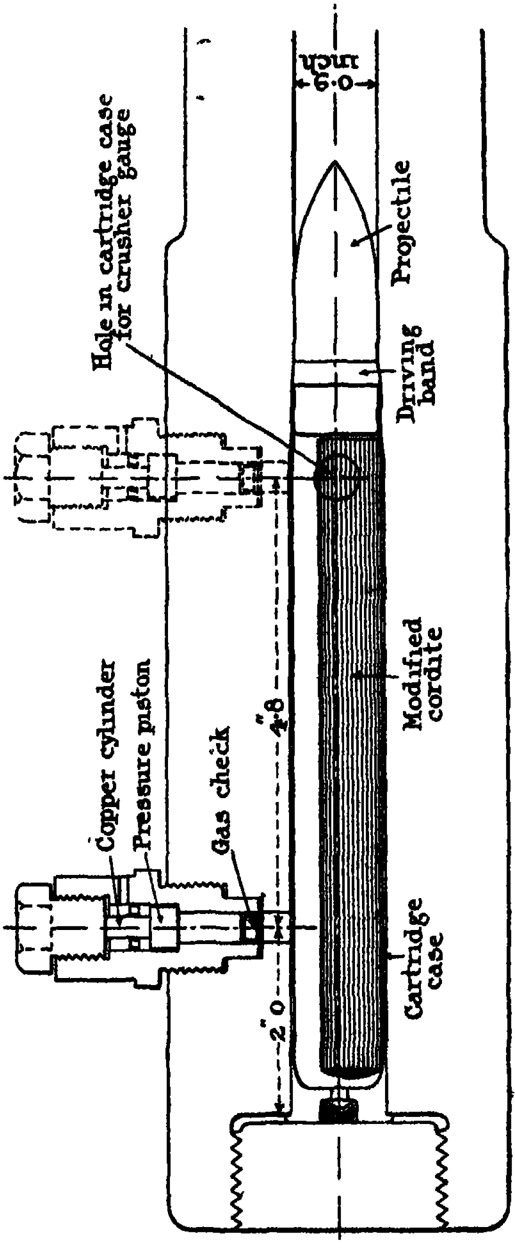

Fig 6.

usually the crusher gauges, placed loose in the gun at the breech, are carried forward with the rush of gas, so that the pressure registered by them may be that due to some intermediate position between the breech and the shot.

This is borne out by an experiment lately carried out with a small bore gun having a long chamber. Fixed crusher gauges were inserted in the chamber (*a*) at 2 inches from the breech end, and (*b*) at 4·8 inches from the first-named gauge, as shown in fig 6

The mean pressure of three rounds (charge 1 oz. 11 drams modified cordite, diameter = 0·039 inch) was at (*a*) 16·45 tons per square inch, at (*b*) 15·15 tons With a charge of 1 oz 12 drams modified cordite, diameter = 0·0298 inch, the mean of three rounds was at (*a*) 18·8 tons, at (*b*) 17·3 tons per square inch The projectile was fitted with a copper driving band and weighed in all the experiments 6 oz.

The maximum pressure in this experimental gun will take place at about double the distance (*b*) is from (*a*), so that applying the above rule for the difference of pressures, those actually obtained are closely approximated to

Hefferman* finds from experience formulæ equivalent to

$$\frac{\chi-\alpha}{v_0'-\alpha} = 1·5 \text{ for cylindrical cordite,}$$

$$= 2·0 \text{ for tubes}$$

While Lissak† finds with powder with a decreasing surface like cylindrical cordite, a value of 1·45

The values found in the above examples are

$$\text{cylindrical cordite} \qquad \frac{\chi-\alpha}{v_0'-\alpha} = \frac{2·3463}{1·4557} = 1·61,$$

$$\text{for tubular} \qquad = \frac{2·98143}{1·4031} = 2·12$$

They will, however, vary in every case

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* See 'Internal Ballistics,' 1907

† 'Ordnance and Gunnery,' New York, 1907.

The Absorption of X-Rays.

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(Communicated by Prof. A. W Porter, F R.S. Received March 13, 1918)

Introduction —Previous to the discovery of the behaviour of X-rays with regard to crystals, the most homogeneous radiation obtainable was that of the characteristic radiation of an element which is excited when that element is exposed to X-radiation of suitable hardness. These characteristic radiations are now found, however, by the new method of analysis, to be constituted of a number of radiations of different wave-lengths Moseley,* shortly after the discovery of the reflection of X-rays, showed that the characteristic radiations of most of the metals he examined consisted of two prominent wave-lengths, Bragg† later found that, in the case of rhodium, palladium and silver, each of these lines could be further resolved into two components Hence the spectra of the characteristic radiation of the K series of these elements consist of at least four different wave-lengths. The analysis of a beam of X-rays into its constituent radiations by reflection at a crystal face provides a means, therefore, of obtaining radiation of a definite wave-length and of such intensity as to enable its absorption coefficient in different materials to be accurately measured.

Bragg and Pierce‡ have already measured the absorption coefficients of the two most prominent lines in the spectra of the elements Rh, Pd and Ag, in a number of metals. To make the absorption coefficient more directly comparable with other atomic characteristics, they gave their results in the form of atomic absorption coefficients · the atomic absorption coefficient expresses the proportion of the energy of an X-ray pencil which is absorbed in crossing a surface on which lies one atom to every square centimetre. The ordinary mass absorption coefficient can be calculated from this quantity by dividing it by the mass of the absorbing atom. The experimental results showed that the ratio of two absorption coefficients is independent of the wave-length of the radiation over considerable ranges, a result previously deduced by Barkla from his experiments, also, that the atomic absorption coefficient is proportional to the fourth power of the atomic number of the absorber.

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* Moseley, 'Phil. Mag,' vol. 26, p. 1024 (1913)

† W. H Bragg, 'Phil. Mag.,' vol. 29, p. 407 (1915)

‡ Bragg and Pierce, 'Phil Mag.,' vol. 28, p 626 (1914)

Combining this last result with the existing relation connecting the absorption coefficient with the wave-length of the radiation, namely—

$$\text{Mass absorption coefficient} \propto \lambda^{5/2},$$

Bragg and Pierce arrived at the following very general law

$$\text{Atomic absorption coefficient} = \text{C}\text{N}^4\lambda^{5/2},$$

where N is the atomic number of the absorber, and λ the wave-length of the radiation. The quantity C is constant over considerable ranges but changes abruptly when we get into the region of selective absorption.

Some doubt has been cast on the validity of the above relation connecting atomic number and atomic absorption coefficient by the results of Auren,[*] who has recently made an extensive investigation of the absorption of a great number of solutions On the assumption that the molecular absorption coefficient can be obtained by adding together the atomic absorption coefficients of the atoms composing the molecule, it has been possible, from the observed values of the molecular absorption coefficients of the different substances used, to calculate the relative atomic absorption coefficients of the elements from hydrogen upwards In the experiment, rays direct from a bulb with tungsten anticathode were employed which had been passed through an aluminium screen 1 25 mm thick to filter away rays of the longer wave-lengths in the beam. A measurement of the wave-length of the radiation transmitted gave a mean value of $0\,35 \times 10^{-8}$ cm

To determine the absorption coefficient of a certain molecule, solutions of different concentrations containing that molecule were taken, from which a fairly accurate mean value of the molecular absorption coefficient could be obtained. From these observed results the relative values of the atomic absorption coefficient of the elements were calculated by assuming the additive law just mentioned The final results led to the conclusion that the atomic absorption coefficient is not proportional to the fourth power of the atomic number, but that the elements divide up into groups, for each of which the atomic absorption coefficient is directly proportional to the atomic number.

The object of this paper is to endeavour to account for these two discordant results. A general law is arrived at which differs slightly from that stated by Bragg and Pierce, by the aid of which the molecular absorption coefficients of all the substances used by Auren have been calculated for the wave-length employed by him and the values thus obtained are all found to agree, within the limits of experimental error, with those actually observed in Auren's experiment.

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[*] Auren, 'Phil. Mag.,' vol. 33, p 471 (1917).

Experimental—In the experimental part of the present investigation, particular attention is paid to the absorption of the α-line of palladium in different substances This line was chosen because its mean wave-length is known very accurately, and its intensity, when reflected from the (111) face of a crystal of carborundum, is so great as to enable measurements of its absorption in different substances to be made with accuracy. Also, the intensity of the superposed second and third order spectra of the shortest wave-lengths in the spectrum of the radiation emitted by the bulb was so small compared with the intensity of the α-line itself that it had no appreciable effect upon the purity of the α-radiation *

The carborundum crystal was placed on the revolving table of the X-ray spectrometer, and fixed in the position in which it reflected the α-line of palladium into the ionisation chamber. By the arrangement of slits adopted in the experiment, the reflected beam was confined to a very narrow pencil, so that the width of the chamber slit could be made very narrow without appreciably cutting down the intensity of the beam that entered the ionisation chamber Its width was fixed at 0 4 mm. The employment of a slit so narrow as this adds to the accuracy of the determination of the absorption coefficient. In the measurement of absorption coefficients, it is important to prevent secondary radiation, either scattered or fluorescent produced by the primary radiation, from entering the ionisation chamber. When the rays traverse the absorbing screen, it is required to find the intensity of the radiation which proceeds as primary radiation after passing through the screen. By using a narrow slit, the experimental conditions necessary for this are approximately obtained These conditions may be further improved by removing the screen a little distance away from the chamber slit, in which case only the secondary radiation contained in the small solid angle subtended at the slit by the illuminated portion of the absorbing screen enters the chamber, and in this particular case this would be negligible. Experiment showed that there was no difference in the value of the absorption coefficient obtained when the absorbing screen was moved from a position about 2 cm in front of the chamber slit to a position near the crystal In the present investigation, the absorbing screens were all placed about 2 cm in front of the chamber slit when their absorption coefficients were being measured.

The substances whose absorption coefficients are measured range from zinc down to light substances containing carbon and oxygen, such as filter paper and water, it is the values of the absorption coefficients of the lightest substances that suggest a possible explanation of the disagreement for which we are seeking to account

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* See Owen, 'Roy Soc. Proc.,' A, vol. 94, p. 347 (1918).

It will be assumed that water and filter paper behave like elements whose atomic numbers are 8 and 7 respectively In water oxygen absorbs most of the radiation, the absorption due to hydrogen being almost negligible in comparison with it, in filter paper, whose constitution may be represented by the formula $n(C_6H_{10}O_5)$, the mass of the carbon atoms is not much smaller than that of the oxygen atoms, so that the substance may be taken to behave towards X-rays as an element whose atomic number is the mean of those of carbon and oxygen, *e g*, of atomic number 7 These approximate assumptions are made here to enable us to plot the absorption coefficients of these substances on the same graph as those for the elements employed, in order to get a rough idea of the shape of the absorption curve in this region Later, the actual value of the molecular absorption coefficient of each substance will be calculated, in which case account will be taken of the absorption of the hydrogen atom whose effect is being ignored in making the above assumptions

The values of the absorption coefficients are obtained from the usual relation $I = I_0e^{-\mu x}$, where I_0 is the intensity of the primary radiation before, and I its intensity after traversing a screen of thickness x cm

The final mean values experimentally obtained are tabulated in Table I, they are the mean of a large number of observations The results previously obtained by Bragg and Pierce with the α-line of palladium are also included in the Table for comparison, and it will be observed that the values obtained now agree fairly closely with their values for the elements used in common In view of this corroboration of Bragg and Pierce's results, it was not considered necessary to extend the investigation to other wave-lengths, as the information obtained with the α-palladium radiation sufficed for the immediate purpose in view.

Table I

Absorber	Atomic number	μ/ρ			
		Owen.	Bragg and Pierce	Hull and Rice	Mean values
Zn	30	34 7	34 0	—	34 3
Cu	29	29 5	30 9	30 ·28	30 2
Ni	28	26 0	28 9	—	27 4
Fe	26	—	23 1	—	23 1
Al	13	3 11	3 16	3 12	3 13
Mg	12	2 43	—	—	2 43
Water	[8]	0 844	—	—	0 844
Filter paper	[7]	0 664	—	—	0 664

Column 5 of Table I contains the values of μ/ρ for copper and aluminium calculated from the following formulæ arrived at by Hull and Rice* in their experiments on the absorption of the end radiations of a Coolidge tube.—

$$(\mu/\rho)\text{Al} = 14.9\,\lambda^3 + 0.12.$$

$$(\mu/\rho)\text{Cu} = 1.50\,\lambda^3 + 0.12.$$

where λ is the wave-length of the radiation (0.586×10^{-8} cm. for the α-palladium line). These values of μ/ρ are also in agreement with those in the other columns.

The means of all the values are given in column 6 of the Table. The figures for the elements zinc to magnesium are more accurate than those for paper and water. The absorption in the case of these two substances is small, and in consequence the coefficient is not so easy to measure with accuracy.

In the diagram the logarithms of the mean value of μ/ρ are plotted as ordinates against the logarithms of the atomic numbers as abscissæ. A curve through the points is very nearly a straight line between the elements zinc and magnesium, but, beyond this point, as we proceed to the region of the lightest elements, the curve departs more and more from its straight-line course. If the absorption coefficients as determined by Auren be plotted in a similar manner, the curve obtained over the range of atomic numbers here employed shows exactly the same characteristics, but the divergence in the case of the light elements from the approximate linear relation existing between log μ/ρ and log N for elements between magnesium and zinc is rather more pronounced than in the present instance.

Before considering the effect which the scattering of the rays might have in determining the shape of the absorption curve as plotted here, it would be advisable to summarise briefly the laws of scattering as experimentally determined by Barkla.

The Scattering of X-rays.—When a beam of X-rays passes through matter it may lose its energy in two ways, namely, (1) by the production of corpuscular radiations and the characteristic radiations that accompany them, and (2) by the production of scattered radiation. In the former case, the quality of the radiation is changed, but in the latter case the quality remains unaltered. As a general rule these two processes take place simultaneously, but they are quite independent of each other.

Barkla has carried out extensive experiments on the scattering of X-rays, from which he has deduced many important laws governing the process of scattering. It is found that the scattering is dependent upon two main factors, (1) the atomic weight of the scatterer, and (2) the wave-length of the radiation.

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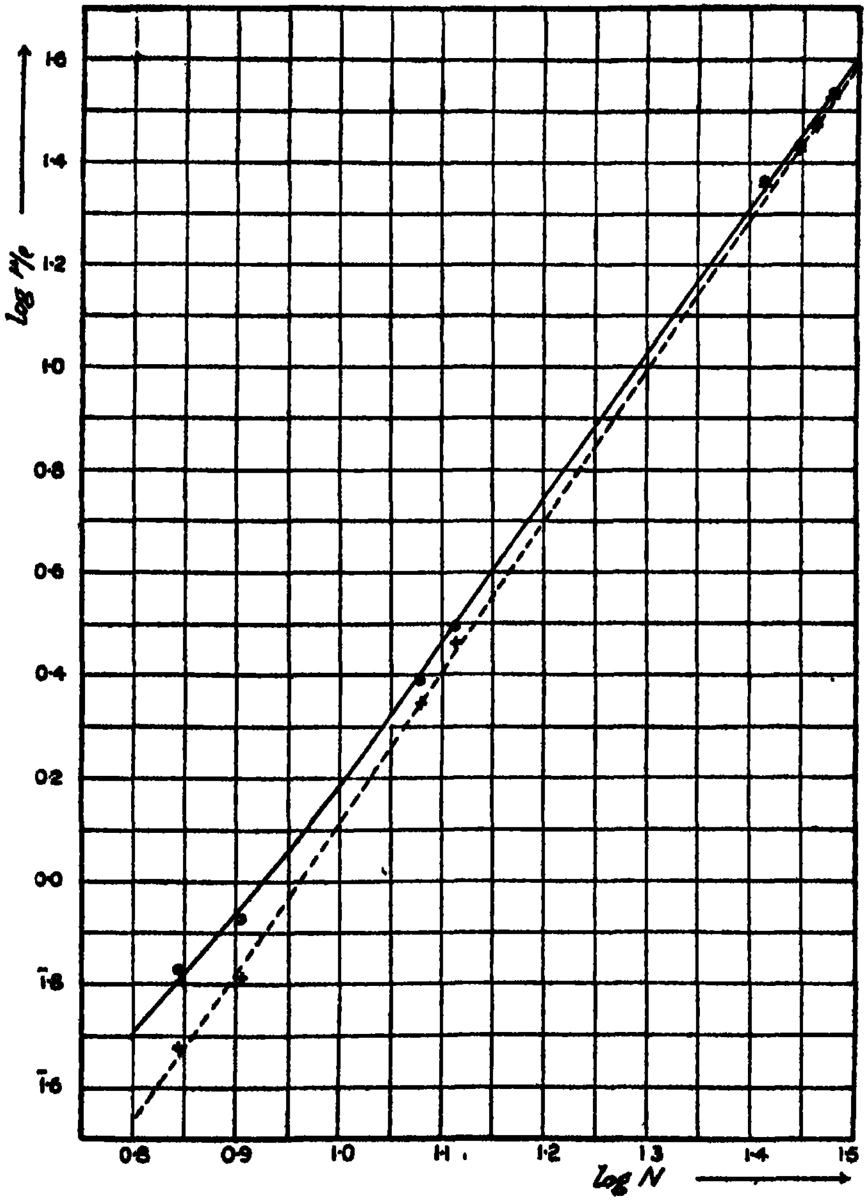
* Hull and Rice, 'Phys. Rev.,' vol 8, p. 326 (1916).



Elements of low atomic weight scatter the same amount approximately mass for mass for radiations over wide limits of wave-length. There seems, however, to be a slight increase in the scattering with an increase in the wave-length of the radiation, this increase becomes more marked when the wave-length becomes so great as to be comparable with the size of the atom

In the case of the heavier elements when the wave-length of the radiation is long, the scattering increases many-fold, but with short waves the mass scattering even in the heavier elements approaches equality with that in the light elements

For the light elements when the wave-length of the radiation is short, the chief cause of loss of energy of the X-radiation may be put down to scattering; in general, it may be stated that the lower the atomic weight of the scatterer the greater is the proportion of loss of energy by scattering to the total loss of energy.

The scattering coefficient experimentally determined by Barkla in the case of light elements was found to have the value 0 2, and, as stated above, was found to be constant over wide limits of wave-length This value 0 2 agrees almost perfectly with that obtained by calculation on the assumption that the number of electrons per atom is equal to the atomic number

According to the simple electro-magnetic theory, the scattering coefficient should be a universal constant, independent entirely of the wave-length of the radiation scattered Barkla's experiments show, however, that the coefficient is constant only for very short wave-lengths Hull and Rice confirmed this result in their work on the absorption of X-radiations of very short wave-lengths in aluminium, copper, and lead, but the value of the scattering coefficient deduced by them from their experiments was somewhat lower than the above value found by Barkla. The wave-lengths employed ranged from $0\,147 \times 10^{-8}$ cm to $0{\cdot}392 \times 10^{-8}$ cm The greater of the two wave-lengths dealt with in the present work, namely, $0\,586 \times 10^{-8}$ cm., is a little outside this range, but it will be assumed that the scattering coefficient for this wave also is constant for all the elements used, and that its value is 0 2, the value found by Barkla for the light elements.

Let us now find what effect the correction for scattering produces on the shape of the absorption curve. The absorption coefficients plotted in the diagram are the total absorption coefficients, which include both the scattering and the fluorescent coefficients If τ/ρ is the fluorescent absorption coefficient and σ/ρ the scattering absorption coefficient, then the total absorption coefficient is given by the relation $\mu/\rho = \tau/\rho + \sigma/\rho$ In the present case, σ/ρ is assumed to be constant and equal to 0 2. Plotting the logarithms of the values of τ/ρ obtained from this relation on this assumption, against the logarithms of the atomic numbers of the absorbing atoms, we find that the points all lie very approximately along a straight line. The line is dotted in the diagram above. This line gives the index 3 approximately, so that the relation between τ/ρ and N may be approximately expressed by the equation $\tau/\rho = C'N^3$, where C′ is a constant. If, however, the atomic fluorescent absorption coefficient is calculated in each case, it is found that the results are more accurately expressed by the equation

$$\text{Atomic fluorescent absorption coefficient} = CN^4$$

This result is slightly different from that of Bragg and Pierce, the difference between the two relations is not so marked in the case of the heavier atoms, because the scattering coefficient is small compared with the fluorescent coefficient, but it becomes very appreciable for elements of low atomic weight, in which case the absorption by scattering becomes the prominent part of the total absorption

To test the validity of this law, it will be used to calculate the molecular absorption coefficients which have been experimentally determined by Auren To do this, we have to consider in detail the transformation of absorption coefficients obtained with one wave-length to those which should be obtained with a radiation of another wave-length This transformation can be carried out without much difficulty by the use of the relation connecting absorption coefficient with wave-length

The Relation between Absorption Coefficient and Wave-length of the Radiation Absorbed—It was pointed out by the writer* a few years ago that the connection between the absorption coefficient and the atomic weight of the radiator whose characteristic radiation was employed, could be approximately expressed by a very simple relation, the absorption coefficient was found to be approximately inversely proportional to the fifth power of the atomic weight of the radiator. This relation was arrived at before definite wave-lengths were assigned to X-rays Since the wave-length of the radiation characteristic of a certain element has been found to be approximately inversely proportional to the square of the atomic weight of that element, the above relation expressed in terms of wave-length takes the form $\mu/\rho \propto \lambda^{5/2}$, which accounts for the factor $\lambda^{5/2}$ appearing in the general law written down by Bragg and Pierce Further experiment† has shown that the relation $\tau/\rho = a\lambda^3$ is a more accurate representation of the observed results (where τ/ρ is the fluorescent absorption coefficient). This law is found to hold for all wave-lengths except those in the immediate vicinity of an absorption band, when the value of the constant a changes suddenly. If the fluorescent absorption coefficient of a certain substance is known for one wave-length, by means of this relation it can be calculated for any other wave-length so long as that substance does not selectively absorb any radiation in the region between the two wave-lengths considered. Also, since it is assumed here that the scattering coefficient is constant for all substances in the case of the two wave-lengths with which we are concerned, the total absorption coefficient is also determined.

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* Owen, 'Roy. Soc. Proc.,' A, vol. 86, p. 426 (1912).

† Hull and Rice, *loc cit.*, Barkla and White, 'Phil. Mag,' vol 34, p 270 (1917); Owen, 'Roy. Soc. Proc.,' A, vol. 94, p 339 (1918)

Calculation of Results.—Table II contains the values of τ/ρ for the most common elements from hydrogen to bromine for the α-line of palladium ($\lambda = 0·586 \times 10^{-8}$ cm.). The values have been taken direct from a straight line corresponding to the dotted line in fig. 1, which had been drawn on a large scale. The mass of the atoms and the calculated values of the atomic total absorption coefficients are also included in the Table.

Table II.

$\lambda = 0·586 \times 10^{-8}$

Element	Atomic number	Fluorescent absorption coefficient (τ/ρ)	Total absorption coefficient (μ/ρ)	Mass of absorbing atom (w).	Atomic total absorption coefficient ($\mu/\rho \cdot w$)
H	1	0·0015	0·2015	$1·64 \times 10^{-24}$	$0·331 \times 10^{-24}$
C	6	0·301	0·501	19·67	9·85
N	7	0·468	0·668	23·00	15·34
O	8	0·687	0·887	26·25	23·28
F	9	0·966	1·166	31·15	36·28
Na	11	1·726	1·926	37·74	72·6
Mg	12	2·239	2·439	39·84	96·9
Al	13	2·805	3·005	44·45	133·7
Si	14	3·516	3·716	46·4	172·1
P	15	4·295	4·495	50·9	228·6
S	16	5·212	5·412	52·6	284·5
Cl	17	6·194	6·394	58·1	371·6
K	19	8·551	8·751	64·1	561·0
Ca	20	9·977	10·18	65·8	669·0
Cr	24	16·98	17·18	85·3	1463
Mn	25	19·05	19·25	90·0	1732
Fe	26	21·58	21·78	91·6	1992
Co	27	24·10	24·30	96·7	2346
Ni	28	26·79	26·99	96·3	2600
Cu	29	29·72	29·92	104·1	3112
Zn	30	32·81	33·01	107·1	3533
As	33	43·05	43·25	122·9	5315
Se	34	46·88	47·08	129·9	6115
Br	35	51·40	51·60	131·0	6755

In order to calculate the molecular absorption coefficient of a complex molecule from the atomic absorption coefficients of the atoms composing that molecule, we shall assume, as Auren did, that the total absorption coefficient of the molecule may be obtained by adding together the total absorption coefficients of its atoms.

Let $(\mu/\rho)_1$, $(\tau/\rho)_1$, and $(\sigma/\rho)_1$ be the total absorption coefficient, the fluorescent absorption coefficient, and the scattering coefficient respectively for the wave-length $0·586 \times 10^{-8}$ cm., for any element, and let the same notation with the suffix 2 denote the same quantities for the wave-length $0·35 \times 10^{-8}$ cm., which is the mean wave-length of the radiation used by Auren in his investigation

Since $$(\tau/\rho)_1 \lambda_2^3 = (\tau/\rho)_2 \lambda_1^3,$$

then $$(\tau/\rho)_2 = 0·213\,(\tau/\rho)_1,$$

and since $$(\sigma/\rho)_1 = (\sigma/\rho)_2 = 0·2,$$

then $$(\mu/\rho)_2 = 0·213\,(\mu/\rho)_1 + 0·1574,$$

which is the relation connecting the total absorption coefficients of any element for the wave-lengths $0·586 \times 10^{-8}$ cm. and $0·35 \times 10^{-8}$ cm. Let t_a' and t_a'' denote the atomic total absorption coefficients for these two wave-lengths respectively.

Then $$t_a'' = 0·213\, t_a' + 0·1574\, w,$$

where w is the mass of the atom

If t_m' and t_m'' denote the molecular total absorption coefficients, then, by the additive law that we are assuming, it follows that

$$t_m'' = 0·213\, t_m' + 0·1574\, m \qquad (1)$$

where m is the mass of the molecule

Knowing the molecular total absorption coefficient (t_m') of any substance for the wave-length $0·586 \times 10^{-8}$ cm., we can calculate by the aid of equation (1) the molecular total absorption coefficient for the wave-length $0·35 \times 10^{-8}$ cm.

Consider, for example, the molecule NaCl

From Table II we find that the atomic total absorption coefficients of the atoms Na and Cl for the wave-length $0·586 \times 10^{-8}$ cm are $72·6 \times 10^{-24}$ and $371·6 \times 10^{-24}$ respectively.

The molecular total absorption coefficient (t_m') is, therefore, $444·2 \times 10^{-24}$

The mass of the molecule (m) = $58·46 \times 1·64 \times 10^{-24}$

Substituting these values in (1) we get

$$t_m'' = 109·59 \times 10^{-24},$$

which is the figure that appears in Table III

The molecular total absorption coefficients of all the substances investigated by Auren (except those containing atoms heavier than bromine, which were omitted because of their proximity to a critical region of the α-palladium line) were calculated in this manner A list of the calculated values is contained in Table III. Column 5 of this Table contains the ratio of the calculated values of the molecular total absorption coefficient of any substance to that of water These figures are directly comparable with the experimental values obtained by Auren, which are included for comparison in Column 6 of the Table

Table III

Absorbing substance	Formula weight	Molecular total absorption coefficient $\lambda = 0\ 586 \times 10^{-8}$ cm	Calculated molecular total absorption coefficient $\lambda = 0\ 35 \times 10^{-8}$ cm	Molecular total absorption coefficient relative to that of water	
		$t_m' \times 10^{24}$	$t_m'' \times 10^{24}$	Calculated.	Observed (Auren)
H_2O	18·01	23 94	9 749	1 0	1·0
$NaClO_3$	106·46	514 2	136·9	14 02	14 2
$NaCl$	58 46	444 2	109 6	11 24	11 5
$NaOH$	40·01	96 21	30 82	3·09	2 77
C_6H_6	78 05	61 09	33 16	3 40	2 97
CH_4O	32·03	34 45	15 60	1 60	1 56
$C_4H_{10}O$	74 08	65·99	33 18	3 40	3 32
C_3H_6O	58·05	54 82	26·65	2 73	2 58
C_3H_8O	60 06	55 48	27 32	2 80	2 69
$(C_2H_5)_4Si$	144 46	257 52	92 18	9 44	8 56
$(C_6H_5)Si(CH_3)_3$	150 43	265 38	95 39	9 78	9 06
HNO_3	63 02	85 51	34 49	3 54	3 86
NH_4NO_3	80 05	101 84	42 36	4 34	4 17
$C_6H_5NO_2$	123 05	122 66	57 89	5 94	5 40
C_5H_5N	79 05	66 24	34 52	3 54	3 14
$C_6H_5NH_2$	93·09	76·76	40 37	4 14	3 63
$MgCl_2$	95 24	840 1	203 54	20 87	22 5
$Al_2(SO_4)_3$	342 41	1400 3	386 64	39 66	41 0
H_3PO_4	98 06	322·71	94 05	9·65	9 43
H_2SO_4	98 09	378 29	105 90	10 86	11 48
HCl	36 47	371 93	88 61	9·09	9 55
KCl	74 56	932 6	217 89	22 34	20 7
$CaCl_2$	111 01	1412 2	329 45	33 8	33 4
$CrCl_3$	158 38	2577 8	589·95	60 5	52·6
$MnCl_2$	125 85	2475 2	559·70	57 4	60 4
$KFe(SO_4)_2$	287 09	3308 2	778 76	79·9	79 4
$NiCl_2$	129·60	3343 2	745·56	76 4	78 4
$CoCl_2$	129 89	3069 2	691 58	70 9	72 9
$CuCl_2$	134 49	3855·2	855 87	87 8	84·1
$ZnCl_2$	136 29	4281 2	947 20	97 1	96 3
$ZnSO_4$	161·44	3915·6	875 40	89 8	84 8
Na_2HAsO_4	185 97	5553 7	1230 0	126 1	136·0
KBr	119 02	7316 0	1587 7	162 8	165 0
$FeSO_4$	151 92	2369 6	544 20	55 8	—

It will be noticed that for all the substances investigated the agreement between the experimental values and the values of the molecular absorption coefficients calculated along the lines indicated here is remarkably good, in view of the possible error in the determination of the mean value of the wave-length of the direct radiation used by Auren, and also the uncertainty attaching to the value of the scattering coefficient, which has been assumed to have the value 0·2 over the whole range of elements from hydrogen to bromine for the wave-lengths concerned.

From the examples cited by Auren in his paper, it would appear that the difference between calculated and observed values is well within the range of variation of the experimental values of the molecular absorption coefficient of a given molecule as determined from solutions of different concentrations. For instance, the observed values of the relative molecular absorption coefficient for the molecule KCl range from 18 2 to 23 6, the calculated value is 22 3

The salt $FeSO_4$ is not included in the list of substances contained in Auren's Table, it is excluded because a general rise was found in the values of the relative molecular absorption coefficient for this salt as the concentration of the solution was decreased This variation is accounted for by the fact that the salt, being oxidised by contact with air and hydrolysed by dilution, passes into a colloidal state The values of the relative molecular absorption coefficient for this molecule, given by Auren in a separate Table, vary from 53 0 for a solution of the concentration 1 08 gramme-molecules per litre to 58 4 for that of concentration 0 133 gramme-molecule per litre The value obtained for this molecule from the present results by calculation is 55 8, and is almost exactly the mean of all the observed values given

The validity of the relation that has been arrived at may be further tested in the case of paraffin wax, which is a substance containing a comparatively high percentage of hydrogen by weight The chemical formula of this substance may be written as $C_{22}H_{46}$ or $C_{32}H_{66}$. Either formula gives by calculation the value 0 456 for the total absorption coefficient of this substance for the wave-length 0 586 × 10^{-8} cm The value obtained by experiment is 0 425 The agreement between these two figures is good in view of the fact that it is difficult to measure the absorption coefficient of this substance with great accuracy

Table IV gives the observed and calculated results for the three substances water, filter paper, and paraffin wax, for the wave-length 0 586 × 10^{-8} cm.

Table IV

Substance.	Formula weight	μ/ρ for $\lambda = 0\ 586 \times 10^{-8}$ cm	
		Observed.	Calculated
Water	H_2O	0·84	0 81
Paper	$x\,(C_6H_{10}O_5)$	0·66	0 67
Paraffin wax . .	$C_{22}H_{46}$ or $C_{32}H_{66}$	0 48	0 46

Conclusions.—The experimental values of the mass absorption coefficients of elements ranging from aluminium to zinc for the wave-length

$0{\cdot}586\times10^{-8}$ cm observed by Bragg and Pierce have been confirmed, and it is shown that the values of the relative molecular total absorption coefficients of different substances experimentally determined by Auren for a radiation of wave-length $0{\cdot}35\times10^{-8}$ cm., may be calculated from these values by making a suitable allowance for the scattering of the radiation

The results indicate that the atomic fluorescent absorption coefficient is directly proportional to the fourth power of the atomic number of the absorbing atom, which may be represented by the following equation :—

$$f_a = C_1N^4, \tag{2}$$

where f_a is the atomic fluorescent absorption coefficient, N the atomic number, and C_1 a constant over ranges outside critical regions

It is known also that the fluorescent absorption coefficient varies as the cube of the wave-length of the radiation absorbed, so that

$$\tau/\rho = C_2\lambda^3 \tag{3}$$

Combining these two relations we get the following general relation connecting atomic fluorescent absorption coefficient, atomic number of the absorber, and wave-length of absorbed radiation :—

$$f_a = CN^4\lambda^3, \tag{4}$$

where C is a constant over certain ranges but changes abruptly at critical points. This relation is independent of the scattering coefficient and refers only to the loss of energy of X-radiation by the production of corpuscular radiations and the fluorescent X-radiations that accompany them

The calculations carried out in this paper are based on equation (4) To arrive at the molecular total absorption coefficient of any molecule, it has been necessary to make the two following assumptions :—

(1) The molecular total absorption coefficient of a complex molecule is additively determined from the atomic total absorption coefficients of its constituent atoms

(2) The mass scattering coefficient of all the elements from hydrogen to bromine has a constant value of 0 2.

The first assumption implies that the absorption of radiation by an atom is the same whether or not that atom is in combination with other atoms This assumption is fully justified by the results of different investigators

There seems to be some uncertainty in regard to the second assumption For instance, Barkla and Dunlop* find for the wave-length $0\,316\times10^{-8}$ cm the following values —

$$\left(\frac{\sigma}{\rho}\right)_{Cu}\Big/\left(\frac{\sigma}{\rho}\right)_{Al} = 1\,12, \qquad \left(\frac{\sigma}{\rho}\right)_{Pb}\Big/\left(\frac{\sigma}{\rho}\right)_{Al} = 2{\cdot}65,$$

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* Barkla and Dunlop, 'Phil. Mag.,' vol. 31, p. 229 (1916).

the relative scattering for lead being more than twice that for copper (For longer waves these relative values show marked increases.) Hull and Rice,[*] on the other hand, find that the relative scattering with this radiation has the same value in each case, and is equal to unity In view of this irregularity it was deemed advisable to assume the scattering coefficient to be constant for all the elements employed for the two wave-lengths dealt with, namely $0{\cdot}35 \times 10^{-8}$ cm. and $0\,586 \times 10^{-8}$ cm. There is some ground for making this assumption in the fact that on the electro-magnetic theory this quantity should be an universal constant. The absolute value 0 2 was taken because Barkla has made many determinations of this quantity for light elements and has invariably found this value for it Furthermore it is the value that, when substituted in Sir J J Thomson's expression for the loss of energy by scattering, gives the value at present accepted for the number of electrons in the atom. The agreement found here between calculated and observed values lends some support to the validity of the assumption. It is possible, however, that the agreement between calculated and observed values might be even better than in the present instance if it were known exactly how the scattering coefficient varies with the wave-length of the radiation scattered, and also the mass of the atom, and an accurate allowance be made for this in the calculations [†]

A method of measuring coefficients of scattering is suggested by the results of this investigation We have found that the atomic fluorescent absorption coefficient is proportional to the fourth power of the atomic number of the absorber, we know also the value of the scattering coefficient for light elements. If then an accurate method of measuring total absorption coefficients be employed, by the aid of these two facts, the variation of the scattering coefficient can be investigated for any wave-length over any range of atomic numbers

Summary.

(1) The absorption coefficients of a number of substances for a radiation of wave-length $0\,586 \times 10^{-8}$ cm (the α-line of palladium) have been determined, and the values obtained confirm those of Bragg and Pierce in the case of elements used in common

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[*] Hull and Rice, *loc cit*

[†] It might be mentioned that when the value of the scattering coefficient for hydrogen was taken to be twice that of other light elements, and the calculations proceeded with in a manner similar to that described here, the agreement between the observed and calculated values of the molecular total absorption coefficient was not so good as in the case investigated when the value of the scattering coefficient of hydrogen was taken to be the same as that of other light elements.

(2) The atomic fluorescent absorption coefficient is proportional approximately to the fourth power of the atomic number of the absorber.

(3) The following relation exists between the atomic fluorescent absorption coefficient, atomic number of the absorber, and the wave-length of the radiation absorbed

$$f_a = CN^4\lambda^3,$$

where C is a constant over certain ranges, but changes abruptly at critical points

This relation is independent of the scattering coefficient, it deals only with the loss of energy of X-radiation by the production of corpuscular radiations and the fluorescent X-radiations that accompany them

(4) Calculations based on the above general relation show that the molecular total absorption coefficients of different substances observed by Auren with radiation of wave-length $0{\cdot}35 \times 10^{-8}$ cm. may be deduced very approximately from the atomic total absorption coefficients obtained for different elements with radiation of wave-length $0{\cdot}586 \times 10^{-8}$ cm, if the coefficient of scattering be assumed to have a constant value of 0 2 for all elements from hydrogen to bromine for both these radiations.

I desire to express my indebtedness to Prof. A W Porter, F.R.S., for the help he has given me throughout the course of the investigation

The Magnetic Storm of December 16–17, 1917, *as Recorded at Kew and Eskdalemuir Observatories.*

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(Received March 19, 1918)

§ 1 In a recent paper,* I described a magnetic storm occurring on August 22, 1916, in which the disturbances recorded at Kew and Eskdalemuir Observatories were closely similar in type The magnetic storm on December 16–17, 1917, affords in many ways a remarkable contrast, the changes recorded at the two observatories differing notably in type The disturbance was the largest of the year. In the Kew declination (D) and horizontal force (H) curves, a prominent feature was a succession of oscillations with periods of about 20 minutes, and it was the hope of identifying these with corresponding oscillations at Eskdalemuir that suggested the present investigation Dr Crichton Mitchell, the Superintendent of Eskdalemuir Observatory, kindly sent the original curves, so the comparison was made under favourable conditions

One of the obstacles to the identification of corresponding movements at the two stations is that, while H and D are recorded at Kew, the north (N) and west (W) components are recorded at Eskdalemuir. On instituting a minute comparison between the Kew H and D curves, it was found that, while oscillations with periods not far from 20 minutes were prominent in both, the times of the turning points by no means always agreed. A turning point in the H curve, for instance, might be represented by a short arrest or slight temporary reversal of movement in the D curve, the original direction of the D movement being almost immediately resumed. Again, towards the end of an oscillation, the trace might be nearly level for some minutes, or there might be a small short period oscillation, so that more than one choice was possible when assigning a time for the end of one major oscillation or the beginning of the next Thus, when one had to compare Kew H with Eskdalemuir N curves, and Kew D with Eskdalemuir W curves, the fact that corresponding movements were difficult to identify is not surprising. Both observatories record vertical force (V), but the Kew V curves are so much disturbed by artificial electrical currents that they cannot be relied on for minor details. There were also difficulties special to the particular storm Some of the N and W movements at Eskdalemuir were so rapid that the photographic trace was in places almost invisible. While the time marks occur on the base lines as well as on the curve lines, there is sensible parallax, and when the marks on the curve lines become invisible, the time cannot be fixed with quite the usual accuracy

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* 'Roy Soc Proc.,' A, vol. 93, p. 177.

§ 2 The disturbance was preceded by a fairly quiet time, so its commencement may be accepted without hesitation as occurring between 8 h and 9 h on December 16 But the subsidence was as usual more gradual Measurements were made of the curves from 8 h. on the 16th to 8 h on the 17th They were taken at four-minute intervals, because four minutes answers very nearly to 1 mm in the base line The curves were not smoothed, and the arithmetic mean of the measurements made throughout each hour G M T. was accepted as the hourly mean Hourly values were thus obtained from 8·5 h to 23 5 h. (11½ P M.) on December 16, and from 0 5 h to 7 5 h on December 17 The arithmetic mean of these 24 hourly values was accepted as the mean for the "day" which commenced at 8 h on December 16

Table I —Mean Hourly Values (Unit 1γ ≡ 1 × 10⁻⁵ C G S)

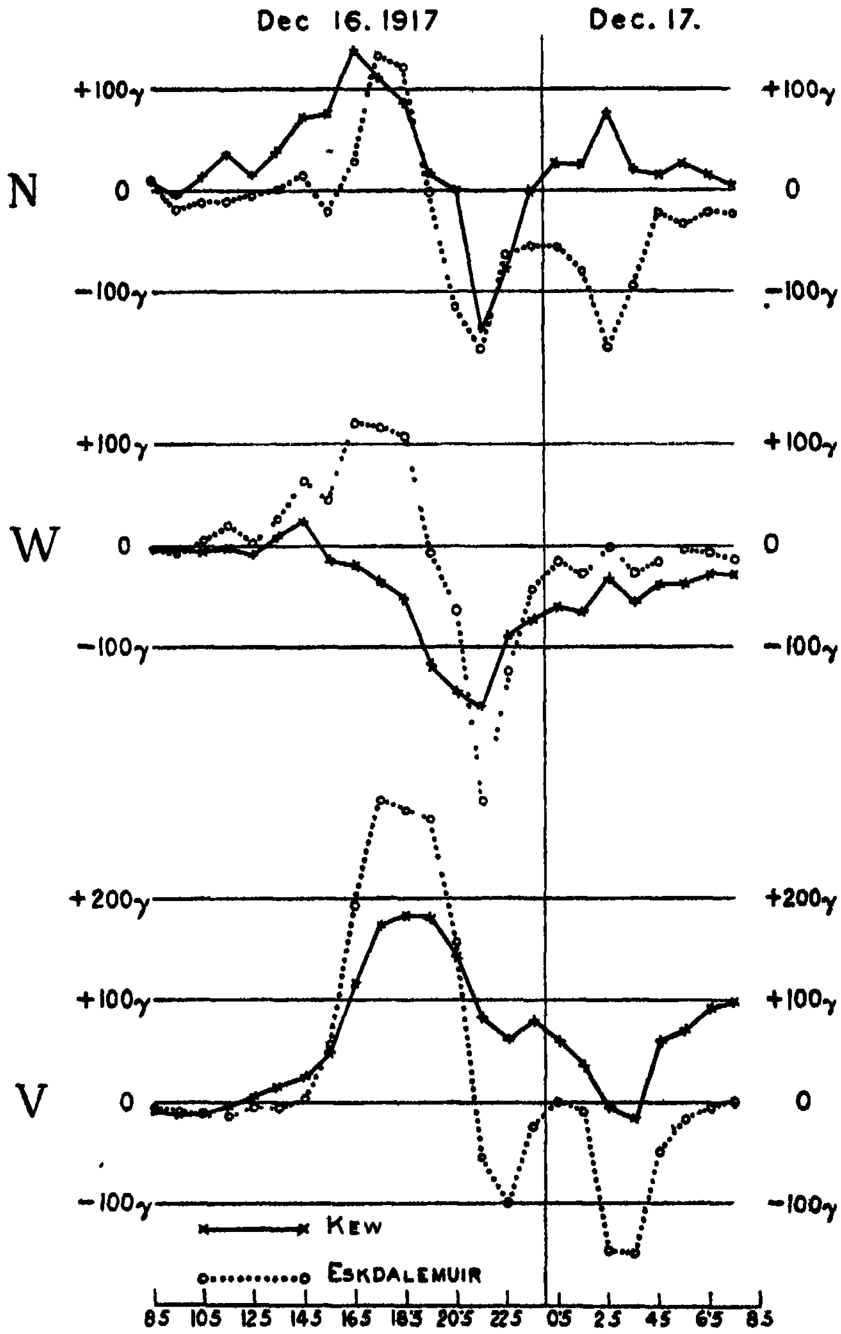
Time, G M T		Kew			Eskdalemuir		
		H	D	V	N	W	V
		γ	γ	γ	γ	γ	γ
Dec 16	8 5 h	+ 9	− 8	− 8	+ 9	− 3	− 6
	9 5	− 6	− 2	− 13	− 18	− 7	− 10
	10 5	+ 10	− 8	− 13	− 12	+ 4	− 13
	11 5	+ 34	− 10	− 5	− 11	+ 20	− 14
	12 5	+ 12	− 13	+ 6	− 4	+ 2	− 5
	13 5	+ 38	− 2	+ 14	+ 1	+ 27	− 5
	14.5	+ 75	+ 6	+ 22	+ 15	+ 64	+ 2
	15 5	+ 70	− 34	+ 48	− 22	+ 46	+ 55
	16 5	+ 181	− 54	+ 114	+ 28	+ 121	+ 194
	17 5	+ 100	− 68	+ 174	+ 189	+ 118	+ 297
	18 5	+ 71	− 78	+ 184	+ 122	+ 108	+ 288
	19 5	− 15	− 118	+ 182	− 2	− 6	+ 279
	20 5	− 36	− 139	+ 144	− 117	− 65	+ 158
	21.5	− 175	− 117	+ 82	− 156	− 252	− 54
	22 5	− 100	− 65	+ 59	− 63	− 123	− 99
	23 5	− 21	− 70	+ 80	− 55	− 43	− 24
Dec 17	0 5	+ 11	− 65	+ 61	− 55	− 14	+ 1
	1 5	+ 6	− 69	+ 35	− 79	− 27	− 8
	2 5	+ 68	− 51	− 8	− 153	0	− 147
	3 5	+ 4	− 59	− 19	− 95	− 27	− 149
	4.5	+ 4	− 40	+ 59	− 21	− 15	− 47
	5 5	+ 18	− 43	+ 69	− 33	− 3	− 18
	6 5	+ 8	− 31	+ 91	− 21	− 8	− 6
	7.5	− 1	− 28	+ 98	− 22	− 18	+ 1

The Kew D trace records angular changes, but at present a rise of one minute in westerly declination is equivalent to a change of 5·36 γ in force directed 15° south of true west The Eskdalemuir V trace unfortunately went off the sheet, in the direction of increasing force, several times between 16 h. and 19 h on the 16th, there being no record for some 55 minutes in all, and the longest continuous absence of trace being about 20 minutes When the curve was off the sheet, the value answering to the margin of the sheet was accepted Judging by the trend of the curve, the margin was not much exceeded in most cases But between 17 h and 18 h, when the trace was off the sheet for 20 minutes at a time, the margin may have been considerably exceeded An error x in one hourly mean implies an error $x/24$ in the mean for the day

The existence of this uncertainty in the mean value of V at Eskdalemuir was one of the reasons which led to the adoption as standard values of quiet day means derived from the previous 24 hours Table I shows the departure of the hourly means from these quiet day means, the unit being for all the elements $1\gamma \equiv 1 \times 10^{-5}$ C G S magnetic unit Maximum and minimum values are in heavy type Every hourly value of D at Kew, except that for 14 5 h, falls short of the quiet day mean, *i e*, represents a position of the declination needle to the east of the normal The existence of this phenomenon at Kew, while there is no corresponding abnormality in W at Eskdalemuir, is one of the most remarkable differences between the two stations. Another remarkable feature is that while N at Eskdalemuir shows the depression customary in that element and in H towards the end of a magnetic storm, there is no corresponding sensible depression in H at Kew V was much enhanced at both places in the afternoon of the 16th, a customary phenomenon of magnetic storms This elevation of V was followed by a rapid fall before midnight, also a customary feature, but subsequent to this there was another considerable rise, and a second rapid fall, more especially at Eskdalemuir

§ 3 Hourly mean values of N and W at Kew were calculated from the hourly means of H and D, and the departures of these from the quiet day means are shown in the accompanying diagram, with the corresponding departures in N and W at Eskdalemuir and in V The scale of ordinates is the same for the six curves

Up to 22 h. on December 16 there is a general resemblance between the changes in N at the two stations, but later, especially from 1 h to 3 h on the 17th, the changes are more nearly diametrically opposite. The N ranges for the whole 24 hours, as given by the hourly means, were 288 γ at Eskdalemuir as compared with 279 γ at Kew This gives a very inadequate idea of the excess in the activity at Eskdalemuir. A better idea is given by the values of the absolute range, *i e*, the excess of the largest over the least value



recorded at any instant during the 24 hours. These absolute ranges were 620 γ at Eskdalemuir in N, and 408 γ at Kew in H.

The resemblance between the W diagrams for Kew and Eskdalemuir is not close, but it is not less close than usual between 1 h. and 3 h on the 17th, when the N diagrams from the two stations were opposed. At Eskdalemuir W was markedly enhanced from 13 h to 19 h on the 16th, whereas at Kew W was markedly depressed after 15 h The ranges from the hourly means were 373 γ at Eskdalemuir and 183 γ at Kew The difference between the absolute W range at Eskdalemuir and the absolute D range at Kew was even greater, the former being 588 γ, the latter only 189 γ

The ranges from the V hourly means were 446 γ for Eskdalemuir and 197 γ for Kew, the corresponding absolute ranges being respectively 552 γ and 259 γ. As already explained, the values for Eskdalemuir are necessarily under-estimates

§ 4 Table II gives the results of measurements of the oscillations of longer period already referred to The different curves were considered and measured quite independently. A minimum reading was taken as the commencement of each oscillation The second minimum for one oscillation served in many cases as the first minimum for the next. In other cases an interval was left, answering to a portion of the curve which was level or showed one or more short period oscillations, the inclusion of which in either of the two adjacent longer period oscillations would have been arbitrary. If we started with the assurance that we had to do with a continuous train of oscillations, confused by the superposition of waves of shorter period, we should naturally assign the doubtful piece of curve to one or other of the adjacent longer period oscillations If there had been a very pronounced resemblance between Kew and Eskdalemuir curves, and the hypothesis of a small differential time error had markedly improved the coincidence of turning points, it might have been fairly justified But, as matters stood, it seemed best to leave the figures exactly as the curve measurements made them

The result is that 14 longer period oscillations were recognised in both the H and D curves at Kew, the first 13 of which proceeded with little or no interruption, whilst the 14th followed after an hour's interval The 13th differed somewhat in type from the others It possibly represented two oscillations rather than one. The mean duration for the first 12 oscillations, accepting the figures in Table II, is in H 20·4 minutes, and in D 21·7 minutes If we assume that there was a continuous train of oscillations, and accept the times given for the beginning of the 1st and the end of the 12th oscillation, the mean durations become 21·8 minutes for H and 22 5 minutes for D

At Eskdalemuir, between 15 h. and 20½ h., on the 16th, 16 oscillations of longer period were recognised in N and 13 in W Some, however, were of

Table II —Oscillations of Longer Period

No	Kew H.									Kew D								
	1st min.		Max.		2nd min.		Interval	Rise.	Fall.	1st min.		Max.		2nd min		Interval.	Rise	Fall
	h	m.	h	m	h.	m	min.	γ	γ	h	m	h.	m.	h	m	min.	γ	γ
1	15	22	15	28	15	36	14	22	14	15	12	15	28	15	36	24	19	42
2	15	36	15	50	15	56	20	50	21	15	36	15	39	15	56	20	19	31
3	16	0	16	13	16	24	24	72	54	15	56	16	5	16	18	22	26	33
4	16	24	16	40	16	44	20	50	45	16	22	16	28	16	44	22	26	32
5	16	48	16	59	17	7	19	59	94	16	44	16	52	17	3	19	28	31
6	17	7	17	17	17	30	23	81	100	17	3	17	17	17	23	20	21	34
7	17	34	17	42	17	48	14	60	24	17	23	17	40	17	46	23	56	22
8	17	48	17	55	18	6	18	27	28	17	46	17	51	18	4	18	17	13
9	18	8	18	12	18	26	18	21	83	18	4	18	6	18	21	17	5	46
10	18	29	18	39	18	52	23	77	67	18	21	18	33	18	50	29	48	43
11	18	52	18	58	19	22	30	27	92	18	50	18	58	19	15	25	5	37
12	19	22	19	26	19	44	22	32	77	19	20	19	26	19	42	22	13	37
13	19	44	20	9	20	25	41	103	77	19	42	19	52	20	13	31	33	34
14	21	13	21	18	21	29	16	83	68	21	10	21	18	21	30	20	65	61

No.	Eskdalemuir N									Eskdalemuir W								
	1st min.		Max.		2nd min.		Interval.	Rise.	Fall.	1st min.		Max.		2nd min.		Interval.	Rise.	Fall
	h.	m.	h.	m.	h.	m.	min.	γ	γ	h.	m.	h	m	2nd	min	min.	γ	γ
1	15	11	15	26	15	32	21	64	86	15	13	15	31	15	36	23	53	38
2	15	32	15	40	15	55	23	64	90	15	36	15	51	15	58	22	68	60
3	15	55	16	4	16	7	12	89	25	15	58	16	6	16	11	13	134	37
4	16	7	16	11	16	17	10	61	39	16	11	16	18	16	23	12	80	121
5	16	20	16	24	16	28	8	35	34	16	23	16	41	16	47	24	70	78
6	16	28	16	37	16	44	16	49	34	16	47	17	1	17	7	20	150	192
7	16	44	16	58	17	1	17	143	35	17	7	17	17	17	34	27	203	192
8	17	1	17	4	17	8	7	149	153	17	34	17	42	17	48	14	100	43
9	17	8	17	11	17	14	6	37	45	17	48	17	54	18	8	20	52	48
10	17	19	17	25	17	35	16	184	197	18	8	18	11	18	24	16	22	60
11	17	42	17	50	17	57	15	79	65	18	24	18	35	18	40	16	89	31
12	17	57	18	4	18	14	17	21	105	19	18	19	34	19	46	28	78	145
13	18	18	18	34	18	39	21	250	102	19	46	20	8	20	25	39	108	102
14	18	39	18	45	19	15	36	67	253									
15	19	15	19	34	19	49	34	107	143									
16	19	49	19	53	20	5	16	16	74									

decidedly shorter period than the others, and may have answered to portions only of the oscillations recognised at Kew, large short-period oscillations occurring sometimes at Eskdalemuir while the Kew curve showed almost stationary conditions.

The Eskdalemuir W oscillations could be brought into fair parallelism with the H and D oscillations at Kew by assuming that Nos. 3 and 4 at Eskdalemuir together answered to No 3 at Kew, whilst Nos. 5–11 at Eskdalemuir corresponded respectively to Nos 4–10 at Kew, Nos 12 and 13 at Eskdalemuir corresponding to the oscillations similarly numbered at Kew We could also bring the earlier N oscillations at Eskdalemuir into fair parallelism with the others by regarding the pairs Nos 3 and 4, 5 and 6, 7 and 8, and 9 and 10 as each but a single oscillation, answering respectively to Nos 3, 4, 5, and 6 at Kew

The Eskdalemuir curves contained some exceedingly large rapid oscillations, practically unrepresented at Kew These must presumably have arisen from some source of disturbance which was either purely local, or was situated very much nearer to Eskdalemuir than to Kew

As regards the size of the oscillations recorded in Table II, the mean values of the rise and fall at Kew were respectively 55 γ and 60 γ for H and 27 γ and 35 γ for D The H movements were thus, on the average, decidedly the larger If we accept the arithmetic mean of the rise and fall as a measure of the amplitude, we find that the first six oscillations in D were very similar in amplitude, while the corresponding H oscillations differed widely There was, in short, little, if any, parallelism between the amplitudes of the oscillations in the two elements If we accept the problematical scheme of correspondence between Kew and Eskdalemuir oscillations given above, we find that the Eskdalemuir ranges were almost invariably the larger, but the numerical results, from a comparison of Kew and Eskdalemuir curves, are too uncertain to be worth recording.

§ 5 During many of the rises and falls recorded in Table II, the rate of change of the element varied greatly Rates of change derived from Table II would thus give an inadequate idea of the rapidity of the more rapid changes. An independent set of measurements was accordingly made, the results of which appear in Tables III–VII. They give the central time of the movement, *i.e.* the arithmetic mean of the times of beginning and ending, the number of minutes the movement lasted, and the estimated rapidity of the change of force in γ's per minute. In most cases, where the rises and falls recorded in Table II showed a sudden marked acceleration or retardation, measurements were made on shorter portions of curve having a nearly uniform slope. In a few cases, however, mean rates are given for longer periods of time, when there was a conspicuous movement in one direction, interrupted by only trifling arrests or reversals. No measurements were attempted of the shorter period movements, mostly small, occupying one or two minutes or less, unless they were quite exceptionally large. The slope of the curve then affords a ready check on the result derived from the time measurements.

Table III.—Rapid Changes in H at Kew.

Mean time	Duration	Change	Rate	Mean time	Duration	Change	Rate	Mean time	Duration	Change	Rate
h m	min	γ	γ/min	h m	min	γ	γ/min	h. m	min	γ	γ/min
15 8·5	3	−32	11	17 38·0	8	+60	7·5	20 1·0	10	+57	5·5
25·0	6	+22	3·5	45·0	6	−24	4·0	11·0	4	−37	9·0
32·0	8	−14	1·8	51·5	7	+27	4·0	22·5	5	−41	8·0
43·0	14	+50	3·5	18 0·5	11	−28	2·5	21 9·0	8	−215	27
53·0	6	−21	3·5	9·5	3	+18	6·0	15·5	5	+83	17
16 2·5	5	+49	10	19·0	14	−83	6·0	21·0	6	−27	4·5
12·0	2	+23	11	34·0	10	+77	7·5	27·0	4	−46	11
21·5	5	−48	9·5	44·0	10	−65	6·5	34·0	2	+25	12
27·5	7	+35	5·0	54·0	4	+21	5·0	54·0	10	+46	4·5
43·0	2	−35	17	19 4·5	5	−32	6·5	22 32·5	39	+89	2·3
53·5	11	+59	5·5	15·5	13	−63	5·0	0 59·0	2	+28	14
17 3·0	8	−94	12	24·0	4	+32	8·0	1 3·0	6	−25	4·0
12·0	10	+81	8·0	38·0	12	−77	6·5	2 14·0	20	+99	5·0
18·5	3	−44	15	48·5	9	+53	6·0	55·5	29	−111	4·0
26·5	7	−55	8·0								

Table IV.—Rapid Changes in D at Kew.

Mean time	Duration	Change.	Rate	Mean time	Duration	Change	Rate	Mean time	Duration	Change	Rate
h m	min	γ	γ/min	h m	min	γ	γ/min.	h. m	min	γ.	γ/min
15 8·5	3	−23	7·5	17 8·0	10	+20	2·0	19 47·0	10	+33	3·5
20·0	16	+19	1·2	20·0	6	−34	5·5	20 6·5	13	−32	2·5
31·0	6	−40	6·5	35·0	10	+46	5·0	21 6·0	8	−35	4·5
37·5	3	+19	6·5	43·0	6	−22	3·5	16·0	4	+54	13
50·0	12	−28	2·3	48·5	5	+17	3·5	21·0	6	−50	8·5
16 0·5	9	+26	3·0	18 15·5	11	−42	4·0	32·5	5	+44	9·0
15·0	6	−30	5·0	27·0	12	+48	4·0	49·0	28	+76	2·5
25·0	6	+26	4·5	38·0	10	−37	3·5	0 59·0	2	+18	9·0
31·5	7	−26	3·5	19 0·5	5	−14	3·0	1 3·0	6	−21	4·0
43·0	2	−14	7·0	11·5	7	−22	3·0	2 41·0	12	+40	3·5
48·0	8	+28	3·5	23·0	6	+13	2·2	58·5	23	−44	1·9
17 0·0	6	−25	4·0	34·0	16	−37	2·3				

Table V.—Rapid Changes in N at Eskdalemuir

Mean time	Duration	Change	Rate
h m	min.	γ	γ/min
15 9 0	4	− 51	13
13·0	4	+ 26	6 5
20 5	11	+ 38	3 5
29 5	5	− 86	17
36 0	8	+ 64	8·0
49 5	11	− 89	8·0
58 5	7	+ 87	12
16 5 5	3	− 25	8 5
9·0	4	+ 61	15
14 0	6	− 39	6 5
22·0	4	+ 35	8 5
25·0	2	− 34	17
35·5	3	+ 40	13
38 0	2	− 22	11
43·0	2	− 30	15
49·0	10	+109	11
56 5	3	+ 40	13
59 5	3	− 35	12
17 2 5	3	+149	50
6·0	4	−153	38
9 5	3	+ 37	12
12 5	3	− 46	15
23 5	3	+161	54
26·5	3	−161	54
32 0	6	− 55	9·0

Mean time	Duration	Change	Rate
h m	min	γ	γ/min
17 41·0	2	− 30	15
47 5	5	+ 64	13
52·0	4	− 64	16
18 8 5	5	− 85	17
26 0	10	+217	22
33 5	1	+ 32	32
36 5	5	−102	20
39 5	1	+ 29	29
43·0	4	+ 46	11
47 5	5	− 38	7 5
19 4 0	22	−217	10 0
18 5	7	+ 64	9·0
29 5	3	+ 35	12
37 0	6	− 79	13
45 5	7	− 84	12
51 0	4	+ 16	4·0
20 2 5	5	− 62	12
9 0	2	+ 30	15
12 5	3	− 30	10
21 5	3	+ 33	11
33 5	5	− 52	10
38 5	5	+ 25	5·0
53 5	5	− 30	6 0
58 5	3	+ 29	9 5

Mean time	Duration.	Change	Rate
h m	min	γ	γ/min
21 19 5	15	−282	19
31 0	8	+193	24
43 5	7	+ 92	13
51·0	6	+ 53	9
22 0·0	2	− 28	14
8 5	5	− 30	6
12·0	2	+ 25	12
24·0	8	− 45	5 5
51 5	7	− 33	4 5
56 5	3	+ 24	8 0
23 12·0	2	+ 28	14
0 0·0	4	− 29	7
59·0	2	+ 37	18
1 2 0	4	− 45	11
51 5	3	− 31	10
56 5	3	+ 47	16
2 7 5	5	− 69	14
20 5	5	− 30	6
36 5	5	+ 62	12
50 5	7	− 74	11
58·0	8	+ 60	7 5
3 3 5	3	− 36	12
8 0	6	+ 42	7·0
48 0	20	+ 84	4 0

Table VI.—Rapid Changes in W at Eskdalemuir

Mean time	Duration	Change.	Rate
h. m	min	γ	γ/min
14 36·0	2	+ 20	10
15 8 5	5	− 59	12
27·0	4	+ 31	7·5
33·5	5	− 38	7·5
38·0	4	+ 32	8·0
54 5	7	− 80	11
16 2 5	7	+128	18
7·0	2	− 35	17
14 5	7	+ 80	11
20 5	5	−121	24
24·0	2	+ 34	17
35·5	3	+ 38	13
43 5	1	− 48	48
46 0	2	− 36	18
49·5	5	+ 88	18
56·5	3	+ 54	18
17 4·0	6	−167	28
12·0	10	+203	20
24 0	4	− 99	25
31·0	6	− 58	9 5
36·0	8	+100	12
45 0	6	− 43	7 0

Mean time	Duration	Change	Rate
h m	min	γ	γ/min
17 51 0	6	+ 52	8 5
18 9 5	3	+ 22	7 5
18·0	8	− 58	7 0
29 0	10	+ 84	8 5
38 0	4	− 31	7 5
43 0	2	+ 26	13
19 4 0	6	− 68	11
15 0	6	− 66	11
25 5	3	+ 40	13
32 0	4	+ 27	6 5
36 0	4	− 49	12
43 5	5	− 92	18
50 0	8	+ 58	7·0
55 0	2	− 27	13
58 5	5	+ 56	11
20 6·0	4	+ 23	5 5
12 5	3	− 58	19
24·0	2	− 35	17
57 5	3	+ 22	7 5
21 4 5	5	−251	50
8 5	3	+142	47
12 0	4	−178	44

Mean time	Duration	Change	Rate
h m	min	γ	γ/min
21 16 5	5	+224	45
23 0	8	−171	21
30 0	6	+ 59	10
37 0	2	+ 54	27
38 5	1	− 35	35
54 5	9	+ 59	6 5
22 5 0	6	+ 42	7 0
17 5	7	+ 36	5 0
23 57 5	3	+ 23	7 5
1 0 5	3	+ 25	8 5
4 5	5	− 32	6 5
22 5	9	− 32	3 5
2 17 0	12	+ 83	7 0
30 5	3	− 24	8·0
38 0	2	+ 28	14
41 0	4	− 32	8·0
50 5	7	− 69	10
59·0	4	+ 22	5 5
3 5·0	6	− 29	5 0
12 0	8	+ 28	3 5
20 5	5	+ 27	5 5
35 5	15	− 42	3 0

Table VII—Rapid Changes in V at Eskdalemuir

Mean time	Duration	Change	Rate	Mean time	Duration	Change	Rate	Mean time	Duration.	Change	Rate
h. m	min	γ	γ/min	h m	min	γ	γ/min	h m	min	γ	γ/min
15 33 5	9	+ 35	4 0	18 8 0	8	− 32	4·0	21 25·0	2	+ 22	11
54 0	8	+ 36	4 5	19 43 0	4	+ 17	4 0	35 0	2	− 19	9 5
16 1 0	6	− 28	4 5	50 0	10	− 74	7 5	38 5	5	+ 47	9 5
6 5	5	+ 32	6 5	20 2 5	9	− 37	4·0	50 5	9	− 30	3 5
20 0	8	+ 130	16	11 5	9	+ 26	3·0	22 1 5	5	− 43	8 5
25 5	3	− 50	17	19 0	6	− 32	5 5	21·0	10	+ 38	4 0
43 0	6	+ 35	6·0	32 5	15	− 56	3 5	2 25 5	55	− 228	4·0
48 0	4	+ 56	14	21 11 0	2	− 89	44	3 12 5	25	+ 91	3 5
51 5	3	− 43	14	15 5	5	− 229	46	42 5	35	+ 87	2 5
17 53 0	4	+ 19	4 5	19 5	3	+ 104	35				

Independent measurements were first made of the absolute times of the beginning and end of each movement A differential measurement was then made giving the duration directly Finally, the slope of the curve was measured with a special scale With times measured only to the nearest minute, no very great accuracy can be claimed for the rates, but the checks mentioned above should be a sufficient protection against large errors. The probable error is naturally greatest where the time interval is least, and the large rates obtained for some of the shortest movements may thus incur suspicion The Kew rates are all much slower than the faster Eskdalemuir rates, and the fastest of them, 27 γ per minute in H, was derived from an 8-minute interval, and so can hardly be much in error. The fastest rate for N changes at Eskdalemuir, viz. 54 γ per minute, is based on a 3-minute interval, but that interval was immediately followed by a second 3-minute interval, the change occurring during which had the same estimated rapidity The two movements constituted an oscillation in which the to-and-fro movements were practically equal The duration of the complete oscillation was thus a particularly convenient one to measure If we accept six minutes for it, but suppose that an error was made in allotting equal times to the two movements, then the duration of one of them must have been over-estimated, and so its rapidity under-estimated A somewhat similar argument applies in the case of the next most rapid N rate, viz., 50 γ per minute.

The most rapid rate for W changes in Table VI, viz. 50 γ per minute, is followed by rates of 47 γ, 44 γ, and 45 γ per minute, and these four rates arise from intervals which form between them the 17 consecutive minutes 21 h 2 m. to 21 h 19 m on the 16th. The error in the total interval is unlikely to have exceeded one minute, and any redistribution of the time amongst the four sub-intervals would almost inevitably have led to at least one rate in excess of any of the four assigned

The fastest rate for V arose from a 5-minute interval It was a particularly favourable case for measuring the slope of the curve, as the trace was bold and the slope uniform to the eye In the case of the longer intervals, the assigned rate must in general have been very sensibly exceeded during part of the time

No V movement was included in Table VII, any part of which was lost through the trace going off the sheet Several rapid movements were excluded for this reason, but none so rapid as the fastest movement given in the Table

No rates were calculated for V changes at Kew, because little reliance could have been placed on them

Table VIII —Aggregates of Rapid Changes and Corresponding Mean Rates.

Element	Rises			Falls			Rises and falls		
	Aggregates		Mean rate	Aggregates		Mean rate	Aggregates		Mean rate
	Duration	Movement.		Duration	Movement		Duration	Movement	
	mins	γ	γ/min	mins	γ		mins	γ	γ/min
Kew H	188	1034	5·5	175	1217	7·0	363	2251	6·2
Kew D	146	529	3·6	158	576	3·6	304	1105	3·6
Eskdalemuir N	183	2214	12·1	196	2430	12·4	379	4644	12·3
Eskdalemuir W	178	2069	11·6	156	2163	13·9	334	4232	12·7
Eskdalemuir V	137	775	5·7	138	985	7·1	275	1760	6·4

Table VIII gives particulars of the aggregates of the rises and falls included in Tables III–VII. But for the loss of trace, the time aggregate for V at Eskdalemuir would have been considerably larger. While the time aggregates for the different elements are not identical, we may fairly conclude that during the most active part of the storm changes in N and W at Eskdalemuir were roughly twice as rapid as changes in H, and 3½ times as rapid as changes in D at Kew The V changes at Eskdalemuir and the H changes at Kew had very similar mean rates, but the fastest V changes at Eskdalemuir were decidedly faster than the fastest H rates at Kew The V traces were, however, of a very different type from the others, oscillations of short period being much less in evidence. Considering that the ranges of the diurnal inequalities in N and W at Eskdalemuir in December are only about 15 γ and 20 γ respectively, it is a little startling to find that during six hours of one particular "day," the mean rates of change per minute in these elements were quite 12 γ The contributions to the aggregates of changes from the other 18 hours of the 24 would have been considerable, and the mean rate of change in N and W for the whole 24 hours could hardly have been less than 4 γ per minute

§ 6 Any complete estimate of the expenditure of energy during a magnetic storm is probably impossible. A magnetic storm is usually accompanied by aurora and earth currents In this particular case, according to observations made by Prof. Stormer near Christiania, the aurora extended from 100 to 400 kiloms above the earth's surface. There is no known method of finding the intensity at any height of the electric currents producing aurora Few stations attempt to measure earth currents, and little, if anything, is known as to the depth to which earth currents extend. Thus the energy expended in aurora, and the energy expended in earth currents, are alike inaccessible to calculation. If the ultimate source of all the terrestrial phenomena is some form of electrical discharge from the sun, the energy represented by the various phenomena in the earth and the atmosphere is probably but an insignificant fraction of the whole

The estimate of the energy represented by the changes shown in magnetic curves is naturally a simpler problem, but even it is fraught with difficulties, practical as well as theoretical. For what has been done to give definiteness to the problem we are mainly indebted to the late Prof. Bidlingmaier [*]

The formula given by Maxwell for the energy in a magnetic field when there are no electric currents is

$$(1/8\pi)\iiint(\alpha^2+\beta^2+\gamma^2)\,dx\,dy\,dz,$$

where α, β, γ represent the rectangular components of magnetic force. The integral is supposed to be taken throughout the whole of the magnetic field It takes, moreover, as point of departure a total absence of force. In the present case we know the absolute values of the three components at a fixed point, and the sequence of changes in these components, but the intensity of the field never vanishes, the changes encountered being in fact small compared with the mean values of the elements. Suppose we confine our attention for the moment to one of the three components The value may be regarded as given by the curve ordinate y at the instant. If y_0 be the accepted normal value, the "activity," as Bidlingmaier has called it, at the instant is

$$(1/8\pi)(y-y_0)^2 \qquad (1)$$

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[*] 'Veröffentlichungen des Kaiserlichen Observatoriums in Wilhelmshaven,' "Ergebnisse der Magnetischen Beobachtungen im Jahre 1911," Neue Folge, Heft 2

What we are concerned with in practice is the mean value of the "activity" for some specified time, whether an hour, a day, or a year. Let

$$y = y_h + \eta, \tag{2}$$

where y_h is the mean value for the hour under consideration. Then, from the properties of the arithmetic mean, we have

$$\text{Mean value of } (y-y_0)^2 \text{ per hour} = (y_h-y_0)^2 + \text{mean value of } \eta^2 \text{ per hour} \tag{3}$$

To determine the mean value of η^2, Bidlingmaier measured the curve at six-minute intervals and took $(1/10)\Sigma\eta^2$ as the mean value for the hour. In the present case the curve was measured at four-minute intervals, viz., at 0, 4, . . ., 56, 60 minutes after the hour, and

$$(1/15)\Sigma\eta^2 \equiv \{\tfrac{1}{2}(\eta_0^2+\eta_{15}^2)+\eta_1^2+\dots+\eta_{14}^2\}, \tag{4}$$

was accepted as the mean. Thus $(1/8\pi)(1/15)\Sigma\eta^2$ answers to what Bidlingmaier called A_h^z for the particular hour. The mean "activity" for the one hour is

$$A_h^z + (1/8\pi)(y_h-y_0)^2 \tag{5}$$

Suppose now we want the mean activity for the day. If y_d be the mean value for the day, and $y_1, \dots, y_{24}$ the mean values for the several hours, then

$$\text{Mean value of } (y_h-y_0)^2 = (y_d-y_0)^2+(1/24)\{(y_1-y_d)^2+\dots+(y_{24}-y_d)^2\} \tag{6}$$

In Bidlingmaier's notation

$$(1/8\pi)\{(y_1-y_d)^2+\dots+(y_{24}-y_d)^2\} \equiv A_d^h, \tag{7}$$

$$(1/8\pi)(y_d-y_0)^2 \equiv A_a^d. \tag{8}$$

His complete expression for the mean "activity" of the day is

$$A_h^z + A_d^h + A_a^d \tag{9}$$

In this connection A_h^z represents the arithmetic mean of the 24 individual hourly values of A_h^z. The curve ordinates are to be regarded as expressed in terms of $1\gamma\,(\equiv 1\times10^{-5}$ C.G.S.), and the unit in the "activity" results is 1×10^{-10} erg per cubic centimetre.

A difficulty at once arises as to A_a^d, viz., what to accept for the normal value y_0. Bidlingmaier apparently thought that theoretically the best plan would be to derive it from the mean value for the whole year, by assuming the rate of secular change uniform. For instance, for Wilhelmshaven in 1911, taking $+16\gamma$ per annum as the rate of change of H, he accepted as the value of y_0 for a day n days subsequent to July 1

$$\text{Mean for year } +(n/365)\,16\gamma.$$

His value of y_0 thus increased gradually throughout the year

The mean value for a year is unknown until the year is completed For the secular change it is desirable to know the mean value for the subsequent as well as the previous year Thus the calculation of A_a^d in the way outlined above would have to be postponed to a somewhat indefinite date Recognising this, though he gave values of A_a^d at Wilhelmshaven for all days from January 1 to June 30, 1911, the period to which he confined himself, Bidlingmaier concluded that in general the only practical course would be to omit A_a^d, *i.e.*, to accept for y_0 the mean value of the individual day This would give for the day's mean "activity" $A_k^x + A_d^h$, the symbols having the same meaning as before

In accepting as the best theoretical value for y_0 one based on the mean value for the year, Bidlingmaier recognised one difficulty The mean value is not the same when we confine ourselves to quiet days as when we include all days of the year This is especially true of H The quiet day yearly mean would seem the more natural one to employ in the present connection, but that is a point on which opinions may differ

Another difficulty is that an annual inequality is believed to exist in the case of all the magnetic elements. If an annual inequality really exists, it should presumably be taken into account in the present problem These considerations support the view that the practical difficulties in the way of applying to individual days a correction A_a^d based on the mean value of the element for the year and the secular change are very serious. When we examine the results for individual days of 1911, which Bidlingmaier found at Wilhelmshaven, we find that A_a^d is by no means trifling The following selection from Bidlingmaier's results should carry conviction on this point —

	D alone			H alone			D + H		
	A_k^x	A_d^h.	A_a^d	A_k^x	A_d^h	A_a^d	A_k^x	A_d^h	A_a^d
Mean from 6 months	2·55	9·72	1·15	1·78	5·40	2·05	4·28	15·12	3·20
April 9, 1911	15·5	53·8	48·8	14·9	35·6	38·2	30·4	89·4	87·0
March 20, 1911	17·4	37·5	0·0	14·6	53·5	2·0	32·0	91·0	2·0

On the average day the value of A_a^d is here about three-quarters that of A_k^x, and on some individual days A_a^d is of the same order as A_d^h. Our view as to the relative "activities" on April 9 and March 20, two of the most disturbed days of 1911, for instance, would be largely determined by the retention or neglect of A_a^d

The adoption of Bidlingmaier's "activities" as an international scheme, in place of the present "character" figures, was under consideration before the war. The question whether a correction of the type of A_a^d should be applied ought, I think, to receive very careful consideration before any final decision is reached

If the present method of selecting the international quiet days continues, perhaps the most satisfactory course would be to accept the mean value from the five quiet days as the normal for all days of the month. At present three months are dealt with together, and the announcement of the quiet days selected, even in pre-war times, did not appear until several months had elapsed, after the end of the quarter No doubt matters could be accelerated by dealing with the months individually If Bidlingmaier's scheme were substituted for the present scheme, the selection of the quiet days would entail the pre-existence of the "activity" statistics In that event, possibly, the best course might be to accept for y_0 the mean value of the last previous day considered quiet at the particular observatory

§ 7. In the case of a disturbance so large as that of December 16–17, 1917, a difference of 2γ or 3γ in y_0 is of minor importance I decided to calculate two sets of "activity" results The first set accepts for y_0 a mean derived from the "day" ending at 8 h. on December 16 The second set adopts Bidlingmaier's suggestion, and neglects A_a^d entirely, accepting for y_0 the mean value for the "day" commencing at 8 h on December 16 For the disturbed day the mean of the 24 hourly means was taken For the previous or quiet day, the mean was taken from measurements made at 14 h and 20 h on the 16th, and at 2 h and 8 h on the 17th The mean of readings taken at these four hours on the average quiet day agrees very closely with the mean from the 24 hours in all the elements With a view to possible future intercomparisons it is desirable to put on record the two sets of mean curve ordinates

	Kew			Eskdalemuir		
	H	D	V	N	W	V
	mm	′	mm	mm	mm	mm
Day of storm	46·2	30·4	73·2	16·8	14·9	61·5
Quiet day	44·0	39·4	69·4	21·8	15·7	55·1
Excess of disturbed day mean	+2·2	−9·0	+3·8	−5·0	−0·8	+6·4
Force equivalent .	$+18\gamma$	-48γ	$+61\gamma$	-26γ	-4γ	$+28\gamma$

At Eskdalemuir the equivalents of 1 mm. of ordinate were $5{\cdot}34\,\gamma$ in N, $4{\cdot}96\,\gamma$ in W and $4{\cdot}33\,\gamma$ in V At Kew the scale value determinations gave as the equivalents of 1 mm of ordinate 5·9 γ in H, 5·36 γ in D, and 16·0 γ in V

In presenting the results obtained from the use of the quiet day mean, I have not followed Bidlingmaier's method of obtaining a correction A_a^d This is convenient when we confine ourselves to the mean "activity" for the whole day, but seems less appropriate when we consider individual hours. In fact it is not altogether clear how Bidlingmaier proposed to treat individual hours, except when A_a^d is neglected In his Table XIII,* when dealing with the diurnal variation of "activity," he takes no cognizance of A_a^d. He gives, however, two sets of figures The first set includes mean values obtained for each hour of the day from the five international quiet days of the month, the second set gives the excess over these values of the corresponding values obtained when all days of the month are included The second set of figures are regarded by Bidlingmaier as giving the diurnal variation of disturbance. They would obviously be unaffected by any correction which was common to the all-day and quiet-day "activities."

I am doubtful whether Bidlingmaier's method of deducing a diurnal variation of disturbance is satisfactory. The differences between the curve ordinates at one and the same hour of the five quiet days of a month must represent in the main disturbance, thus it is only what is common to the five days that can well claim to represent entirely quiet conditions.

In the present case, even if I had entirely approved Bidlingmaier's procedure, it would have been impracticable to adopt it, because the selection of quiet days for December, 1917, is unlikely to be announced until midsummer, 1918.

§8. The best course seemed to be to refer each hourly measurement *directly* to the quiet day mean accepted, so that the complete mean value of the "activity" for the hour becomes

$$A_h^z + (1/8\pi)(y_h - y_0)^2. \tag{10}$$

Tables IX and X give the results thus calculated for individual hours at Kew and Eskdalemuir. The unit employed in the Tables is that used by Bidlingmaier, viz., 1×10^{-10} erg per cubic centimetre.

Bidlingmaier's notation being complicated, the following notation has been employed in the Tables and their discussion

$$\left.\begin{aligned} A_2 &\equiv (1/8\pi)(1/15)\Sigma\eta^2 \\ A_1 &= (1/8\pi)(y_h - y_0)^2 \\ A_1' &= (1/8\pi)(y_h - y_d)^2 \end{aligned}\right\} \tag{11}$$

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* *Loc. cit.*, p. 29.

Here η has the same meaning as in (2), while y_0 and y_h are the mean values for the days which end and commence respectively at 8 h on December 16

The excess in the mean value of A_1 for the day over that of A_1' is equivalent to Bidlingmaier's A_a^d. The values derived for the excess from (8), accepting the differences between y_d and y_0 given in § 7, are as follows, in terms of the unit employed in the Tables

At Kew Observatory ... 6 8 in H, 92 0 in D, 146 3 in V,
At Eskdalemuir Observatory 27 6 in N, 1 4 in W, 30 3 in V

All the preliminary arithmetical operations which went to the construction of the Tables were performed on differences of curve ordinates expressed in millimetres or minutes of arc, the unit 1×10^{-10} erg being introduced only in the final operation, thus an absolute identity was not to be expected between the differences between A_1 and A_1' just calculated, and those derived in Tables IX and X by taking the actual means of the 24 hourly values

It may be well to add that a calculation was made to ascertain the order of magnitude of the corrections that would have been applied if an attempt had been made to allow for the "activity" present on quiet days The calculations were applied to the *mean* diurnal inequalities for December as given by five selected quiet days a month,

The Kew data were for the 11 years 1890–1900 combined, the Eskdalemuir data for the single year 1914 The means of the 24 hourly values thus found for A_1' (Bidlingmaier's A_d^h) were in the usual unit

At Kew, for H, 0 28, for D, 1 02, for V, 0 10, total for 3 elements 1 40,
At Eskdalemuir, for N, 0 45, for W, 0 66, for V, 0 07, total for 3 elements 1 18

No data were available for the calculation of A_2 (or A_h^z), but the contributions from A_2 on a quiet December day must be very small If the quiet days had been treated individually, larger mean values would have been found for A_1'. But, if we may judge from Bidlingmaier's data for January, 1911,* the whole corrections with the quiet days treated individually would not have exceeded double the figures given above, and the largest correction for any individual hour would have been only from three to four times the mean The effect of any such corrections in Tables IX and X would have been insignificant

The calculations were carried to the first decimal place, but it has been omitted in all the values of the total "activities," as well as in all individual cases where the value of A_2, A_1, or A_1' exceeded 100 Results are given for the two horizontal components combined, as well as for the sum from the

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* *Loc cit.*, p 29

Table IX —" Activity " at Kew from 8 h December 16 to 8 h December 17, 1917
(Unit 1 × 10^{-10} erg per cubic centimetre)

Hour G M T	H			D			V			H+D					H+D+V				
	A_2	A_1	A_1'	A_2	A_1	A_1'	A_2	A_1	A_1'	A_2	A_1	A_1'	A_2+A_1	A_2+A_1'	A_2	A_1	A_1'	A_2+A_1	A_2+A_1'
8—9	1·3	3·2	0·6	0·2	2·5	67·0	1·5	2·5	190	1·6	5·8	67·6	7	69	3·0	8·3	257	11	260
9—10	6·2	1·4	14·4	2·4	0·2	88·0	0·8	6·7	218	8·6	1·6	102	10	111	9·4	8·3	321	18	330
10—11	7·5	4·0	0·4	0·9	2·5	67·0	0·9	6·7	218	8·4	6·5	67·3	15	76	9·3	13·3	285	23	295
11—12	5·9	46·1	17·6	0·6	4·0	57·5	2·0	1·0	174	6·5	50·0	75·1	56	82	8·5	51·0	249	60	257
12—13	1·7	5·7	0·0	1·4	6·7	48·8	1·4	1·4	116	3·1	12·5	48·8	16	52	4·5	13·9	165	18	170
13—14	5·4	57·5	24·9	0·6	0·2	84·3	1·2	7·8	84·3	6·1	57·7	109	64	115	7·3	65·5	194	73	201
14—15	12·8	224	153	2·9	1·4	116	2·2	19·3	57·5	15·8	226	269	241	285	18·0	245	327	263	345
15—16	13·8	195	129	10·4	46·1	7·8	15·1	91·8	6·7	24·3	241	137	266	162	39·4	333	144	372	183
16—17	8·9	684	555	4·7	116	1·4	12·7	518	112	13·5	800	556	813	570	36·3	1318	668	1354	704
17—18	23·3	398	301	10·3	159	7·8	2·0	1206	518	33·7	557	309	590	343	35·7	1763	827	1799	863
18—19	29·7	201	134	10·6	212	24·9	14·9	1348	603	40·3	413	159	453	199	55·2	1762	762	1817	817
19—20	42·7	9·0	31·2	6·0	555	195	4·9	1320	593	49·7	564	226	613	276	54·5	1883	819	1938	874
20—21	29·6	51·6	95·7	2·3	770	330	3·9	827	274	31·9	821	426	853	457	35·8	1648	700	1683	736
21—22	109	1220	1408	49·1	545	190	17·5	265	17·6	158	1768	1598	1924	1756	176	2033	1615	2209	1791
22—23	44·6	398	509	7·7	168	11·5	1·5	139	0·2	52·3	567	520	619	573	53·8	705	520	759	574
23—24	10·8	17·6	46·1	0·9	196	19·3	1·7	255	14·4	11·7	213	65·3	225	77	13·4	468	79·7	481	93
0—1	1·1	4·8	0·2	0·7	168	11·5	5·2	148	0·0	1·8	173	11·7	175	13	7·1	321	11·7	328	19
1—2	4·4	1·4	2·0	1·8	190	17·6	8·2	48·8	26·9	6·1	191	19·5	197	26	14·4	240	46·5	254	61
2—3	51·6	184	121	17·5	104	0·4	19·9	2·5	190	69·1	288	121	357	190	89·0	290	311	379	400
3—4	5·9	0·6	3·2	1·8	139	4·8	14·1	14·4	255	7·7	139	8·0	147	16	21·8	154	263	175	285
4—5	3·7	0·6	3·2	0·2	83·7	2·5	10·8	119	0·2	3·9	84·4	5·8	88	10	14·7	203	5·9	218	21
5—6	1·6	12·9	1·0	0·8	73·7	1·0	1·1	190	2·5	2·4	86·6	2·0	89	4	3·5	276	4·5	280	8
6—7	0·3	2·6	1·0	0·1	38·3	12·9	0·7	330	35·9	0·4	40·8	13·9	41	14	1·2	371	49·8	372	51
7—8	0·4	0·0	7·8	0·7	31·2	15·9	0·2	382	54·5	1·1	31·3	23·7	32	25	1·3	414	78·3	415	80
Mean for " day	17·6	155	148	5·6	150	57·6	6·4	303	157	23·3	305	206	328	229	29·7	606	363	637	392

Table X.—" Activity " at Eskdalemuir from 8 h December 16 to 8 h December 17, 1917.
(Unit 1×10^{-10} erg per cubic centimetre)

Hour G M T	N			W			V			N+W					N+W+V				
	A_2	A_1	A_1'	A_2	A_1	A_1	A_2	A_1	A_1'	A_2	A_1	A_1'	A_2+A_1	A_2+A_1'	A_2	A_1	A_1'	A_2+A_1	A_2+A_1'
8—9	2·0	3·2	51·6	0·7	0·4	0·2	0·1	1·4	46·1	2·7	3·6	51·8	6	54	2·8	5·0	97·8	8	101
9—10	19·0	12·9	2·5	4·8	2·0	0·4	0·2	4·0	57·5	23·8	14·9	2·9	39	27	24·0	18·8	60·4	43	84
10—11	19·5	6·7	7·8	1·0	0·6	3·2	0·1	6·7	63·7	20·5	6·4	11·0	27	32	20·6	13·1	74·8	34	95
11—12	1·4	4·8	9·0	4·6	15·9	24·9	0·2	7·8	70·3	6·0	20·9	33·9	27	40	6·3	28·6	104	35	110
12—13	2·6	0·6	19·3	2·6	0·2	1·4	0·2	1·0	40·8	5·4	0·8	20·7	6	26	5·5	1·8	61·5	7	67
13—14	2·4	0·0	39·0	5·0	29·0	38·3	0·0	1·0	43·4	7·3	29·1	67·3	36	75	7·3	30·1	111	37	118
14—15	7·7	9·0	70·3	14·3	163	194	1·0	0·2	26·4	22·0	172	255	194	276	23·0	172	281	195	304
15—16	30·8	19·3	0·6	18·7	84·3	100	39·0	121	29·0	49·5	104	100	153	150	38·5	224	129	313	218
16—17	64·9	31·2	121	41·6	583	623	145	1499	1098	107	615	743	721	850	253	2114	1841	2366	2093
17—18	131	694	1007	115	555	593	3·1	3514	2904	246	1249	1600	1495	1846	249	4763	4504	5012	4753
18—19	236	593	873	24·2	465	500	8·0	3305	2888	260	1058	1372	1318	1633	268	4362	4065	4631	4334
19—20	90·1	0·2	22·9	86·1	1·4	0·2	23·9	3101	2530	176	1·6	23·1	178	199	200	3103	2553	3303	2753
20—21	9·7	545	330	50·2	168	148	52·5	995	673	60·0	714	478	774	538	112	1708	1151	1821	1264
21—22	326	970	683	143	2530	2431	239	116	268	469	3500	3094	3968	3563	708	3616	3362	4324	4070
22—23	64·3	158	54·5	45·1	603	564	24·7	390	632	109	761	619	870	728	134	1151	1251	1285	1385
23—24	2·0	121	31·2	8·5	73·7	60·6	11·4	22·9	108	10·6	194	91·8	205	102	22·0	217	200	239	222
0—1	1·9	121	31·2	1·4	7·8	4·0	0·5	0·0	29·0	3·3	128	35·2	132	39	3·8	128	64·3	132	68
1—2	11·1	249	112	8·8	29·0	21·1	4·7	2·5	51·6	19·9	278	133	298	153	24·5	280	185	305	209
2—3	34·0	933	643	44·9	0·0	0·6	227	861	1220	78·9	933	643	1012	722	306	1794	1863	2099	2169
3—4	42·9	360	184	9·0	29·0	19·3	119	885	1248	51·9	389	203	441	255	171	1273	1452	1444	1623
4—5	0·9	17·6	1·0	4·6	9·0	4·8	5·5	88·0	218	5·5	26·5	5·8	32	11	11·0	115	224	126	235
5—6	0·8	43·4	2·0	1·2	0·4	0·0	1·0	12·9	84·3	2·0	43·7	2·0	46	4	3·0	56·7	86·3	60	89
6—7	0·2	17·6	1·0	0·3	2·5	0·6	0·4	1·4	46·1	0·5	20·1	1·6	21	2	0·8	21·4	47·7	22	49
7—8	0·6	19·3	0·6	0·2	6·7	3·2	0·1	0·0	29·0	0·8	26·0	3·9	27	5	0·9	26·1	32·9	27	34
Mean for ' day	45·9	205	178	26·5	223	222	37·8	622	592	72·4	429	400	501	472	110	1051	992	1161	1102

three components, because it was the only way of intercomparing the Kew and Eskdalemuir results from the horizontal plane, and because Bidlingmaier omitted vertical force

The relative importance of A_2 and A_1' is of special interest in connection with the question of the calculation of A_2, a very laborious process when carried out strictly according to the formula If we express A_2 as a percentage of A_2+A_1' in the case of the mean values for the day, we obtain the following results —

At Kew, in H, 11, in D, 9, in V, 4, in H+D, 10, in H+D+V, 8,
At Eskdalemuir, in N, 21, in W, 11, in V, 6, in N+W, 15, in N+W+V, 10

With these we may compare Bidlingmaier's mean results at Wilhelmshaven for January to June, 1911, viz, in H, 24, in D, 21, in H+D, 22 It would appear, however, that most of Bidlingmaier's results were derived not from actual measurement of $(1/10)\Sigma\eta^2$, but from measurements of the hourly ranges, and certain numerical relationships which he believed to hold between the value of A_h^x and the square of the corresponding hourly range In a recent paper,* I arrived at the conclusion that Bidlingmaier's relationships were not generally satisfactory If they had been accepted in the present case, we should unquestionably have got a higher mean value for A_2. If $A_2/(A_2+A_1')$ were always as small as the above values for Kew suggest, it could not be claimed that any very high degree of accuracy is essential in the method of calculating A_2

Bidlingmaier found that on the average day the "activity," whether A_2, A_1, or A_1', was considerably greater for D than for H On the present occasion, the "activity" at Kew, especially A_2 and A_1', is conspicuously larger for H than for D The A_2 and A_1' results for N and W at Eskdalemuir differ less, which suggests that the horizontal direction in which the changes of force were most active made a fairly close approach to the magnetic meridian

In temperate latitudes the range of the regular diurnal inequality in V is considerably less than in either horizontal component Thus, undoubtedly, on the average day at any European station, the contribution from V will be small compared with the sums of the contributions from the horizontal components But, on the present occasion, the contribution from V to A_2+A_1 or A_2+A_1' at Kew is nearly equal to the combined contributions from H and D, while at Eskdalemuir the contribution from V considerably exceeds that from N and W combined. This shows that the question of the exclusion of V data from an international scheme possesses an importance which is not suggested by a study confined to ordinary days

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* 'Terrestrial Magnetism and Atmospheric Electricity,' vol 22, p 57 (1917).

§ 9. It is interesting to note, confining ourselves to the results for H + D that the mean value of $A_2 + A_1'$ at Kew on December 16–17 was 11·8 times that of the average day from January 1 to June 30, 1911, at Wilhelmshaven, and 15 times that of the average January day Also, the mean value of $A_2 + A_1'$ at Kew between 21 h and 22 h. on December 16 was 52 times that of the corresponding hour of the average January day at Wilhelmshaven It is obvious that, when we confine ourselves to data from a single year, a few highly disturbed days might exercise a swamping influence in statistics as to the annual or diurnal variation of magnetic "activity"

During the earlier part of the storm the "activity" at Kew was not inferior to that at Eskdalemuir, but after 16 h on the 16th the Eskdalemuir "activity" was much the greater At both stations, so far as the horizontal field is concerned, 21 h–22 h on the 16th was decidedly the most disturbed hour This was the time at which aurora was brightest, according to reports from meteorological stations, several Scottish stations reporting a corona not far from the zenith Kew and Eskdalemuir also agree in showing a maximum of "activity" in V between 17 h and 20 h on the 16th, and a marked recrudescence of "activity" between 2 h and 4 h. on the 17th, which was presumably associated with auroral streamers observed at Southport between 2¼ h and 2¾ h

§ 10. In view of the great labour involved in applying Bidlingmaier's scheme, and the various uncertainties, I suggested in 'Terrestrial Magnetism' (*loc cit.*) that a much simpler scheme, if applied at the numerous observatories which participate in the present international scheme, might suffice for the magnetic classification of days. The suggestion was that, if R_h, R_d, R_v represent the absolute ranges in the three magnetic elements, $R_h^2 + R_d^2 + R_v^2$, or, if the horizontal field only is considered, $R_h^2 + R_d^2$ might be accepted as a measure of the mean "activity" for the day If the horizontal components recorded were N and W, then $R_n^2 + R_w^2$ would take the place of $R_h^2 + R_d^2$

I have compared the mean monthly values of the "activity" at Wilhelmshaven for three of the six months dealt with by Bidlingmaier, with the corresponding arithmetic means of the squares of the Kew daily ranges. The following were the results obtained —

	January	March	May	Mean
$\dfrac{\text{Mean value of } A_2+A_1' \text{ at Wilhelmshaven}}{\text{Mean value of } R_h^2+R_d^2 \text{ at Kew}}$	0·000196	0·000224	0·000194	0·000205
$\dfrac{\text{Mean value of } A_2+A_1 \text{ at Wilhelmshaven}}{\text{Mean value of } R_h^2+R_d^2 \text{ at Kew}}$	0·000221	0·000251	0·000235	0·000236

Taking now the "activities" and ranges found for December 16–17, we get the following results:—

At Kew—

$$(\text{Mean } A_2+A_1' \text{ for } H+D)/(R_h^2+R_d^2) = 0{\cdot}000113,$$

$$(\text{Mean } A_2+A_1' \text{ for } H+D+V)/(R_h^2+R_d^2+R_v^2) = 0{\cdot}000146,$$

$$(\text{Mean } A_2+A_1 \text{ for } H+D)/(R_h^2+R_d^2) = 0{\cdot}000162,$$

$$(\text{Mean } A_2+A_1 \text{ for } H+D+V)/(R_h^2+R_d^2+R_v^2) = 0{\cdot}000237.$$

At Eskdalemuir—

$$(\text{Mean } A_2+A_1' \text{ for } N+W)/(R_n^2+R_w^2) = 0{\cdot}000065,$$

$$(\text{Mean } A_2+A_1' \text{ for } N+W+V)/(R_n^2+R_w^2+R_v^2) = 0{\cdot}000106,$$

$$(\text{Mean } A_2+A_1 \text{ for } N+W)/(R_n^2+R_w^2) = 0{\cdot}000069,$$

$$(\text{Mean } A_2+A_1 \text{ for } N+W+V)/(R_n^2+R_w^2+R_v^2) = 0{\cdot}000112.$$

The difference between the above figures for the two stations arises from A_1' and A_1. If we confined ourselves to A_2 we should get in place of the above ratios: For the horizontal components, 0·000012 at Kew, 0·000010 at Eskdalemuir; for all three components, 0·000011 at both places.

The means of the absolute daily ranges for the whole year 1914 were:

At Kew, 44γ in H, and 55γ in D;

At Eskdalemuir, 60γ in N, and 58γ in W.

This makes

$$(R_n^2+R_w^2)/(R_h^2+R_d^2) = 1{\cdot}40.$$

Wilhelmshaven being very nearly midway as regards latitude between Kew and Eskdalemuir, the absolute ranges at Wilhelmshaven probably exceed those at Kew. Thus if we had employed Wilhelmshaven instead of Kew ranges for comparison with the Wilhelmshaven "activities" of 1911, we should most likely have got smaller values than those obtained above for the ratios.

The scheme which I suggested does not postulate that the ratio borne by "activities" to squares of ranges should be the same for all stations. It might well be different at a quiet station like Helwan, and at a disturbed station like Sitka. Everything considered, however, it is probable that on any scheme which assumed "activity" proportional to the square of the absolute range, the mean "activity" for December 16–17 would have been sensibly over-estimated at Kew, and considerably over-estimated at Eskdalemuir. On the other hand, the large difference between the values of A_1 and A_1' in Table IX, and the uncertainty this implies, shows that even with the elaborate procedure entailed by Bidlingmaier's full scheme, there is a large probable error at individual stations.

Under these circumstances a simple scheme, even if admittedly imperfect, deserves consideration.

A Method of Avoiding Collision at Sea.

By J. Joly, Sc.D., F.R.S., a Commissioner of Irish Lights.

Part I.

(Received May 8, 1918.)

The following method of avoiding collision at sea depends on the use of synchronised signals, transmitted in different media. Such signals, travelling at different rates, enable the distance of their source to be inferred by observation of the gain in time of the faster upon the slower travelling signal. Thus, if signals be simultaneously emitted by wireless and by submarine bell (or Fessenden oscillator), the former being transmitted with practically infinite velocity, the latter arrive with a lag which is the time the submarine sound requires to traverse the intervening medium. The rate of propagation of sound in water being closely 4800 feet per second, the lag is 0.62 second for one-half sea-mile.

In practice the signals may be so ordered as to dispense with the stop-watch or chronograph. This is accomplished by sending out the wireless ticks in groups of, say, 20 "dots" spaced to intervals of 0.6 second. The stroke of the bell precedes the first of these dots by one of these intervals. Thus, when the sailor is half mile from the source he hears the first wireless dot along with the bell stroke. If he is 1 mile distant the bell stroke comes in with the second dot, and so on. He has, in fact, only to count up the dots till he hears the bell, and the number of the dot coincident with the bell is the number of half sea-miles intervening between his ship and the source of the signals. It is possible to estimate the quarter mile by noting a want of coincidence between bell stroke and dot. This method of estimating distance is in actual operation in assisting mariners to navigate the approach to New York Harbour, the signals being emitted from the Fire Island Light Ship It is of special value in coastal navigation.

I shall assume that readings can be effected no more accurately than described above—although more exact and yet practical methods of determining the lag of the sound wave are probably feasible. In what follows, however, the reading to the quarter sea-mile will be assumed as the limit of accuracy attainable, in determining the distance between ship and ship The emission of the signals may be entirely automatic, a mechanically driven contact maker accurately spacing the signals This would be in charge of the wireless operator on board, and in fog, thick weather, or darkness, would be set in operation according to (future) Board of Trade regulations

In the successful working of the method now to be described these synchronous signals are all sufficient. The groups of signals might be spaced half a minute apart That is at the beginning of each half minute the emission of 20 wireless dots of such low power as to avoid unnecessary distance would be commenced The group would finish in 12 seconds There would then be a pause of 18 seconds, when a second group would be started

I have already suggested a method* based on synchronised signals, and involving a knowledge of the course and speed of each ship, each vessel transmitting by wireless code to the other ship information as to her own course and speed The present suggested method does not involve this knowledge, nor any other communication between the ships beyond the synchronised signals described above.

The rate of mutual approach of ships which are moving so as to collide is constant and remains so to the moment of collision If they are advancing so as to pass clear, the rate of mutual approach diminishes as the vessels draw near one another, beomes zero when the vessels are at the least distance apart, and then becomes negative. I shall assume for simplicity of treatment that the relative velocity of the ships is transferred entirely to one of the ships, which I shall call B, the other ship, A, being stationary. The suggested method of avoiding collision involves the discrimination of what may be called the constant danger rate of approach from the variable rate denoting safety

Let the initial distance of B be 10 nautical miles or knots. The sailor on A is not aware of the bearing of B He detects her synchronised signals (which at such a distance would be those of a Fessenden oscillator and low power wireless), and observes the decrease in the distance of B indicated by the successive signals. His primary and essential purpose is to ascertain *if there* is a diminution in this loss of distance, as shown by succeeding signals

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* 'Roy. Soc. Proc.,' A, vol. 92, p. 252 (1916).

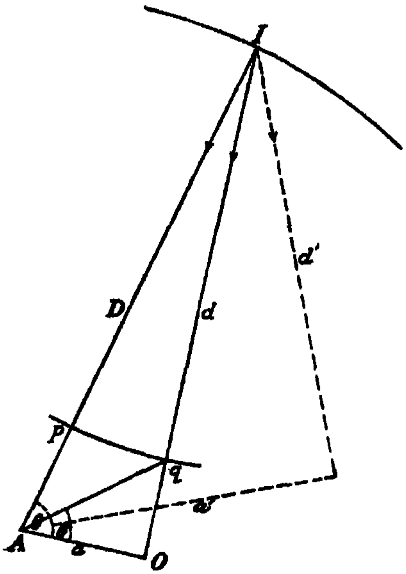

FIG 1

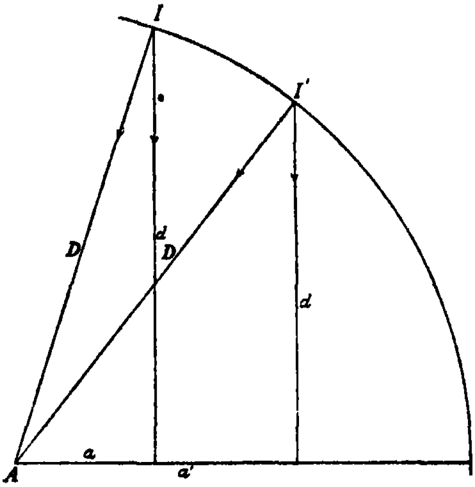

FIG 2

A may be regarded as placed in the centre of a circle of, say, 10 miles radius, upon the circumference of which B is somewhere placed, say at I Then, if D is the radial distance of B from A, D is the danger course of B Let d be an alternative course of B which passes clear of A at the minimum distance, or clearance AO $= a$ Then a is perpendicular to d, and $a \sec \theta = \mathrm{D}$ Assuming some value of a, the angle θ is known, and $d = a \tan \theta$

B may be advancing along D or along d The sailor can discriminate between these courses when B has moved a distance Ip = Iq from the initial point I, such that he can detect with certainty a difference in the distance of B according as to whether she is at p or at q. For if B is on the danger course, Ap is the residue of the 10 miles, if she is on the safety course her distance must be greater than Ap. B at successive instants is all along further from A on the safety than on the danger course. The point q is such a point as permits this fact to be estimated by the synchronised signals Now $q\mathrm{O} = d - \mathrm{I}p$, and $q\mathrm{O}/a = \tan \theta'$, giving θ' Then A$q = a \sec \theta'$ Accordingly the comparison is between the distance Ap and $a \sec \theta'$

It is evident that with increasing values of a the difference between Ap and $a \sec \theta'$ increases, the latter distance, for a given value of Ap, increasing as θ' diminishes The discrimination between safety and danger can, therefore, be effected at an earlier moment for a wide clearance of A and B than for a "close shave"

I have described the most general case where the course of B may radiate from the initial point I in any direction such that she tends to approach A. This construction holds till a becomes equal to D, when her course is tangential to the circle at I, and she continues all the time to recede from A

The various courses which may be run by B as radiant from I, such as d or d', fig 1, may also be considered as initiated at points on the circle so placed that the nearest approach to A is measured along a common radius, as in fig 2. The distances shown by the successive signals will be the same in this as in the more general case for any given value of a or θ, but the compass bearings of B from A will be different in the two cases In short, it is evident that the triangle IOA may be supposed rotated into any position round A, the problem remains essentially the same

The following Table shows the distance of B from A corresponding to the danger or direct distance shown in the first column The successive columns after the first refer to various values of a, or the least passing distance between the vessels The initial distance is taken as 10 miles.

Table I —Danger and Safety Distances (Initial Distance 10 Miles)

Danger Distances	Safety Distances Miles						
	$a = \frac{1}{2}$	$a = 1$	$a = 2$	$a = 3$	$a = 4$	$a = 5$	$a = 6$
miles							
8	8 0	8 01	8 05	8 11	8 21	8 33	8 49
6	6 00	6 08	6 18	6 30	6 52	6·83	7 21
5	5 00	5 05	5 20	5 44	5 77	6 20	6 71
4	4·02	4 07	4 30	4·64	5 10	5·66	6 32
3	3 08	3 11	3 44	3·93	4 55	5 27	6·08
2	2·02	2 19	2·69	3 37	4 16	5 04	6·00
1	1 11	1 38	2 15	3·05	4·00		
½	0 70	1 10	2 02	3 00			
⅓	0 59	1 04	2·00				
¼	0 55	1 02	2·00				

At a passing distance of ½ mile B must approach A to within ½ mile measured on the radial or danger course, before any indication of her real course would be obtained. At ½ mile distance the danger and safety distances differ by 0 20 mile. This is nearly ¼ mile, and some indication of want of coincidence in the signals would be detected by an experienced observer At ⅓ mile the increase of distance on the safety course is more than ¼ mile, and should also be detected At ¼ mile the difference is

0·30 mile, and the want of coincidence in the signals could not be missed. We may say, then, that, given a reliable transmitting apparatus, timing the signals with accuracy (a condition perfectly attainable), and a careful observer, alert and practised in listening to such signals, there should be no need to alter course for a passing distance of ½ mile I describe further on certain aids towards accuracy, which should render this quite certain

Looking to the next column, giving the distances for the course passing at 1 mile from A, we see that already at 2 miles there must be want of coincidence in the signals, and at 1 mile this must be quite conspicuous. *At ½ mile the sound signals are displaced by one radio dot, and only an* error in the count could lead to its being overlooked The aids towards accuracy just referred to will be found to eliminate the possibility of this error, the counting being rendered unnecessary

Looking along the successive columns of the Table, we perceive that the difference between safety and danger becomes ever more conspicuous, and may be detected earlier as the value of the passing distance a increases With a passing distance of 2 miles, when the ships are 3 miles, or even 4 miles, apart, the fact of safety should be apparent At 1 mile apart the difference between danger and safety increases to two radio dots. For 3 miles passing distance two radio dots measure the difference of safety and danger at 2 miles, and so on

If we plot these results, the distances, as determined by synchronous signals, being taken as ordinates, and the danger distances or successive units of time as abscissæ, we obtain the group of curves shown in fig 3 The diagonal right line is the curve for danger Above it, and ever more differentiated from it, lie curves for increasing values of a The slope of the danger curve—*i e* the rate of approach or relative velocity—is constant for this line only. In the other curves the relative velocity diminishes, first slowly, then more rapidly, and finally decreases to zero, when the curve becomes parallel to the axis of x

It is evident that, if the sailor plots his observations, he would obtain one of these curves, the precise slope at any point depending upon the value of a and the relative velocity of A and B

We have considered the case of the signals of B being first heard at a considerable distance from A. As B draws near, the divergence between the danger course and the safety course separates more and more the two possible loci of B. When, therefore, B has travelled some considerable distance from the initial point, say 2 miles, the value of $a \sec \theta'$ attains a certain excess above the value of $D - Ip$. In this excess the discrimination of safety and danger is founded. But, suppose, now, that the distance at which the ship B is first heard is considerably less than 10 miles, say 5 miles, we have to ascertain if a mile run either on danger or safety

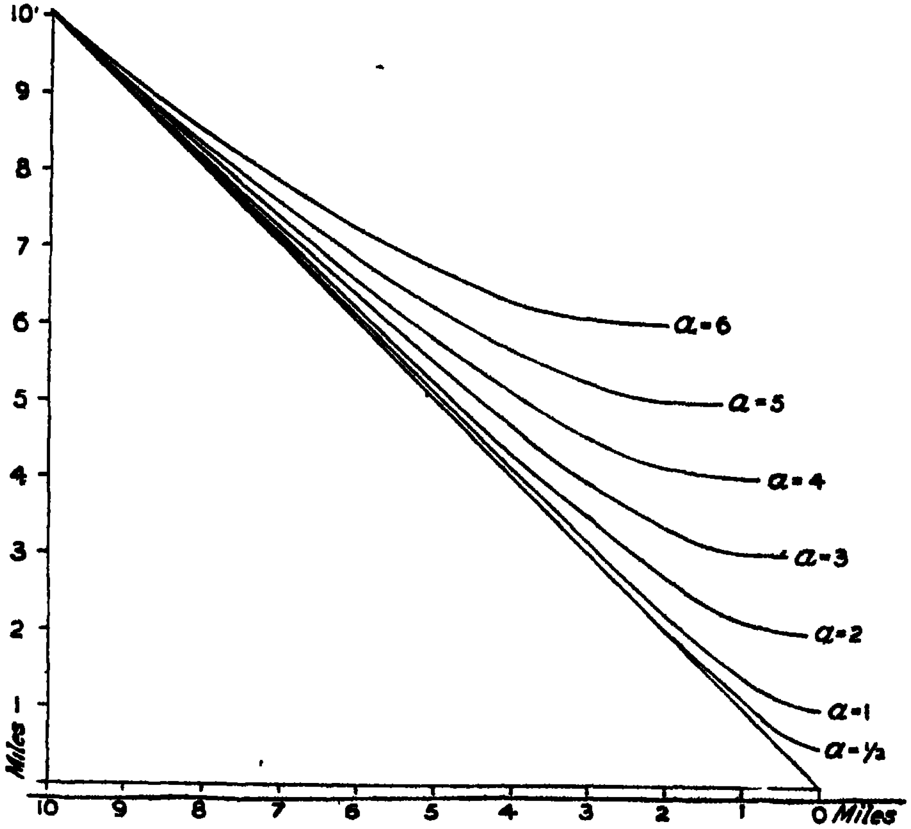

FIG 3 —Safety and Danger Distance

course (keeping the same value of a) affords an equally good basis for discrimination between the courses There is, in fact, a certain loss of discriminating distance in this case. The following Table (II) assumes an initial distance of 5 miles and various values of a. If we compare this with Table I, we see that the excess of safety over danger distance is less, although not much less. Five miles may be taken as a fair value for the initial distance where submarine bell is used But the initial distance might be even less Table III, calculated for a value of a of 1 mile and initial distances of 10, 6, and 3 miles, shows that there is but little difference when a is small The reason for the difference attending greater and lesser initial distances may be most briefly referred to the fact of the lesser separation of the danger and safety loci for a given run on the two possible courses when the start of B is supposed to be close to A. At increased values of *a* this effect is diminished.

Table II.—Danger and Safety Distances (Initial Distance 5 miles)

Danger Distances	Safety Distances Miles			
	$a = 1$	$a = 2$	$a = 3$	$a = 4$.
miles				
4	4 03	4 10	4 24	4 47
3	3 07	3 27	3 61	4 12
2	2 15	2 55	3 16	4 00
1	1 85	2·08	3 00	4 12

Respecting large values of *a*, it will be apparent that these insure in all cases an early discrimination between safety and danger. For the observation must then reveal at an early stage, and while there is still a large separation between the vessels, the fact of safety by attainment of the zero rate of approach. We are therefore, in judging the value of the method, more concerned with the discriminating distances for small values of *a* than for large.

Table III.—Influence of Initial Distance ($a = 1$ mile)

Danger Distances	Initial Distances		
	10 miles	6 miles	8 miles
miles			
6	6 08	6·0	
4	4·07	4 04	
2	2 19	2 16	2 08
1	1 88	1 36	1 30
½	1 10	1·08	1·05
¼	1 02	1 01	1 00

From the foregoing Tables it is, I think, evident that a practical method of avoiding collision is to be found in the simple observation of distances separating the vessels. The method possesses the very great advantage over the old method by observation of bearing, that the distance separating the ships, which is the really important element in all cases, is being watched throughout in the mere taking of the observations.* We have to consider now how the method may best be applied so as to diminish the danger of errors and increase its sensitiveness so far as practicable

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* *Note added in the Press.*—Observation on A of ordinary sharp sound signals emitted by B at uniformly spaced, regulated intervals, would suffice to determine, by Döppler's principle, whether the ships were receding from, or approaching, one another; an estimate of the relative velocity could be made, and it could be ascertained whether this

There is no doubt that a practised operator would readily estimate the distances to the ¼ mile by simply counting up the radio dots and observing whether there was coincidence between bell stroke and dot, or whether the former fell between the successive dots

A suitably designed stop-watch would, however, afford him much aid, and effectually guard against errors of counting This stop-watch would carry a central hand, which completed a rotation in exactly ½ minute, or, for a more open scale, ¼ minute. The number of its revolutions, and the minutes or half-minutes, are recorded on a smaller dial The operator starts this watch into action on hearing the first radio dot of a group Thereafter, the watch keeps time with the emission of signals on the other ship. This is no great demand The instruments for sending out signals would, of necessity, be standardised under proper supervision

As the watch keeps time with the emission of the signals on B, it is evident that the operator on A has only to listen for the bell stroke, and read on the watch dial the instant at which they reach him To enable this to be done, the watch carries the usual stop-hand, which moves with the central timing hand, and is stopped by the operator on hearing the bell. It then shows on the dial the position of the timing hand when the bell was heard The dial is divided into divisions corresponding to the time taken for the travel over ½ sea-mile of the sound in water

Thus the whole circumference would be divided into about twenty-five parts, of the value of 0 6 second each, and these would be again subdivided into four or five parts, *i e* reading ⅛ or $\frac{1}{10}$ mile separating the vessels A numbering showing sea-miles and ½ sea-miles is carried round the dial. On the release of the stop-hand it returns to zero, and automatically starts afresh when the timing hand begins a fresh revolution. It will be seen that the operator's duties, in taking in the signals, consist (*a*) in starting the watch, by the first radio dot of a group, into unison with the signals emitted by B, and (*b*) in pressing the stop and reading the dial when the bell strokes come in

In order to apply his readings to the discrimination between safety and danger, he has before him a paper pad or card ruled with a fine, sharp, diagonal line sloping downward from left to right This line is the danger

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velocity remained constant or varied Thus, if the relative velocity was 24 knots, B approaching, sounds in water emitted every 60 seconds by B would appear to observation on A to be spaced 59 5 seconds. This system, however simple it may appear, is under the grave disadvantage of leaving the distance between the vessels indeterminate.

line A horizontal axis beneath is divided into centimetres and millimetres, and the centimetres are numbered from right to left A vertical scale is carried on the edge of a T-square, which moves across the pad, in guides, so that, when moved, the points on the edge of the T-square describe lines accurately parallel with the horizontal. The divisions on the edge of the T-square are numbered to correspond with the numbering on the dial of the watch, *i e* for miles and fractions

When the first distance is read on the stop-watch the T-square is shifted till the corresponding point is upon the sloping line The second distance obtained by observation is treated in the same manner, the ordinates being ruled on by pencil There is now a certain horizontal spacing proportional to the velocity of approach of the vessels, read between the ordinates of these points This distance may be confirmed or checked by a third observation being also plotted in the danger line So far the assumption is that collision is threatened, and, as we have seen, at a considerable distance the rate of approach will differ but little from the danger rate if the value of a is small

The rest is obvious As distance after distance comes in they are plotted to their appropriate ordinates as read upon the T-square and to the abscissæ given by the first two or three observations plotted on the danger line If now collision is coming, these points will continue to lie on the danger line Otherwise the points at the near distances will lie above the danger line, following the course of such curves as are shown in fig 3 The departure of the points from the danger line indicates safety If observations are spaced half a minute apart it would be desirable that an assistant operator should plot the results as they come in

The little trouble of plotting the distances, as described above, may be eliminated by use of a chronograph designed for the purpose Here the diagonal of the danger curve describes a spiral on a cylinder driven by clockwork. A pen travels parallel with the axis of the cylinder Its motion is uniform and describes the length of the cylinder in 12 seconds It marks the surface of the drum when the operator makes a contact on hearing the bell. After completing its course it returns to the base and starts afresh 18 seconds later The travelling pen replaces the stop-watch

The first mark is adjusted to the diagonal line by setting the cylinder round on its axis. The second mark is also so adjusted. The cylinder is now started into motion and automatically rotates through the angular separation of these marks just before the pen starts on its journey The operator, after the first adjustment of the angular displacement of the cylinder, has only to make contact at each bell stroke. Many variants of such a machine may be suggested It is not, in fact, necessary to plot the observations. It is only essentially necessary to ascertain if the distances as given by the earlier observations are maintained Hence any mechanical or optical means which will submit these distances for comparison with those given by subsequent observations will afford the sailor all that he requires

If collision seems threatened some rough indication of the bearing of B is finally necessary in order to ascertain which vessel must give way This may be readily effected by radio-goniometer, using the wireless dots emitted by B or in many cases by aerial sound signal The navigator on A knows now in what direction B is approaching and this enables him to decide whether action is incumbent on him or not

Throughout these operations proceeding on A, similar observations are proceeding upon B A is emitting synchronous signals timed so as not to clash with those emitted by B Thus they would be displaced ¼ minute upon those of the other ship. If necessary a code signal by wireless might pass between the vessels, regulating this procedure

PART II

(Received June 6, 1918)

For the practical application of the method referred to above, it is desirable to so order the emission of the signals that the time interval between the arrival of the signals should be capable of observation with the utmost accuracy attainable To this end the listener must be placed in a state of preparedness for the coming signal. Hence the important signal should be a certain one of a series Thus the important synchronised radio dot might be the third of an equally spaced series of dots, and the bell-stroke, or oscillator blast, the third of three equally spaced signals, or three successive synchronised radio and submarine signals might be issued, the third being preferably taken for observation The observer gets into stride by counting up to the important signal.

An instrument whereby the signals may be emitted and received is shown in fig 4. We are looking at the dial of an accurate clock, having one central hand, which completes a revolution in nearly 15 seconds. This assumes that a ship may be under observation 12 miles off, and that sound in sea-water travels 1 sea-mile in 1 24 seconds. (Good experiments on this important datum, made with such sound-producing instruments as would be used, are to be desired) The complete rotation of the hand, therefore, occurs in the time taken for sound to travel between vessels 12 miles apart. A smaller hand records the number of revolutions.

The clock can be started and stopped by electric contacts At the outer extremity of the central hand is attached a stamp capable of printing a fine radial line about ½ cm. long on the enamelled dial or face of the clock This stamp comes into operation on making a contact As it is necessary to secure a sharp line, and as the extremity of the hand may be moving at the rate of, say, 3 cm per second, the action must be brief To facilitate good definition, the stamp, which I assume charged with aniline ink, sweeps very close above the surface of the dial It may, moreover, be hinged where it meets the supporting hand, so that, during the instant of contact, it is not dragged laterally, but is slightly deflected from the perpendicular The hinge axis is, of course, parallel with the radius of the dial

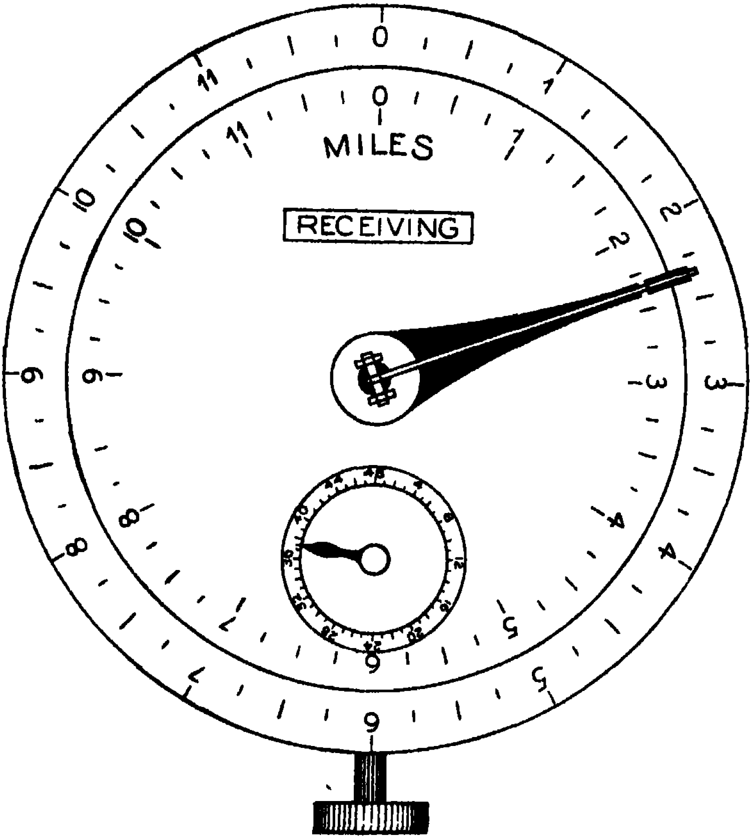

Fig 4

The outer rim of the dial can be rotated smoothly round the inner part on turning the thumb-screw in front A notch and spring normally retain it in its position of coincidence with the central dial. The radial lines struck by the stamp cross the meeting line of the outer and inner dials, about one-half the line appearing on each dial. Both dials are numbered in miles as shown

It will now be understood that this machine acts as a chronograph Upon hearing the third wireless dot the operator by making contact starts the hand into rotation and then listens for the submarine sound signals When these arrive he again makes contact at the third sound signal The stamp then instantly strikes the radial line on the dial For better distinction it might be well to have the permanent subdivisions on the dial in red and the signal records in black or purple The hand continues its uniform rotation and begins a second rotation About 15 seconds have now elapsed

During the second period of 15 seconds sound signals are travelling from ship A to ship B The clock begins to make radio-contacts some time before the hand completes the first rotation First these contacts spell out the characteristic code signal proper to the vessel, then there are three radio dots in succession and the third dot is made just as the hand completes the first rotation, at the instant on which it is crossing the zero line Three submarine bell strokes (or oscillator blasts) are made coincident with the three radio dots Thus, the third submarine signal is emitted along with the third radio dot The third dot and the third submarine signal are the signals which will be used on the other ship, B, for finding distance While the hand of the clock is describing its second revolution the semaphore signal shown on the dial and carrying the word "receiving" is replaced by one with the word "emitting" upon it. During this second rotation the operator on the other ship, B, is taking in the signals sent out by A

During the third revolution the operator on A has only to wait for the sound signal to come in and to again make contact in order to secure the record of the second distance. He now has before him two records of distance If the second shows a greater distance than the first, the other ship is receding and further observations are unnecessary

If, however, the second distance is less than the first, B is approaching, and it is now the business of the observer to watch the rate of approach with a view to finding whether the rate is constant or diminishing, as explained above. And here is where the use of the rotating outer dial comes in. The operator can at any time compare the earlier differences of distance shown by successive readings with the later differences If the lines are sharp the comparison is a sensitive one He brings one of the earlier lines, *i.e.* that part of it which is on the outer dial, into coincidence with a later line on the inner dial The ships may be five miles apart when the earlier line was struck He places the outer line *in directum* with a line on the inner dial which was struck when the ships were, say, two miles apart. Is the next succeeding line on the outer dial now coincident with the next succeeding line on the inner dial? This comparison may be readily made in the time interval—about half a minute—between the bell strokes If there is coincidence the danger of collision still threatens and further observations must be made The outer dial is returned to its original or zero position. Observations can be carried on up to a point when it is no longer safe to defer attending to the rule of the road This point will depend on the rate of approach of the vessels. The bearing of the other ship may now be got roughly by taking in the radio dots on the radio-goniometer or by aerial sound-signal and the procedure as regards altering course or holding on regulated accordingly At the conclusion of the observations the aniline ink marks are removed from the dial by a cloth or sponge

In this system the duties of the operator during a series of observations are confined to taking in the sound signals as they arrive, and to examining these for signs of diminished rate of approach The first takes but an instant to perform The observer may, therefore, give a great part of his attention to the latter There is no need for him to attend to the wireless dots once the clock has been synchronised with that on the other vessel A small error in this synchronisation is of little importance It merely increases or diminishes by a little the distances read throughout The important question as to whether there is a diminished rate of approach or whether this remains constant is unaffected Nor is any extraordinary accuracy of time-keeping required of the clock An accuracy of one minute in the month is easily attained in watches and clocks This would mean a + or − error of less than 1/100 of a second during the five or six minutes in which the observations were in progress

There can be no overlapping or confusion of signals controlled in the manner suggested The synchronisation of clocks would be effected by attention to simple rules of procedure When fog, etc, comes on, every ship must set in motion the apparatus described above This means that every 30 seconds ships A and B are sending out their characteristic code signals, followed by three successive synchronised radio dots and bell-strokes. Now, when A hears B it is the business of the operator on A to secure synchronisation, but as B may have already secured this, he begins by observing his clock when the radio dots of B are coming in If the third radio dot is received when the hand of his clock is just at zero, he knows that B has already synchronised the clocks, and he need not interfere If, however, he observes that there is not this coincidence, or that the emission of his signals are obviously out of symmetry with those coming in, he sets his clock to synchronise with the signals from B, as already described.

The problem presented by the presence of three vessels in an area of mutual audition is deserving of consideration, although it is doubtless one of rare occurrence on the high seas It is, of course, quite possible for all three vessels to hold their courses and speeds, each ship observing the distance of two ships and emitting her own signal It would be easy to do this, using a modified form of the chronograph described above But, in general, it may be assumed that A and B are already synchronised before C comes in, and they may have reached a critical stage in their observations, when interruption might create confusion On the whole, therefore, the safest course would seem to involve that C keep out till A and B are clear This would mean that C emits no synchronised signals, although she might avail herself of those proceeding from A and B The onus of keeping clear would fall on C, and she must slow down if requisite In general, she will early find herself clear of A or B, and need only attend to preserving a certain specified radius of safety respecting one of these vessels She will assist her navigation by taking in the radio signals of A and B on her radiogoniometer It would be allowable for her to emit the usual aerial sound signals

The fundamental condition of safety will always involve that each ship preserve a certain radius of isolation or radius of safety, within which no other vessel must enter Taking the dimensions and speeds of modern vessels into account, ½ mile would not be too great a radius, 1 mile would seem to be better But many points arise in connection with this subject which can only be handled by experienced sailors, and this is one of them

Note on Certain Coloured Interference Bands and the Colours of Tempered Steel

By A. MALLOCK, F.R.S.

(Received April 25, 1918.)

But few people can have failed to notice the equally spaced dark bands which are seen when parallel rows of palings are viewed from a passing train, and which appear to advance at the same speed as the observer.

The general explanation of such bands is obvious, depending, of course, on whether the uprights of the nearest row are in line with those of the more distant, or whether both are visible at the same time. In the first case less obstruction is offered to the passage of light from the background than in the latter, so that the brightest part of the band occurs when, in the line of sight, the uprights of the two rows coincide, and the darkest, when both series are in view at the same time.

In 1874 I examined, with some care, the production of this class of interference bands, and there is a good deal of Italian work on the same subject. If the interfering lines are equally spaced, and parallel to the intersection of the planes in which they lie, the bands are not uniformly spaced, but come closer together as the line of sight departs from the normal to the mean plane. If the lines are not parallel to intersection of the planes, the bands become hyperbolic, and if the surfaces containing the lines are curved, the bands may assume very complex forms. Of this the wood-grain-like figures seen when looking through two folds of muslin or gauze are an example.

If, keeping the eye stationary, the distance between the two folds is altered and the form of the bands is expressed in terms of the distance between them, it will be found the curves lie on a family of quasi-cylindrical surfaces, one within the other, of which surfaces the bands themselves may be considered as plane sections. Hence there is a real analogy between the interference patterns seen through the gauze curtains and the grain patterns of tangential sections of wood, where the annual rings also form quasi-cylindrical surfaces.

Instead of using two sheets of gauze, the single sheet may be placed on looking glass, and the interference observed between the sheet and its reflected image. While making some experiments of this sort in 1874, I noticed that in some cases the bands so produced were rather strongly coloured. I have mentioned this fact to many people, but it does not seem to be generally known, and though the explanation is very simple, it may be worth recording

In fig 1 let AB be the section of a grating of pitch a, composed of opaque lines of width b, with transparent interval of width c, and let these lines cover the surface of a glass plate of thickness T, the back of which is silvered Let μ be the mean refractive index of the glass, and $\Delta\mu$ the difference between the refractive indices for red and violet

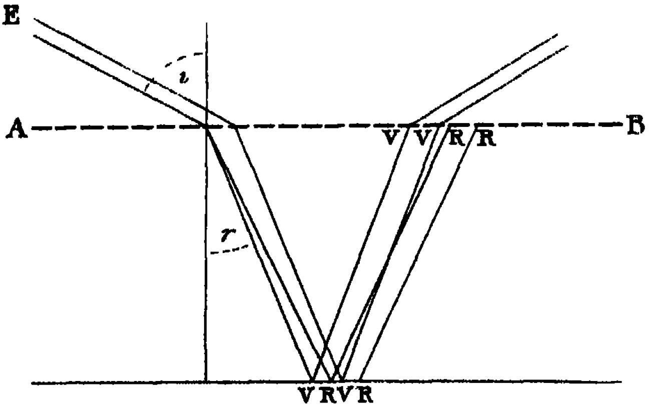

FIG 1

Let a beam of light fall on the grating at an angle of incidence i, and consider the course of the ray which passes the edge of one of the lines The distance L_R between the red and violet rays at the point where, after reflection, they again meet the upper surface is $2T\Delta\mu \frac{\tan r}{\cos^2 r}$, or, in terms of i,

$$L_R = 2T\Delta\mu \frac{\sin i}{(\mu^2-\sin^2 i)^{3/2}}$$

Spectra of this length will be formed by the rays from every point in the width of c, and thus each aperture of the grating will produce a beam of width $L_R + c$ at the place of emergence, part of which will be stopped by one of the opaque lines Hence the emergent light will be coloured The purity of the colour in beam, before emergence, depends on the ratio c/T, but in any case the margins show pure spectrum colours For the formation of the coloured interference bands now under consideration, c should be large compared to the wave-length (in order that diffraction effects may be small), and T should be large compared to c, to secure adequate separation of the colours

If the grating is illuminated by diffused light and is observed from some point E, it will be seen that the combination of glass and grating is rather a crude form of Maxwell's colour box in which the slit used is wide, and the part of the spectrum which is suppressed is equal to the difference between the pitch of the grating and the width of the slit

To estimate the colours which will be seen from E for different values of i, the method used by Maxwell and by Lord Rayleigh in his paper on the colours of these plates* is most convenient The coloured strip formed by dispersion which contains all wave-lengths is to be replaced by three strips of primary violet, green, and red, each of width c, so disposed that their margins are coincident with the proper position which those colours would appear in spectrum formed by the glass by light passing the edge of b

The quantity of each colour not prevented from reaching the eye by b is to be found, and the centre of gravity of weights proportional to these quantities placed at the angles of the chromatic triangle then indicates the colour which will be observed at E for a given value of i

The primary colours used by Maxwell and Lord Rayleigh are —

(1) A violet from between the G and F Frauenhofer lines,

(2) A green close to the E line, and

(3) A red near the C line

A mixture of these in the proportions 30 V, 31 G, and 18 R gives white light

In fig 2 a part of the grating and primary colour strips are shown in plan

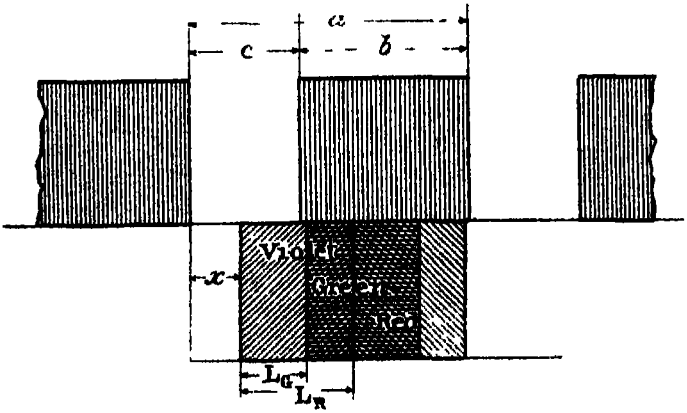

Fig 2

Let x be the distance of the left-hand margin of the violet from the right-hand edge of one of the opaque lines of the grating, L_G+x and L_R+x being the corresponding distances for the green and red

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* See Lord Rayleigh, 'Collected Papers,' vol 2, p 498, *et seq*

In passing from one interference band to the next, x will vary from 0 to a, and it is easy, from Diagram I, to write down the quantity of each colour which reaches the eye for any given value of x.

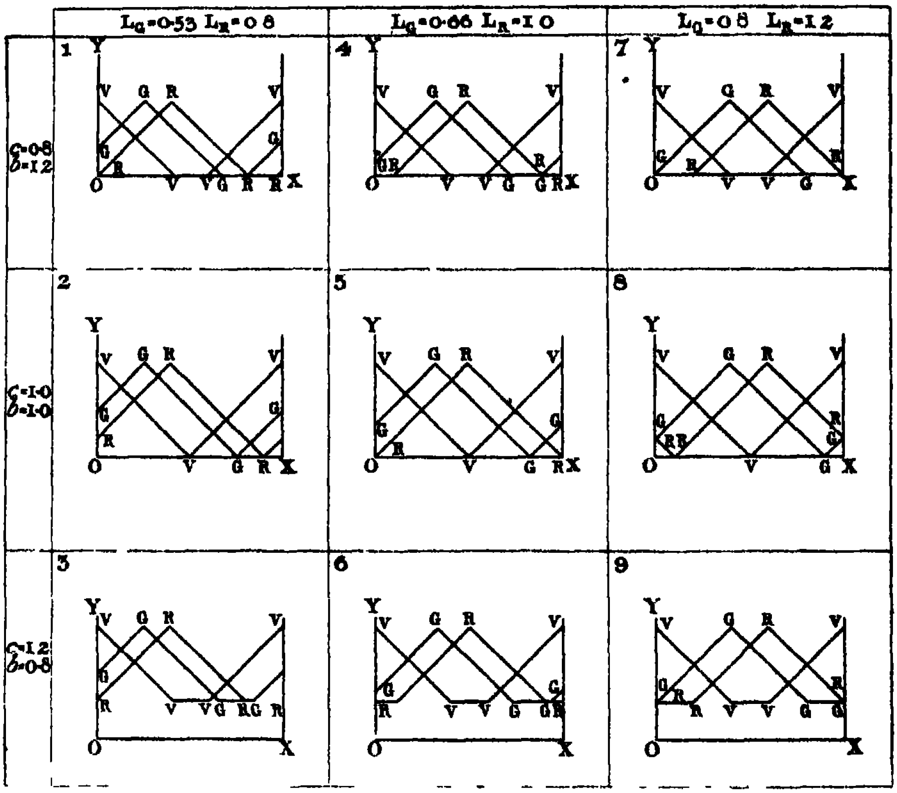

DIAGRAM I —The ordinates (y) of the figures 1–9 represent the quantities of violet, green, and red, which reach the eye at F, in terms of the distance (x) between left-hand edge of the violet (see fig 2) and the right-hand edge of the opaque part (b) of the grating The values of b, c, L_G, and L_R, are given for each line and column of the diagram As the angle of view (i in fig 1) changes from the edge of one interference band to the corresponding point of the bands adjacent, x increases or diminishes from 0 to a or $-a$

Diagram II gives the locus of the centres of gravity of these quantities in the colour triangle, while x varies through one complete cycle The curves are curious and irregular, as might be expected from the discontinuity of the conditions which govern the mixture of the colours, but they all agree in one respect, namely that no green shows itself in any part of the band In fact, green cannot appear in the resultant colour unless the dispersion, or thickness of the plate, is sufficient to separate the primary colour patches more or less completely from one another, and this agrees with observation.

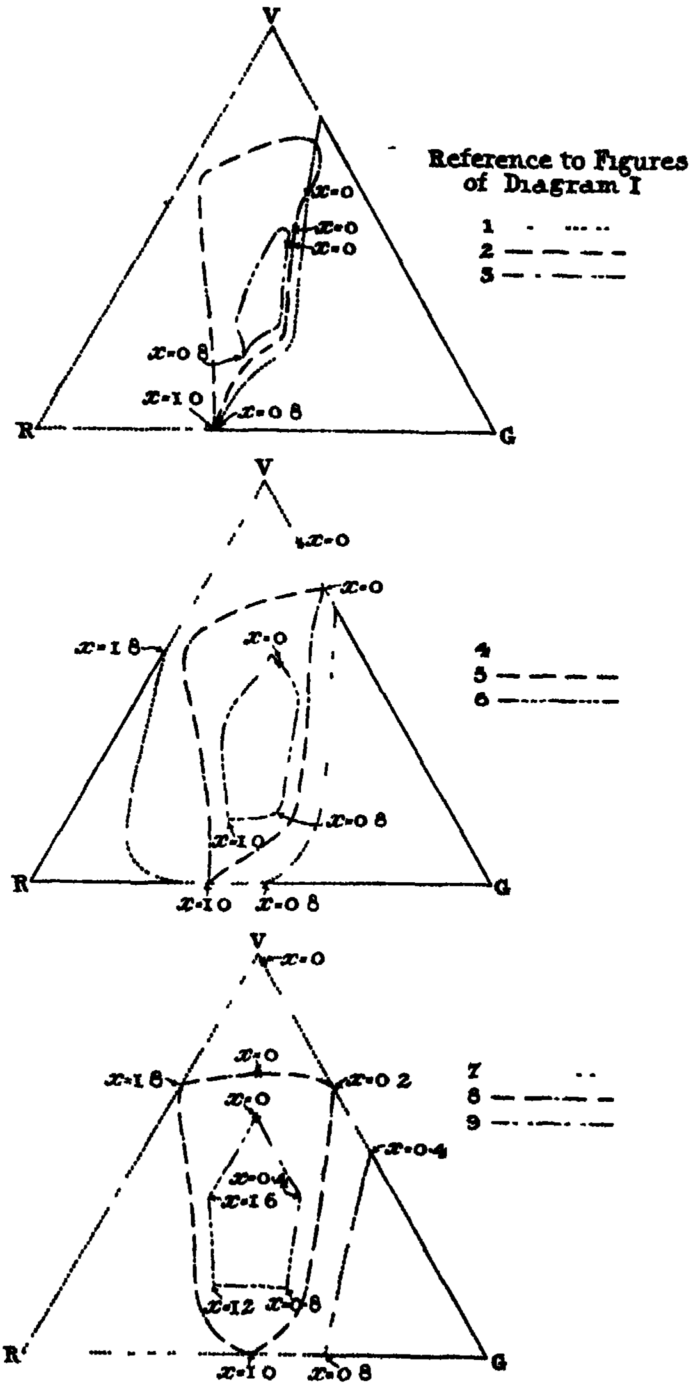

DIAGRAM II.—The curves within the chromatic triangles are the loci of the centres of gravity of weights proportional to the quantities of the primary colours reaching the eye, corresponding to the examples in Diagram II, whilst x varies from 0 to a Values of x are written against the most marked features of the curves.

The actual colours in the cases figured range from good blues through reddish-purple and orange and a straw-yellow to nearly white And in some instances the sequence is so close to the colours of tempered steel that I was led to examine the latter with some care

I had always thought, and I believe it has generally been assumed, that the colours of tempering were instances of the ordinary interference colours of thin plates, but the following simple experiment seems to prove conclusively that this cannot be the true explanation

If the colours were due to a film of appropriate thickness, a reduction of that thickness ought to change the colour, the blue should change to green, orange to yellow and so on I found, however, that if the tempered steel surface was gently polished, until the clean surface of the metal was reached, there was no change of colour during the process, the blue remaining blue, and the yellow yellow, until the whole of the colour was removed The intensity of the colour decreased as the film became thinner, but its character remained the same.

The colours of tempering are best seen by polarised light, and their intensity is greatest at the angle of maximum polarisation When so observed the blue changes, as the angle of incidence increases, through reddish-purple to a dark orange and finally to a straw yellow The yellow and orange parts, on the other hand, change but little, becoming rather more intense at the angle of maximum polarisation, but when the incidence is large, the whole surface which has been coloured by the tempering assumes a nearly uniform yellow

Thus the blue moves towards the red end of the spectrum, while the orange does the opposite, and although similar changes occur in the case of the higher order of Newton's rings, the thickness of plate required is far greater than that which produced the colours of tempering It would seem probable, therefore, that the latter colours are due to some form of selective opacity depending on damped molecular periods comparable with the wave period,* rather than on a structure comparable with the wave-length

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* This must be true of all pigment colours, but something more is required to explain the dependence of the colours of tempering on the angle of incidence, a feature which is strongly marked also in the case of many of the aniline colours when examined as dry films by reflected light

*On the Absorption of X-Rays in Copper and Aluminium.**

By C. M Williams, B Sc , Fellow of the University of Wales

(Communicated by Principal E. H. Griffiths, Sc D , F R.S Received April 20, 1918)

Much experimental work has been directed towards the determination of the absorption coefficients of X-rays in elements—especially with respect to the relation existing between the absorption coefficient and the wave-length—owing to the importance of the bearing of the results on the theories of electromagnetic radiation and atomic structure Nevertheless, on account of the experimental difficulties encountered in this work, serious discrepancies appear among the results of different observers. The method described below appears to offer a reliable and accurate means of measuring the absorption coefficients of homogeneous X-rays in various materials and the wave-lengths of the rays employed

Among the difficulties experienced in the experimental arrangements, two of the chief are —

(*a*) The heterogeneity of the source

(*b*) The variations in intensity of the source

By making use of the reflection of X-rays by crystals, it is possible to analyse a beam of X-rays into its component wave-lengths According to the classical equation of Bragg, we have, if d = spacing of the crystal grating, θ = glancing angle of the incident ray, λ = wave-length of the reflected ray, n = order of the spectrum,

then $$2 . d \sin\theta = n\lambda.$$

It is to be observed that the reflected ray may contain, in addition to the primary wave corresponding to $n = 1$, other waves of lengths $\lambda/2$, $\lambda/3$, etc , *i e*, submultiples of the primary wave-length λ, corresponding to $n = 2$, 3, etc , respectively

In these experiments the interference method of obtaining homogeneous beams was employed, a rock-salt crystal being used to analyse a beam of X-rays generated by a Coolidge tube. The difficulty introduced by the possibility of the reflected beam being mixed with waves of a higher order than the first was obviated in the following way. It has been shown by Duane and Hunt[†] that the voltage V, required to excite a wave of frequency n is given by the equation $eV = nh$ (e = electronic charge,

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[*] An account of research work carried out in the Memorial Physical Research Laboratory of the University College of South Wales and Monmouthshire, Cardiff

[†] 'Phys. Rev.,' vol. 6, No 2, p 169

h = Planck's constant) Thus by carefully adjusting the voltage applied to the tube, keeping it so as to be—for any particular setting of the crystal—just below that required to produce the reflected wave of the second order, homogeneous rays were obtained This principle is illustrated in fig 1, which shows the change in the log-absorption curve for aluminium

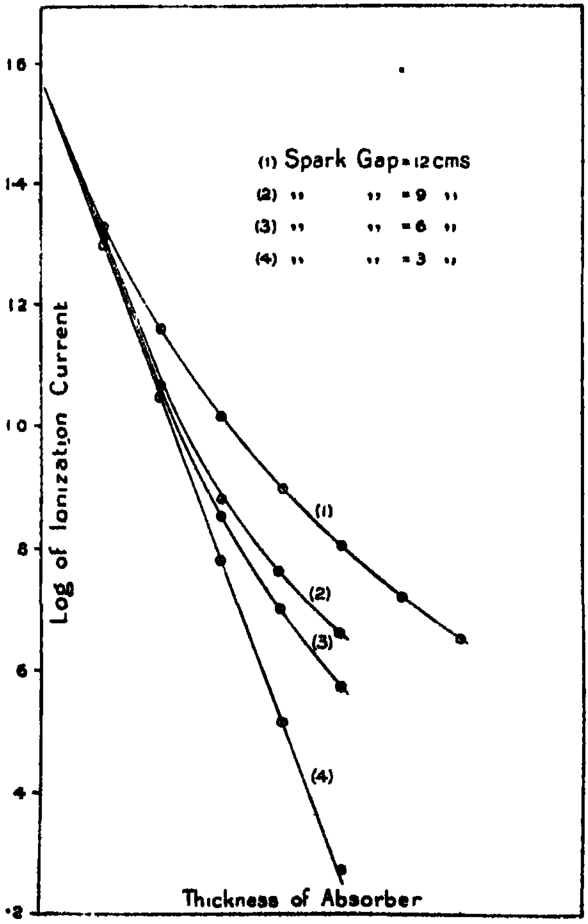

Fig 1

with the variation in the voltage applied to the tube, as measured by the alternative spark-gap between two bright metal spheres 1 cm in diameter, the wave-length of the X-ray was approximately 0·58 Å U The log-absorption curve becomes a straight line (4) when the waves of higher order than the first are eliminated by a sufficient reduction of the voltage.

In order to correct for the variations in intensity, which were pronounced, a special device was used. Instead of employing an ionisation chamber of the ordinary pattern, a double one was constructed.

This consisted of a rectangular metal case AB (fig. 2), divided longitudinally into two by the metal plate CD. Through the upper and lower compartments of this ionisation chamber were passed respectively the insulated electrodes E and F, which could be placed separately in connection with an electroscope. The beam of X-rays reflected from the crystal was thus split into two parts, and the ionisation produced by that entering the lower compartment provided the data to correct for the variations in the beam proceeding into the upper one.

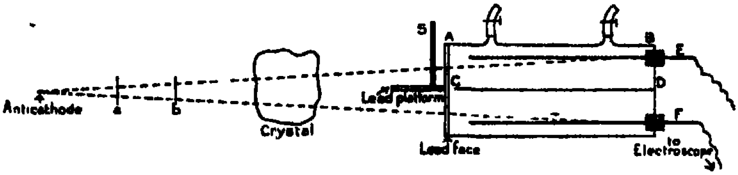

FIG. 2

Thus in making a determination of the absorption coefficient of a substance, the electrodes E and F were first connected to earth and afterwards insulated. The X-ray bulb was then excited for a certain period, at the expiration of which E and F were connected in turn to the electroscope, and the ionisation produced in each chamber measured. Several sheets of the absorbers were placed before the upper slit in succession, and the above process repeated in each case. After making the necessary corrections and reducing all the readings of the upper chamber to a standard reading of the lower, a curve was plotted showing the relation between the log of the ionisation current and the thickness of the absorber. Very consistent results were obtained in this way, the points all lying very evenly on a straight line—indeed, the results for the absorption coefficient obtained in independent experiments rarely varied by more than 1 per cent., or very occasionally by 2 per cent.

As an illustration of the method of working, the actual readings taken in an experiment are given below in Table I., while fig. 3 is the log-absorption curve obtained from these readings, and is typical of the curves given by this method. The results obtained are summarised below in Table II., the wave-lengths were standardised by observing the position of reflection of the β peak of the platinum spectrum.

These results present some interesting features. It is well known, for example, that the ratio of the absorption coefficient of one substance to that of another is approximately independent of the wave-length, provided the

Table I

Reading of spectrometer table θ	No of sheets of absorber	Actual readings taken of		Readings corrected for calibration		Readings corrected for charging up		Reduction of readings of E to F = 20	Mean	Calibration of electroscope		
		E	F	E	F	F	F			Deflection	Volts	
93° 30′ (λ=0 480 A U)	0	27·0	12·0	24 5	12·0	23 3	11 4	40 8	40 4	0	0	Mass of aluminium per unit of area =0 08412 grm
		41·0	17 0	33·0	16 5	31 4	15 7	40		8·0	0 2	
		42·0	18·0	34·0	17 0	32 3	16 2	40		16 5	0 4	
		26 5	12·0	24·0	12·0	22 8	11 4	41 7		26 5	0 6	
		31·0	13·0	27·0	13·0	25·7	12 3	40 3		39·0	0 8	
		30 5	13 0	26 5	13·0	24 8	12 3	40		45 0	1·0	
		33·0	14 5	28·0	14·0	26·6	13 3	40				
		29 5	12 5	25 5	12 5	24 3	12·0	40 4				
	1	37 0	19·0	31·0	18·0	29 2	17 4	33 5	34·0			
		41·0	21·0	34 0	19 5	32 1	18·6	34 5				
		39 0	20 0	32 0	18 5	30 2	17 7	34 1				
		21·0	11 5	19 5	11 5	18 4	10·9	34·0				
		29 5	15 5	25 5	15 0	24·0	14 3	33·6				
		21 5	11 5	20·0	11 5	18·8	10 9	34 5				
	2	25 5	16 5	23·0	16·0	21 4	15 2	28 1	28 6			
		36·0	21 5	30·0	20·0	28 0	19·0	29 4				
		25·0	16·0	23·0	15 5	21 5	14 8	29·0				
		26 0	16 5	23 5	16 0	21·9	15 2	29·0				
		21·0	14·0	19 5	18·8	18 1	13 1	28·0				
	3	14 0	11·0	13·8	11	12 7	10 5	24·2	23·9			
		19 5	15 5	18 5	15	17·0	14 3	23 7				
		17 5	14·0	16 5	13 8	15 1	13 1	23 5				
		21 5	17·0	20 0	16·0	18 4	15 2	24·2				
		24 5	19·0	22 5	18	20·7	17 1	24 2				
	4	15·0	14·0	14 7	13 8	13 3	13 1	20 3	20 1			
		14 0	12·8	13 8	12·8	12 5	12 2	20 4				
		16·0	15·0	15 5	14 7	14·0	14·0	20 0				
		17·0	16·0	16 4	15 6	14 8	14 8	20·0				
		17·0	16·0	16 4	15·6	14 8	14 8	20·0				
	Lead 2 mm thick	2 0	23									
		2·0	24									
		2·0	23									
	Both E and F screened with lead	2	1									
		2	1									
		2	1									

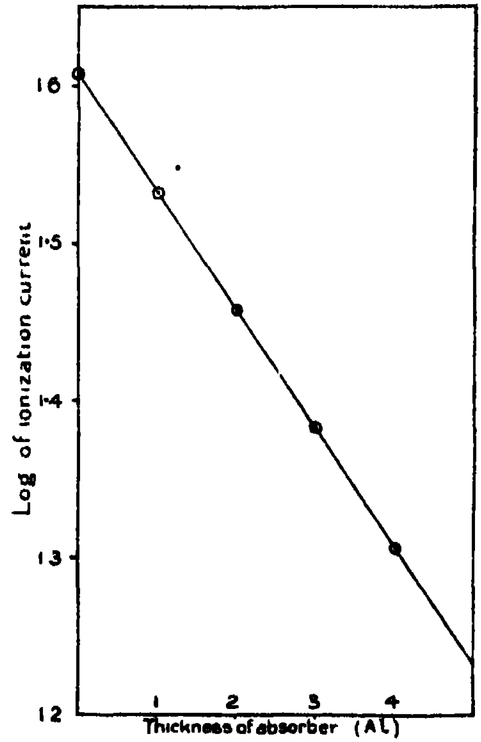

Fig 3

Table II

Wave length × 10^{-8}	μ/ρ	
	Aluminium	Copper
0 431	1 483	14 82
0 456	1 654	16 67
0 480	2 036	20 65
0 504	2 ·050	23 ·00
0 529	2 600	26 63
0 553	2 748	29 57
0 577	3 102	32 70
0 627	4 039	41 17

wave-lengths do not include any near those characteristic of the absorber If now we plot μ/ρ_{Al} against μ/ρ_{Cu} in a graph, we get the curve shown in fig 4 A striking feature about this curve is that a break appears in it at a wave-length λ = 0·49 Å.U (mass absorption coefficient, 2), it is significant that Barkla* obtained evidence of the emission by aluminium of a J radiation, the wave-length of which was approximately equal to this (mass absorption coefficient, 1·9) The results obtained by Pierce, who has also measured absorption coefficients over this range, are also shown in fig. 4. The slope of the line representing Pierce's results agrees with that found in these experiments, while there is also evidence of a discontinuity at $\lambda = 0{\cdot}49^{-10}$ cm. Unfortunately, Pierce did not extend his observations far enough in the direction of the shorter waves to establish this with certainty

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* See "Bakerian Lecture," 1916, 'Phil Trans.,' A, vol 217, p 352

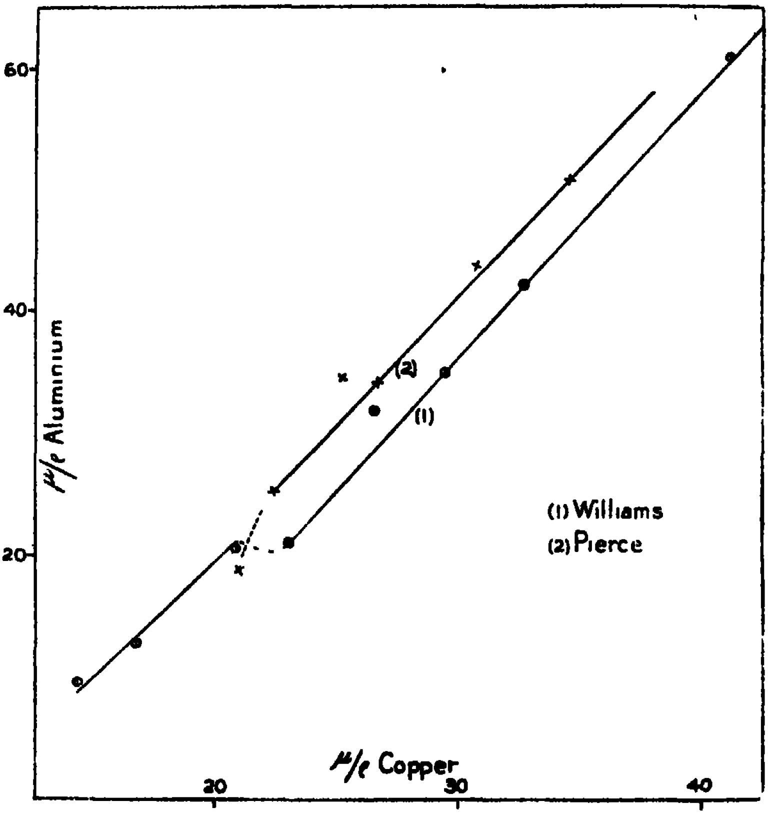

Fig 4

Barkla,* from his most recent results, considers that the wave-length of the characteristic J radiation of aluminium is about 0·37 Å U It is to be noticed, however, as he himself points out, that the radiations used in his absorption experiments were not homogeneous, and hence the wave-length as deduced from the mass absorption coefficient in aluminium is subject to error *e g*, Bragg found that the silver characteristic rays consisted of two waves of wave-length 0 491 and 0 554 Å U, and with absorption coefficients

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* 'Phil. Mag,' October, 1917, p. 273.

1·94 and 2·7 respectively, while the absorption coefficient given by Barkla was 2·5—a mean between these two numbers.

Previous results have indicated that the relation between the absorption coefficient and the wave-length may be expressed, at least approximately, by the formula $\mu/\rho = a\lambda^n + C$, where a, n and C are constants, C representing the scattering coefficient. The value $n = 3$ appears to be the most satisfactory for the results obtained hitherto.

The following curves (fig. 5) were obtained by plotting μ/ρ against λ^3.

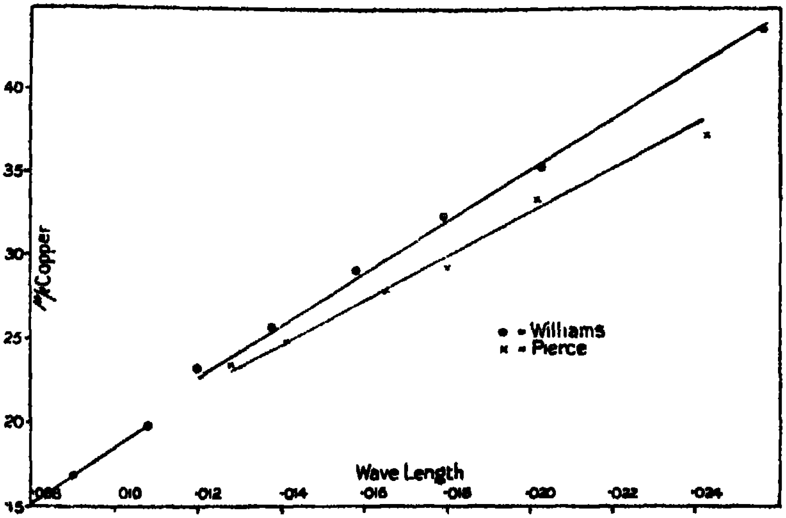

Fig. 5A. Wave Length should read (Wave Length)^3.

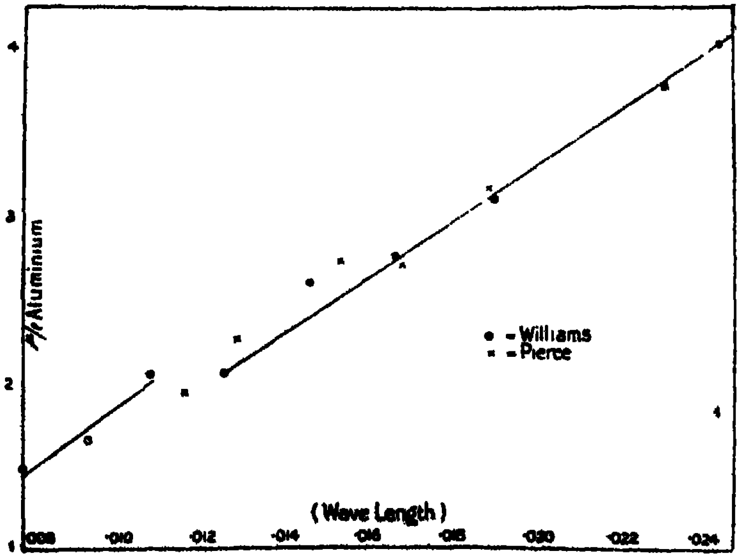

Fig. 5B. (Wave Length) should read (Wave Length)^3.

It will be seen that the points lie approximately on a straight line, though, here again, the break in the aluminium curve at a wave-length $\lambda = 0·49$ Å.U. is pronounced, while there is also evidence of a discontinuity in the copper curve at a wave-length a little below that at which the break occurs in the aluminium curve—a result similar to that obtained by Barkla.

Fig 6 shows the curves by plotting log μ/ρ against log λ for aluminium and copper respectively In the two cases the graphs are straight lines, but while the slope for the aluminium line is 3, that for copper—in which case the points are particularly regular—is almost exactly 5/2—a result in agreement with Owen's 5th power absorption law These curves also bring out very clearly the fact that discontinuities occur both in the case of copper and of aluminium at wave-lengths 0·448 and 0·49 respectively

Thus it appears that the value of n which best satisfies the results beyond a discontinuity is 3 in the case of aluminium, and 5/2 in that of copper It should be remembered, however, that the scattering coefficient C has been neglected We should, therefore, more accurately have plotted log $(\mu/\rho - C)$ against log λ. The values given for C range from 0·1 to 0·2, and it will be found that the slope in the case either of copper or of aluminium is not materially affected by the inclusion of the scattering coefficient

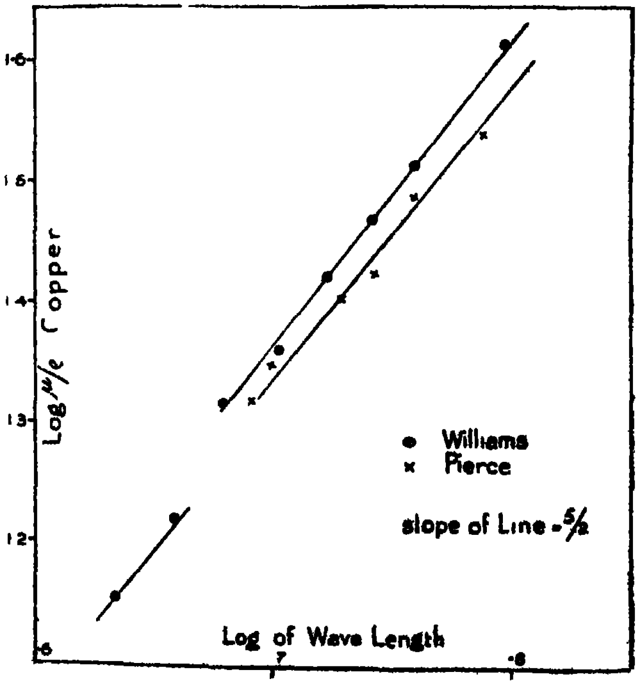

Fig. 6a

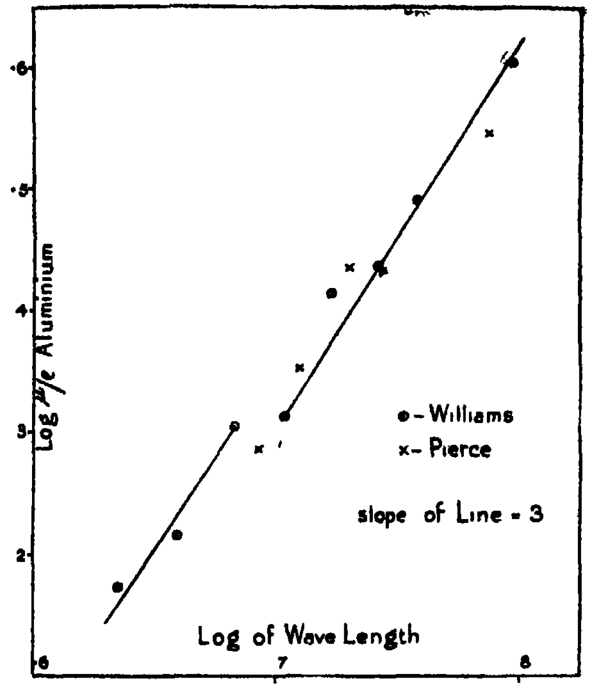

FIG 6B.

I desire to express my thanks to Principal Griffiths for the interest he has shown in the work, and for his kind consideration in regard to my wants I am also indebted to Captain J H Shaxby, for much valuable advice and criticism Part of the expenses incurred in the research has been defrayed by a special grant from the University of Wales

A Statistical Survey of Colour Vision.

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(Communicated by Prof A Gray, F R S Received April 8, 1918)

It is well known that there are many people with slightly abnormal colour vision, who would not be differentiated from the normal by any of the usual tests for colour-blindness The object of this paper was to make a statistical study of such people, such a study is of interest because it enables us to say whether normal colour vision passes gradually into colour-blindness, or whether the colour-blind form a well-defined class by themselves This is important from the point of view of colour-vision theory.

For, suppose we test the colour vision of 1000 persons, chosen entirely at random, and assign marks to it, ranging from 1 to 30, taking 1 for total colour-blindness and 30 for the most perfect colour vision it is possible to have We could thus divide them into 30 classes. Suppose next that the number of persons in each class is plotted against the number of the class Then we shall get a frequency curve Such a curve might have various shapes, if it were to turn out of the shape shown in fig 1 it would show we were dealing with two classes of persons, and it would be natural to identify the large maximum with the trichromats, and the small one with the dichromats of the Young-Helmholtz theory

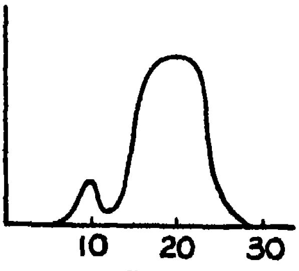

Fig 1

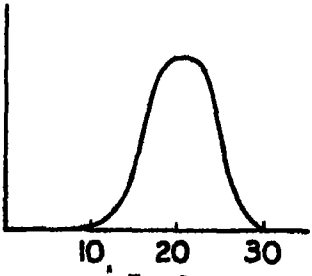

Fig 2

If, however, the curve turns out of the shape shown in fig 2, and if, after the methods of biometricians have been applied, no trace can be found of the little maximum on the left, then we would have evidence that colour-blindness is not due to the absence of one of three colour sensations, as maintained by the Young-Helmholtz theory of colour vision. The point at issue then is: Has colour-blindness a maximum of its own, or is it merely an outlying portion of the main curve?

How are we to divide colour vision into classes? The method which occurs most naturally to one is that of the Edridge-Green colour perception spectrometer This instrument is a spectrometer, in the focal plane of the telescope of which are two adjustable shutters with vertical edges the shutters can be moved into the field from right and left respectively, each by its own micrometer screw To each of these screws is attached a drum, which gives directly in wave-lengths the position of the edge of the corresponding shutter Thus the limits of the region of spectrum exposed can always be ascertained.

In using the instrument the observer first ascertains the exact position of the red end of the spectrum, placing the edge of the right-hand shutter at that point and reading its position The left-hand shutter is then moved in and out, until he obtains the largest portion of red which appears monochromatic to him, no notice being taken of variations in brightness, but only in hue The position of the edge of the left-hand shutter is then noted The edge of the right-hand shutter is next placed at the point previously occupied by the edge of the left-hand shutter, and the latter moved towards the violet, until the next monochromatic patch is marked out. In this way the whole spectrum is traversed, until finally its violet end is reached

The number of monochromatic patches recorded by the observer would give the number of the class to which he belongs.

It is not possible to use Dr Edridge-Green's results for the purposes of my test, as they are not a random distribution, but specially selected cases I resolved, therefore, to acquire data of my own As colour perception spectrometer I used a Browning spectroscope with a single 60° flint prism. The eyepiece was fitted with shutters which could be moved into the field from right and left in the focal plane, each by its own micrometer screw. The collimator was provided with a new slit, and diaphragms were fitted inside both collimator and telescope in order to reduce stray light to a minimum A scale of wave-lengths was fixed up on the wall of the room at a distance of 4 metres from the instrument, this scale was seen in the field just below the spectrum, by reflection in the second face of the prism As source a 16-candle-power carbon glow lamp enclosed in a tin with a ground glass window was used. The wave-length scale was illuminated by a glow lamp, and both this lamp and the other one were controlled by switches close to the observer, so that the spectrum and scale could be viewed either simultaneously or separately. The method of reading the wave-length was thus similar to that employed in the old Bunsen and Kirchhoff spectroscopes, but, as the scale was in Indian ink on drawing paper and quite 33 cm. long, it was, of course, much more accurate. It should be noted that, as the telescope was focussed on this scale at 4 metres, the collimator had to be put slightly out of focus for parallel light, and the rays did not go through the prism parallel, but this was not an appreciable source of error.

To compare the colours at the edges of a patch, the eye has to travel backwards and forwards from the one edge to the other Lord Rayleigh has pointed out* that, while, when working in this way, he marks out as monochromatic the region 586–595 $\mu\mu$, or 15 times the distance between the sodium lines, yet, when the colours bound each other sharply, he can detect a difference of colour between the sodium lines themselves In the former case the gradual character of the transition is an obstacle to the recognition of the difference

Dr Edridge-Green takes exception to Lord Rayleigh's results,† stating that the difference the latter obtains between the two halves of his field is not one of colour, but is due to stray light or to a difference in intensity If a double-image prism is placed between the eye and eyepiece, so as to form two images of the rectangular monochromatic patch, and these images are arranged side by side, so that the more refrangible side of the one touches the less refrangible side of the other, then, according to Dr Edridge-Green, the double patch appears monochromatic the whole way across

I have repeated this experiment with the double-image prism, using both a calcite Wollaston prism with a separation of 1° and a simple calcite wedge with a smaller separation, and find that by this means I can reduce the width of the monochromatic patch in the neighbourhood of the D lines to $\frac{1}{4}$, and in the neighbourhood of 480 $\mu\mu$ to $\frac{1}{2}$ These were the only regions in which I tried the experiment. The result is quite decisive, and, to me at least, admits no manner of doubt. The reason why Lord Rayleigh was able to reduce the difference of colour to $\frac{1}{15}$ was doubtless owing to the more favourable conditions in his experiment. He was working with two fields, which were probably as broad as they were deep, with the double-image prism and colour perception spectrometer I was comparing two very narrow rectangles, in one case the apparent width of each being only 15′

I am therefore unable to agree with Dr Edridge-Green that the monochromatic region is a "fundamental physiological fact." Its width depends to some extent on the dispersion of the spectrometer and magnification of the eyepiece. Thus, in Dr Edridge-Green's instrument, the average number of patches is about eighteen ; in mine it is 14·86 In my instrument the whole length and height of the spectrum subtend apparent angles of 27° and 8° at the eye.

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* 'Scientific Papers,' vol. 5, p. 621 ; 'Roy Soc. Proc.,' A, vol 84, p 464 (1910).

† 'Roy Soc Proc.,' B, vol 84, p 116 (1911)

The brightness of the spectrum has also an influence on the number of patches visible, but this can be neglected as far as the slight alterations of intensity that occur under ordinary working conditions are concerned For example, the brightest part of my spectrum has an illumination of 500 metre-candles. I then see 17 patches When the illumination of the brightest part of the spectrum is reduced to 64 metre-candles by putting a fogged photographic plate in front of the slit, I see 12 patches. When it is reduced to 2 metre-candles, I see 7 patches, at this illumination the violet is so faint that I cannot see it at all From measurements made at different times, it seems that the voltage of the lamp did not vary 5 volts on 250. This would cause only a 10 per cent change in the intensity, and would not affect the number of patches appreciably The slit was always kept at the same width, when illuminated with sodium, it appeared 10 $\mu\mu$ wide

The following Table gives my results Of the 79 observers, 74 were students, the remaining five included two professors of physics, a lecturer on physics, a head science teacher, and myself. Of the students 38 were women, the men and women are indicated respectively by the letters M and W in the Table The number immediately after this letter gives the number of patches The other entries give the limits of the patches in 10^{-6} cm. When the decimal is not printed, it is understood to be 0 Of the students, 8 were in their Honours course, 12 in their Higher course and the remainder in their Intermediate course, so they were all well experienced in physical measurements The distribution was purely a random one, I knew nothing of the colour vision of any of the observers before asking them to make a set of readings, and no one who was asked declined The observers included all the laboratory students available, except one or two foreigners with a defective knowledge of English.

The Monochromatic Patches Marked out in the Spectrum by 79 Different Observers.

1	M.	22	73·5–69–66–64–62–60 7–59 8–58·5–57 8–56 3–54·5–52·5–51 3–50–49·2–48–47–46–44·7–43·3–43–42–41
2	M.	18	76·5–70 5–66–63–61–59 8–58–57 5–55·5–54–52·2–50·8–50–49·5–48·5–47 5–46·5–45–42 7–42 2.
3	M.	20	76·5–65–62·5–60 5–59·8–58 1–57–55·8–54·4–52·6–51 4–50 6–49 7–48 9–48 2–47·4–46 8–46 2–45–43 1–41 6.
4	M.	17	74 7–67–64 6–62 1–60·8–60–58·7–57·2–55 7–53 3–51·5–49 8–48 7–47·5–46·4–45 3–43·6–42 0
5	W.	12	77–65–60·2–54 3–51 2–50–48 5–47 5–46 8–46–45·6–43–41 5.
6	W.	26	74–65·6–61·2–59·4–58·5–57 6–56·5–55·5–55–53·5–51–49·5–48·7–48–47·3–46·7–46·2–45 8–45·4–44·8–43·3–43·3–42·6–42·2–41 7–40·6–38·8
7	W.	13	71–65–61–59–57·2–56·2–54 5–52 4–48–47–46 2–45–43·5–41 5.

8	W	18	73-65-63-62-61-60 5-59 5-59-58-56 6-54 9-52 5-50 8-49 4-48 1-47-45·8-42 8-41 8
9	W	20	71 3-65 5 62 7-60 5-59 1-57 8-56 1-54 8-53 4-52-51 2-50 1-49 2-48 8-48 1-47 6 47 2-46 5-45 6-43 6-41 6
10	W	12	75-64 60 3-57 8-56-52 8 51-49 5-48 2-47·2-45 8-43 5-41 5
11	M	17	73-64 8-61 4-60-58·9-57 5-56 8-55 8-51·2-49 4-48 8-48 1-47 4-46 8-45 8-44·6-42·6-41 2
12	M	14	72-66-62 9-60 8-59 2-57 6-54 8-51 6-49 8-48 7-47 4-46 1-44 2-42 0-40 8
13	W	9	71·3-63-58 8-56-53-49-47 4-45 4-41 8-41
14	M	18	74·2 66-63 8 61-59 7-57 8-56 6-53 8-51 1-49 9-48 8-47 9-46 9-46 3-45·1-43 7-42 5-41·8-41 5.
15	M	20	73-66-64 2-61 1-60 0-59-57 8-56 1-54-51 9-50 7-50-48 8-48 3-47 5-46 4-45 4-44-42 7-42 2-41 5
16	W	12	73-65-62-59-56-51 8-50-48 5-47 2-45 3-43 1-41 9-40 8.
17	M	12	77 5-64 8 59 9-57 6-55 8-53-50 8-49 2-47 6-46-43 8-42 8-41 2
18	M	12	72-65-61-59-57-54 3-51 7-50 2-48 7-47 5 46 2-43 6-41 7
19	W	14	73-67-61-59-57 8-55 2-52 5-49 7-49-48-47 2-46-44 8-43-41 8
20	W	16	78-72-66 6-63-60 2-59-57 8-56 2-54 2-52-50-48 5-47 3-46 1-44 2-42·1-40 2.
21	M.	17	71-65-61-59-58-56 6-55-53 1-51 1-50-49-48 2-47 5-46 8-46-45 2-42 9-41 7.
22	M	16	71-63-60 6 59 7 58-56 53 7-51 9-51 4 50 4-49 3-48 4-47 2-46 2-44 4-42·5-41 7.
23	W	24	71-63 8-60 8-59 5-58 5-57 8-57-56-54 2-52 5-51 3-50 5-49 9-49 3-48 8-48 3-47 4-46 7-45 8-44 5-43·7-42 8-42-41 2-39 7
24	W	19	67-63 60 5-59 7-58-57·4-56 5-55 3-53-51 1-50-48 8-48-47 3-46 4-46-45 5-45 1-44 6-41 8
25	M.	12	70 64 5-62-59 8-58 7-57 6-52-49-47 6-46 5-45 4-42 8-41 5
26	M	18	77-64-61-59 7-58 2-56 8-55 1-53 3-51 7-50 2-49 2-48 2-47·4-46 7-46 1-45·5-44 8-43 2-41 3
27	M	11	72-64-61-59 2-56-53-51 2-49 8-48·2-46 8-45 2-41 2
28	W	13	75-61-59 7-58-56 6-53 4-51-50-49 3-48 3-47-45 8-43·8-41 5
29	M	13	72-62 4-60 4 58 8-57·8-56 4-53·6-51 1-49 2-48 5-47 1-45 5-43·2-41 8
30	M	12	76-67-62 2-59 7-57 6-54 6-51 4-49 2-47 7-46 2 44 9-43 2-42
31	W	9	74 2-64-59 2-57-54 8-49 3-47 8-46 8-45-40 8
32	W	15	74-66 60-58 7-57 8-57 1-56·0-53·8-51 7-49 1-48-47 1-46 3-45 3-43 6-41 5
33	W	13	72-63 8-61-59 5-58 1-56 6-52-50 7-49 1-48 1-46 9-45 6-42 6-41.
34	W	12	78-65-61-59 7-58 7-57·3-56 1-54 1-51·1-48 2-47·3-46·2-41·6.
35	W	16	76-66 5-63 1-61 1-60-58 8-57 6-55 4-52 5 50 8-49 3-48 4-47 6-46 7-45 5-44·4-40 8
36	W	13	71 63-60-59-57 8-56 8-55 2-52 3-49 6-48 6-47·8-46·3-43 7-42·2
37	M	14	70-64-61 6-60-58·8-57-54 6-51 7-49 9-48·7-47 7-46 8-45 8-42 9-41·2.
38	M	5	70-62-57 51 8-48·4-42 5
39	M	16	75-63-60 6-58 5-55-52·9-51·6-50 6 49·9-48 8-47 5-46 6-45 3-44·2-42·9-42·3 41·2.
40	M	16	70-65 4-61-60-58 8-57 2-54-52-50·7-49 8-48·8 47·7-46·8-45·6-43·6-42·4-41·5.

41	W.	11	74–63–61–59 2–58 1–56 6–54 5–51 5–49 7–46 2–44 6–42.
42	M	9	71–63–59 9–58 1–51 3–49 7–48 3–46 6–44 3–42
43	M	16	72–66 5–63 8–61 2–60–58 8–57 3–55 8–53 2–51–49 9–48 8–48–46 9–45 7–43 7–41·9
44	M	15	72–68–63–61 8–59 5–58 5–54 9–51 4–50 4–49 3–48 2–46 9–45 9–43 8–43 2–42 3
45	W	12	75–64–60–56 5–52 2–50 7–49 4–48 4–47 4–46 2–44 8–43 3–41 8
46	W	16	73–65–63–60 8–59 5 58 2–56 8–53 4–51 1–49 3–48 2–47–45 6–44–42 4–41 5–40 7
47	W	15	69–65 6–61–59 7–57 6–56 6–54 8–52 4–50–48 6–46 6–45 8–44 9–44–42–41
48	M	12	71–66–63 5–60 8–59–57–51–49 2–48 4–47 2–46–43 2–42 2
49	M.	9	66–58 8–54 5–52–50 5–49 2–48 4–47 5–45 2
50	M.	13	75–69–65–62–60–58 8–57 1–54 8–51 8–49 7–48 4–46–42 5–41 7
51	M	13	71–63–60 8–59 9–58 2–56 4–54 5–52–50–47 6–46 8 45 4–43·9–42 7
52	W	16	77–65–62 2–61 5 59 3 58 8–58 57–55–52 2 50 8 49 4 48 5–47 3–46 2–44 5–40 8
53	W	5	73–62–57 2–50 2–48–41 4
54	W	13	75–69 5–66–62 8–60 3–58 2–55 2–52 50 8–49 47 2–45 7–44–42
55	M	15	73–66–63 5–61 8–60 2–57 2–56 54 5 52 5–51 4–49 9–48 8–47 4–45 5–43 4–41 7
56	M	21	73–67–64–62 5–61 2–59 7–58 8–58–57 2–54·2–51 3–50 1–49 6–49 2–48 8–48 5–47 6–46 7–45 7–43 2–42–41 4
57	W	16	70–65–62 61 4–59 9–58 6–58–57–52 7–51·4–49 5–48 5–47 6–46 5–44 8–42 4–41 6
58	M	19	73–65–62 1–60 5 59 58 4–57 8–56 4–54 9–52 3–51–49 8–49 1–48 4–47 5–46 5 45 9–44 8–42 9–42 3
59	M	15	67 63–62–59 8–58 2 57–55–52 8–51 0–49 8–49 2–48 2–46 6–45 1–43 5–42 3
60	M	13	73–66 5–62–61–59 8 58 8–56 5–52 4–50 7–49 9–48 8–47 1–44 7–41 5
61	W	18	73–63–60 5–59 5–58 8–58–54 8–52 2–50 3–49 6–48 8–48 2–47 2–46 2–45 2–44–42 8–41 6–40 5
62	W	19	70–65–62 8–61–59 7–58 2–57 2–56 2–54 6–52 5–51 4–50 1–49 2–48 5–47 6–46 3–45 5–44 3–42 3–41 4
63	M	16	72–66–63–62–61–60–59 1–57·8–56 2–54–50–49–48 2–47 2–46–42 8–41 8
64	M.	14	77–65 62–60 3–59–57 7–56–52 4–50 6–49 2–47 8–46 7–45 8–44 3–42·5
65	W	12	73–66–61–60–58–55–51 8–50 2–48 8–47 7–46 4–44 8–41 2
66	W	16	71–64 5–61–59 7–58 2–57–56 53 8–51 7–50·2–49 1–48·2–47 0–46–45–43 2–41 4.
67	M.	15	75–63–61 2–60–58–57–55 8–53 9–52–50·3–49·2–47 9–46·6–45 7–43 7–42
68	W.	19	77–67 5–65 5–63 5–61 2–59 5–58–56 9–54 9–53–51 9–51–49 8–48 7–47 5–46 4–45–43 2–41 8–40 9
69	M.	13	72–63–60–59–58–54 3–51 8–50 1–48 8–47·8–46 9–45–42·8–41·8
70	M.	13	73–62–59 5–57 8–56·5–54 8–52 4–50 1–49–48–47–45 8–43 5–42.
71	M	8	75–69–65 4–62–58–53–48 5–44 5–42 3
72	W	14	73–62–60·2–58 8–57 2–55 4–52 7–50 2–48 6–47·5–46 7–45 6–44 7–43 5–42 0.
73	W	17	69–65–63–62–60 5–59 5–58 6–57 3–56 2–53 7–51 3–50 3–49 4–48 5–47 7–46 4–45 2–42.
74	W.	10	74–62–60 5–59 5–57 6–53·5–50 2–48 8–47–45·6–42 5
75	W	16	74–66–61 5–59·5–58–57–55·2–54–52–51 2–49 7–48·8–48–46 7–45 2–43–41·8.

76	W	15	72-63 2-61 60-58 9-58-56 3-54 5-51·6-50 1-49·3-48·1-46·7-45·7-43·9-41 4.
77	W	16	71-66-64-62-60 5-59-57 6-56 5-55 1-53·6-51 7-50-48 8-47 4-46-44 2-42
78	M	17	73-64-61 8-60·5-59 7-58·8-58-57 2-56-54 6-52 9-51·5-50·2-49 2-47 8-46-44 3-42·4
79	M	24	73 65 63 61-59 9-59-58 1-57 2 56·2-55-53 7-52-51-50-49 5-49 2-48 8-48 2-47 5-46 7-45 5-44-42·9-42-41 3

Before making the test the method of taking readings was explained by means of a water-colour drawing of the spectrum and two sliding cardboard screens. Then a start was made from the red end Only one reading was, as a rule, taken for each patch, but generally one or two readings were repeated in each set, after the set was completed It was found they almost always agreed No preliminary practice was given The first two or three observers read their own wave-lengths, and set the telescope in each fresh position themselves Afterwards I did this, as I found it saved time Each complete test required 30 or 35 minutes. I supervised every test personally, and watched the fingers of the observer on the micrometer screw, to make sure he was testing the patch from both sides. Only in 3 of the 79 cases was there any misunderstanding One of the three I discovered, when about halfway through, was marking off differences of intensity as well as of colour, so we had to start again. Another was inclined to ignore degrees of yellow at the brightest part of the spectrum, regarding them as changes of brightness, while a third in one case ignored a thin strip of colour close up to the edge of a patch "because it was a bit of the last patch accidentally left over."

The question arises as to the reliability of the numbers If the observer were to make a second set of readings, would he obtain exactly the same number of patches? Probably not. No 4 in the Table is my own set About a year previously I made a set, obtaining

76-64 7-62 7-61-59 2-58 2-57 2-56 53 3-52 2-50 7-49 8-48 7-47 6-46 7-45 5-43 7-42,

i e, 17 patches, but with different limits. Three weeks later I made two sets on the forenoon and afternoon of the same day, when I was very tired, obtaining

76-65-62-60-59-58-56·6-55-52-49 8-49-47·8-46·6-44·2-41·6,

and

77-65-62 5-60 2-58 7-57·7-56-53-50·3-49·3-47·8-47-45·5-43·5-41·5,

giving 14 patches each time In taking these two sets the left-hand shutter was moved only in one direction, in the first set into the field, and in the second set out of the field, the object of the experiment being to see whether the direction of setting had an appreciable effect on the number of patches Apparently it had not. Four months later than this I made a set when feeling very fit, and obtained 19 patches, namely

76–66–63–61–59 7–58 8–58 1–57–56–54 5–52 5–51–50–49 1–48 4–47 5–46·5–45·4–43 2–41 3

Also seven of the other observers made a second set four weeks or so later than their first set, when No 9 got 24 instead of 20, No 1 got 26 instead of 22, No. 6 got 30 instead of 26, No 38 got 6 instead of 5, No 2 got 16 instead of 18, No 18 got 15 instead of 12, and No 17 got 16 instead of 12 But the most interesting case was that of No 5 She obtained 12 patches originally, 60 2–54 3 being only one patch This observation was repeated at the time and verified Six weeks later she started a second set, obtaining seven patches between the red end of the spectrum and 60, where she had obtained only two before The test was then discontinued, and another test made a week later, when she obtained 24 patches for the whole spectrum, four of these being between the red end and 60 4. As the laboratory work of this student on Fresnel's biprism, polarimetry, etc, was exceptionally accurate and reliable, I have no alternative except to assume that her colour vision had undergone a decided change

Excluding her case as an abnormal one, I believe the number of patches recorded by the same observer can vary within a total range of about 35 per cent, but if the two sets are made on the same day, the agreement will be much better I am inclined to think the general increase in the number of patches at the second determinations recorded above was due to the fact that they were all made on two very dark December days, when the appreciation of colour might be consequently more acute It should, of course, be remembered that variations of individual observers from day to day compensate one another, and do not affect the results obtained from a large number of observers over a long period

Fig 3 gives the results, each observer being plotted, the men as circles and women as crosses, above the number of patches obtained It will be observed that this varies from 5 to 26 The two observers with 5 and the observer with 8 patches were decidedly colour-blind. The observer with 26 was one of the four students who "saw indigo"* The other three have left the University, and so I could not get their number of patches. The two observers with 5 patches did not know they were colour-blind, they had difficulty in naming yellow, calling it red or green The observer with 8 patches knew he was colour-blind, as he could not distinguish a blue from a green tramcar at a greater distance than 100 yards. He called green a dirty yellow, and, as there was some doubt as to his confusing intensity with hue in his first set of readings, a second set was made immediately afterwards, when he obtained 5 patches

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* R. A. Houstoun, "Newton and the Colours of the Spectrum," 'Science Progress,' vol. 46, p. 250 (1917)

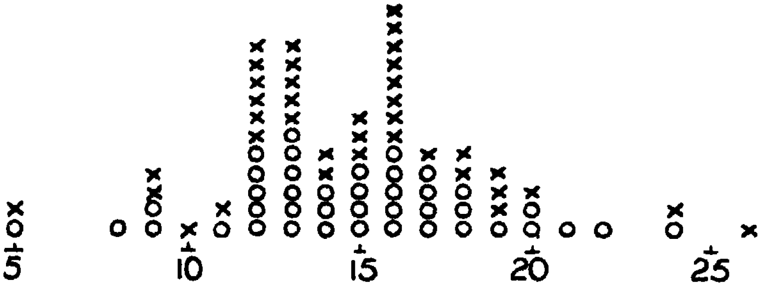

FIG 3.—Circles, men, crosses, women

The observations recorded in fig. 3 show that normal colour vision has quite enough "scatter" to include colour-blindness as an outlying portion of itself. It is true, of course, that the two "fives" are separated by a gap at the one end, but so are the two "twenty-fours" and the "twenty-six" at the other end But it would require a very much greater number of observers than 79 to positively disprove the existence of separate maxima for the dichromats and monochromats

I have been much puzzled by the minimum at 14 in fig 3 It seems to divide the observers into two groups, though two members of the mathematical staff of the University to whom the results have been shown declare that this division is purely fortuitous. If the observers are plotted against the width of the patch they obtain at 589 $\mu\mu$, or the width of the patch they obtain at 480 $\mu\mu$, we get the same two groups with practically the same number of observers to the left of the minimum in all three cases. If the observers are plotted against the width of the patch they obtain at 535 $\mu\mu$, there is not the same well-defined minimum If in the case of an even number of patches the middle boundary is called the spectrum middle point, if in the case of an odd number of patches the centre of the middle patch is called the spectrum middle point, and if the observers are plotted against their spectrum middle points, then we get only one maximum Thus the essential difference between the two apparent groups lies in a greater power of colour discrimination at the points of the spectrum, where the colour changes most rapidly with wave-length I was under the impression that there were two groups forming before I had supervised 20 tests, and formed various hypotheses to explain it, *e g*, that the one group was confusing hue with intensity, etc, but to no purpose. It will be interesting to know if the same division shows itself when tests are made with another apparatus

It occurred to me that the one group might differ from the other in other ways, for example, in the amount of red it added to green to make yellow For testing this, I required a colour-mixing apparatus of the type used by Lord Rayleigh. I was able to improvise one very satisfactorily in the following manner :—

The Cornu prism consists of two 30° prisms of right-handed and left-handed quartz, with their optical axes at right angles to the plane of symmetry of the prism I separated the two halves of a Cornu prism and reversed them, *i e*, mounted them with the hypotenuse surfaces in contact When set at minimum deviation in this position, they gave two spectra with the red of the one falling on the green of the other, these spectra being practically plane polarised (really very slightly elliptically polarised) at right angles to one another The telescope of the spectrometer possessed a slit in its focal plane Between this slit and the eye was a nicol eyepiece, the lenses of which were removed An eye looking into the telescope saw the surface of the prism illuminated with whatever colour happened to fall upon the slit The instrument was set so that the green of the one spectrum and red of the other fell upon this slit, thus, the colour of the prism could be changed from pure green through yellow to red by rotating the nicol To produce the comparison colour, a 60° crown glass prism, which had approximately the same deviation as the Cornu prism, was clamped on the top of the latter, so that it appeared illuminated in sodium yellow It was then itself not far from minimum deviation. The red and green employed with this apparatus were respectively of wave-lengths 620 $\mu\mu$ and 560 $\mu\mu$.

I tested 15 of the observers with this apparatus, each making four settings with the nicol, taking them in turn from the top of each of the groups of fig 3, but found that their settings all agreed within the error of observation

The only previous attempt which I have been able to find to apply statistical methods to colour vision is in a paper by Schuster.* He inclines to the belief expressed by Lord Rayleigh, that differences from normal vision do not seem to follow the law of error We have not, however, as much faith in the Gaussian law of error now as 25 years ago. The experimental psychologists have introduced methods for measuring the different sensations, although they do not seem yet to have established many results, and the past 20 years have been noted for the development of statistical methods. I think some of this activity might be profitably directed to the question of the theory of colour vision, so as to settle the present deadlock; the time is now past for the description of special cases of colour-blindness. It might be advisable to institute a test in colour vision as part of the students' course in certain of the larger physical laboratories in the country, and appoint a committee to deal with the results.

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* "Experiments with Lord Rayleigh's Colour Box," 'Roy. Soc Proc.,' vol 48, p 140 (1890)

It is, of course, very likely that objection may be taken to the nature of the test described in this paper. One has to trust to the judgment of the observer as to when the difference of colour appears. It would be better if the reading were to follow automatically, for example, if upon a deep red background we were to make an orange-red letter appear, keeping the luminosity of the letter the same as that of the background, but gradually increasing the admixture of yellow. The amount of yellow would be controlled by the supervisor, who would note the reading as soon as the observer recognised the letter. Observers with good colour vision would require less yellow. We would have the difficulty here that the luminosity of colours to different observers varies in a different way with wave-length, and it would not be possible to maintain the condition of constant luminosity for all observers. I believe, however, that contrast due to difference of luminosity would be much less effective in causing a letter to appear than contrast due to difference of colour, and might be ignored.

I have attempted to construct a colour perception spectrometer depending on this principle. It consisted of a spectrometer with the eyepiece removed, and a slit substituted. Against the last face of the prism was placed a photographic plate, which had been fixed without being exposed, and so was covered with a uniform film of gelatine. On this film a letter was painted in an aniline dye, the idea being that the dye would be quite transparent to certain colours of the spectrum, but that as the telescope was gradually rotated, a point would be reached at which the letter would stand out. In practice the dye itself was transparent enough, but when it was applied the gelatine swelled out and acted like a lens, and so the letter remained visible when it ought to have disappeared. Another attempt was made by photographing a letter with a Lumière three-colour plate, but I have no time at present to follow the matter up.

OBITUARY NOTICES

OF

FELLOWS DECEASED.

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From the Portrait by
H HARRIS BROWNE

SIR ANDREW NOBLE, 1831–1915

SIR ANDREW NOBLE, who died at Ardkinglas, his Scottish home, on October 22, 1915, was born at Greenock on September 13, 1831 His father was George Noble, a retired naval captain, who lived in a house which at that time was called 65, Union Street, but was afterwards named "Springbank" In earlier days the Nobles had been landowners in Dumbartonshire, but the property was sold at the end of the eighteenth century It was bought back again in 1889 by the subject of this memoir, who, when he was made a baronet in 1902, was able to associate with his name the family estates of Ardmore and Ardardan

George Noble had twelve children, five sons and seven daughters Andrew was the third son The home life was one of strictness and discipline, for the father's ideas of education were very thorough The rudiments of learning were instilled into the boys by various teachers in Greenock, beginning with Peter Murray, who grounded them in English grammar, and ending with the classes of Mr Robert Buchanan, who taught them writing and mathematics. From his early years Andrew always did well in these local intellectual contests, and he became the possessor of several small silver medals, with the word *Dux* upon them. From Greenock he went on to the Academy at Edinburgh, and passed as a cadet into Woolwich in the spring of 1847 His father died of typhus fever in the autumn of the same year.

In June, 1849, he received a commission in the artillery, and went almost at once with his battery to Malta Most of his military life was spent in Canada, where he married in November, 1854, Miss Margery Campbell, the daughter of a notary in Quebec. He and his wife lived to celebrate together in 1914, after 60 years of domestic happiness, their diamond wedding.

About a year after his marriage Noble was ordered to South Africa, where the Kaffir War was still dragging on He had no opportunity of actual fighting, but his experiences in South Africa much interested him, and in later life he was always fond of recalling them All this time he was following with eagerness several branches of science, and lost no opportunity of adding to his mathematical knowledge, which was already considerable He returned from the Cape in January, 1858, and soon found himself involved in those special studies which were to bring him distinction. The inventions of Armstrong had lately turned the attention of military men to the subject of rifled artillery, and had suggested the possibility of arming the services with weapons more powerful and effective than those which survived from Trafalgar and Waterloo. The controversy which was raging was well suited to Noble's particular genius; his talents for patient enquiry, careful experiment, and accurate observation were exactly what was needed to place the various issues on a scientific basis.

After acting for some months as Secretary of the Royal Artillery Institution at Woolwich, he was appointed on September 1, 1858, Secretary to a Special Committee on Rifled Cannon This Committee examined seven different types of gun, Whitworth's, Armstrong's, Vandaleur's, Irving's, Lynall Thomas's, Laurence's, and Baron Wahrendorff's In their report they recommended the immediate introduction of guns rifled upon Mr. Armstrong's system Complaints were made that Noble's attendance on this Committee interfered with his Woolwich work, and he severed his connection with it in November, but in July, 1859, he was again chosen as Secretary of a Committee, described at first as "on Calibres and Guns," but afterwards as "on Plates and Guns" The object of this Committee was to ascertain whether the Armstrong system, already adopted for guns of small calibre, could be applied to larger sizes. Noble was now constantly at the War Office, carrying out experiments and supplying technical information and advice to the Secretary of State. Early in 1860 he was Inspector of Artillery, and he was added to the Ordnance Select Committee as an Associate Member

In August, 1860, during some firing trials conducted against the Martello Towers at Eastbourne, Sir William Armstrong, at that time Director of Rifled Ordnance, suggested to Noble that he should join the Ordnance Company which had just been established at Elswick. Sir William pointed out that, although there was plenty of engineering capacity at Elswick, the company was much at a loss for military technical information as to the appointments necessary for batteries and guns. The proposal evidently attracted Noble, but he asked for time to consider it. He took counsel with his superior officers, some of whom advocated, while others deprecated the step The promise of his prospects in the army (for he was already a noted expert in his own line), and some natural doubt as to the financial stability of the Elswick undertaking, caused him to hesitate. Eventually he was allowed two months' leave, and he went at once to Newcastle to judge for himself and see what sort of duties would be imposed upon him He was so favourably impressed that in October he decided to throw in his lot with the firm, and at the beginning of 1861 he was gazetted out of the regiment At the same time he became a partner in the Ordnance Company Before his death in 1915 he had been for some years the only surviving member of that particular partnership

As Noble's subsequent career was so closely identified with the Elswick Works, it may be of interest briefly to recall the unusual circumstances under which that establishment was gradually drawn into the manufacture of artillery material In 1914, at the outbreak of war, the supply of munitions for several critical months was largely dependent upon the great Arsenal on the banks of the Tyne. Yet the inception of this undertaking was purely peaceful The Elswick Engine Works were started in 1847 to manufacture hydraulic machinery to the designs of Armstrong, who himself was, of course, the moving spirit. After a slow beginning the enterprise succeeded, and in the early fifties it was known as a rising and prosperous concern. At the end of 1854 the Crimean War caused Armstrong to turn his active mind to the subject of guns, and after prolonged trials and experiments his system was, as has been mentioned, officially recommended in 1858.

This recommendation, and the published descriptions of the new artillery, caused, as it well might do, an immense sensation There was an urgent clamour for the re-armament of the services, and a call for immediate sources of supply to satisfy the demand. The pressing question was where the guns were to be obtained All the requirements both of naval and military ordnance had hitherto been met by Woolwich Arsenal It has been assumed sometimes that Woolwich was at once called upon to build to the Armstrong designs, and that the Elswick Ordnance Works were formed as a supplementary establishment. This is hardly an accurate statement of what actually happened Not only were the guns themselves a new thing, but their manufacture was also a complete novelty The method of shrinking wrought-iron coils upon a steel barrel, the accurate machining of the breech mechanism, with its separate and close-fitting wedge-piece, the rifling of the bore, and the delicate workmanship throughout, were all quite new Woolwich at the time was little more than a metal foundry, and was about as capable of building an Armstrong gun as a village blacksmith's shop is capable of turning out a modern motor car.

At Armstrong's own works at Elswick the case was different As far back as 1855 the new ordnance had been made there, and, by the time the system was accepted, a number of trial guns had been delivered In 1857 a moderate plant had been laid down for the manufacture, though there was no idea in the minds of the partners that the demand would be as extensive as 12 months later it became

The Armstrong Works, indeed, were at first the only place where the guns required by the country could be made, and it was Woolwich that had to be transformed to the Elswick model. An expenditure of £200,000 was incurred to provide the Arsenal shops with the necessary equipment. Armstrong's own position was the cause of great difficulties He had already handed over his patents to the Government as a gift, and they were free to do as they pleased with them But two points were clear, the first that his close association with the supply of the material was essential, and the second that, if his own works were used, as they were bound to be, for manufacture, awkward financial questions would certainly present themselves. To meet the first point the position of Director of Rifled Ordnance was created for him, while, to meet the second, a separate company, called the Elswick Ordnance Company, was formed, and, though the ordnance works were set down side by side with the existing engine works at Elswick, the two were kept distinct. Armstrong retained his partnership in the engineering business, but had no share in the Ordnance Company. His own time was divided between London and Woolwich, and he appointed George Rendel, a young man of 26, manager of the ordnance works at Elswick With him he presently associated Noble in the manner and for the reasons which have been already given. The company was, of course, absolutely flooded with Government contracts, and the utmost secrecy was maintained as to all details of design and manufacture

Matters continued in this state until 1863, when the Government orders came to an end This decision has been often attributed to the increasing jealousy occasioned by the official patronage of Armstrong and Elswick. It is true that rival inventors were loud with hostile criticism, but, apart from this, there remained the fact that, with both Elswick and Woolwich in full swing, the needs of the services were more than met No further guns were ordered, for the sufficient reason that none were required

The position was naturally an anxious and disquieting one for the Elswick partners, whose profits had been absorbed in extensions to their works. The capital expended had been £168,000, and the whole plant was offered to the nation for £137,000 This offer was rejected, and finally between £60,000 and £70,000 was paid as compensation for the termination of the existing arrangements Armstrong resigned his position under the Government and returned to Elswick, where the Ordnance Company was at once amalgamated with the Engine Works. The private firm which rose from this combination was called Sir W G Armstrong and Co, and dates from 1863 Noble was one of the original partners. There was great delay in getting permission from the Government to deal with foreign nations, but consent was presently given, and a connection, which became literally world-wide, was in course of time established

In the progress of Elswick, the share which Noble took can be clearly defined. Apart from his industry and administrative ability, to which a later reference will be made, it was to his researches in the sixties and early seventies that a further advance in gunnery was due. The inventions of Armstrong marked a great step, but the modern gun, as we know it, was the direct result of Noble's work Coming to Elswick with considerable experience of the possibilities offered by this department of engineering, he found himself with a free hand He was unhampered by official restrictions, and there were at his disposal funds available for costly experiments and elaborate apparatus Supplementary to his daily labours in the office and workshop, he carried on continuously his careful and scientific research He studied often far into the night, and, gifted with a robust constitution, his powers showed little sign until late in life of the strain which he constantly imposed upon them

His earliest, and, indeed, his chief, enquiries, were into the action of fired gunpowder He followed the lines of pioneers in the same study, of Rumford, Rodman, and others, all of whom had attempted, though without reaching any accepted conclusion, to determine the pressures produced when a charge of gunpowder was exploded in a confined space. But their experiments can hardly compare with those of Noble, so much further did he carry their investigations His treatment of every detail was most thorough. He examined the results of large numbers of explosions, he recorded the temperatures and pressures of the gases, he analysed the solid residues, while no phenomenon, however trifling, escaped him As an instance of this minute attention may be quoted his observation of certain discrepancies which threw doubt on the indications of the Rodman pressure gauge, and the acumen with which he discerned the reasons for these discrepancies He passed from experiments in closed vessels to similar experiments in guns, and invented a chronoscope, by which the velocity of the shot at any point in passing down the bore could be ascertained

The perseverance with which he collected and collated his results was only equalled by the correctness with which he deduced conclusions from them. Satisfied at last as to the soundness of his theories, he gave practical effect to them by insisting on the advantages of slower burning powder and by the adoption of the modern long gun in place of the old short one With the long gun, and assisted by the improvement made in gun steel, breech-loading was gradually, but at last universally, adopted

After severing their connection with Elswick, the English authorities had reverted to the old-fashioned short muzzle-loading guns The experts at Elswick found that, with their longer guns, muzzle-loading was so clumsy as to be almost impossible, and they became in consequence adherents of breech-loading Writing to Sir Frederick Campbell in April, 1878, Noble pressed upon his notice the new Elswick 6-inch, 78-cwt., breech-loading gun. He described the breech-closing mechanism as being on the well-tried French system, with the gas stopped by a steel cup on the Elswick plan He went on to say that, while he was far from recommending the universal substitution of breech for muzzle-loading, yet, for the broadside armament of ships, breech-loading possessed very great advantages

Some impetus was undoubtedly given to the breech-loading system by the accident which occurred on board H M.S. "Thunderer" on January 2, 1879, when a 38-ton gun burst at firing practice in the Sea of Marmora. The enquiry into this disaster forms an interesting episode in Noble's life, for the conclusions reached by the Commission were based upon his evidence. It had been suggested that the bursting of the gun was due to some defect in the hydraulic loading machinery, which had been supplied from Elswick, and Noble went out to Malta to attend the Commission. His interpretation of the mishap caused considerable sensation, for he maintained that the gun had been double loaded. The intention had been to fire both guns in the turret together, and it was not noticed that one had missfired This gun was then reloaded, and a second charge rammed in on the top of the unexploded charge. Subsequent experiments confirmed Noble's views, and the whole incident tended to show that, whatever might be the drawbacks of breech-loading, the accepted system was not without certain attendant dangers of its own.

However this may be, the new Elswick guns were within a year or two adopted, the Company was received back into official favour, and since that date has maintained the closest connection with the British services. In order to complete this summary of artillery progress, with which Noble had so much to do, it should be added that gun mountings owe a great deal to the inventions of Vavasseur, with whom the Elswick firm became connected in 1883 Though his name is not so widely known as that of Armstrong or Noble, it may well rank with theirs in the inner history of naval ordnance

About ten years after the manufacture of guns began at Elswick, the Company interested itself in the building of warships Starting with the gunboat "Staunch," which was in effect a floating platform carrying a single heavy gun, arrangements were made with the firm of Charles Mitchell and Co, shipbuilders, to undertake further work in connection with Elswick George Rendel devoted much of his attention to the new branch, and the management of the ordnance department was left to Noble alone. When Rendel left in 1881 to become a Civil Lord of the Admiralty, Noble became the acting head of the concern, for Armstrong had long since relaxed his close supervision of the works. In 1882, an amalgamation with the Mitchell firm was effected, and a public limited company was formed under the title of Sir W. G Armstrong, Mitchell, and Co, Limited The ordinary share capital was £2,000,000, but only £665,000 was offered to the public Applications were received for more than twice this number of shares Armstrong was of course Chairman, and Noble, who was the largest shareholder, became Vice-Chairman. The further developments of this undertaking are well known, and may be dismissed in a sentence or two A combination with the Whitworth Company of Manchester was made in 1896, and the name was changed in consequence of this to Sir W. G. Armstrong, Whitworth, and Co A large foreign connection was maintained. branches were started in Italy and Japan, while in England extension followed upon extension until the hands employed were numbered by tens of thousands. During the present war the increase of the place has been abnormal, but even before the outbreak of war in August, 1914, the Company had become one of the largest commercial concerns in the kingdom.

On the death of Lord Armstrong in December, 1900, Noble succeeded, in the natural course of things, to the Chairmanship His devotion to Elswick was constant. Not only was his the guiding hand in all large matters of policy, but day after day he was in the offices or shops attending to every detail of the business He travelled all over the world for the Company, and his judgment upon business questions was excellent. He was full of enthusiasm and energy, while no leader could have supported more loyally those who worked with him, or could have inspired more loyalty in his associates. He was especially successful in dealing with industrial problems, and, as Chairman of the Engineering Employers' Federation, handled with discretion difficult negotiations upon labour and wages.

He found time to deliver lectures to various Institutions and to contribute papers to their Transactions. These are full of new facts, and many of them constitute a distinct step forward in their own special subject.

A collection of some of his papers has been published by John Murray for the convenience of those who wish to study his work closely. He himself refers to this collected edition in the following terms —"Having entered the service when rifled artillery was not thought of, having served as Secretary to the Committee which introduced rifled artillery, and having been more or less connected with all the great changes which have taken place, both as regards the guns, their mountings, equipments, and propellants, it may be that the present volume gives, in some respects, a not uninteresting history of the immense changes that have taken place in the Naval and Land Service Armaments."

It is impossible in the space available for this memorial notice to do more than refer briefly to some of his lectures and papers

In his first paper contributed to the Royal Artillery Institution in 1858 entitled "On the Application of the Theory of Probabilities to Artillery Practice," he used Encke's fundamental formula giving the functional form of a "probable error." Much experimental work is involved, and the paper clearly establishes the great accuracy of rifled artillery in comparison with smooth bore ordnance He concludes with the remark "There is, perhaps, no branch of mathematics from which more information of importance to practical artillerymen can be obtained than from the Theory of Probabilities." The paper shows that, at the early age of 27, Noble had thought deeply on the subject that was to dominate his whole life, and that he was already coming to the front in securing its advancement

His second paper (1863), also contributed to the Royal Artillery Institution, is an account of his early researches on the velocities of projectiles fired from guns These were made with apparatus constructed by the Belgian officer, Major Navez In essentials this apparatus was somewhat similar to that now in use, but its details, as may well be imagined, were very inferior to those adopted in our present instruments. Notwithstanding these drawbacks, Noble succeeded in obtaining very valuable results. He investigated the variation in the velocity of the projectile caused by variations in its own weight and in the weight of the powder charge, he also gives a diagram of the actual path of the projectile fired with the gun laid at a small angle of elevation.

The third paper ('Philosophical Magazine,' September, 1863) is a mathematical investigation, based on certain assumptions, as to the effect of rifling on the muzzle velocity of a projectile fired from a rifled gun It was one of the many pieces of work done by Noble in forcing forward the adoption of rifled ordnance to replace smooth-bore guns. He calls on the Government to make experiments to assist in the solution of this matter

In Noble's fourth paper ("On the Tension of Fired Gunpowder," 1871) we come to the first of that series of investigations into the behaviour of explosives and artillery which will render his name famous In this paper he brings under review the work of his predecessors, M. de la Hire, Robins, Count Rumford, Colonel Cavalli, and Major Rodman, of the United States. His investigations led him to the invention of improved crusher gauges, to measure the maximum pressure at any point in the bore of a gun when fired (a subject to which he in his after life gave continued attention). These gauges were essentially the same as those in use at the present time

From the use of these instruments he was led to design the modern chambered gun, and to the adoption in the British Service of the moulded form of powders, which, while giving the same, or even a higher, muzzle velocity to the projectile than that previously obtained, gave in the gun a much lower maximum pressure than was until then believed possible

The following important conclusions were reached in these investigations —

(*a*) "Our old rule of proof for powder, that of the *éprouvette* mortar, seems with our present lights to be specially designed to produce in powder those qualities whose absence we most desire"

(*b*) "The maximum pressure of fired ordinary gunpowder (density being unity) , unrelieved by expansion, is not much above 40 tons to the square inch"

Noble, in his fifth paper, "On the Pressure Required to give Rotation to Rifled Projectiles," contributed to the 'Philosophical Magazine' in 1873, continues and extends the investigations given in his third paper

His sixth paper, "Researches on Explosives," Parts I and II (contributed in collaboration with Sir Frederick Abel to the 'Philosophical Transactions,' 1875 and 1879), was of very great interest and value, and forms the basis of all modern internal ballistics

It dealt with explosives (principally gunpowder) from every point of view, historical, chemical, thermal, physical, mathematical, and mechanical

The objects set out to be obtained were stated to be as under:—

1st To ascertain the products of combustion of gunpowder fired under circumstances similar to those which exist when it is exploded in guns or mines

2nd To ascertain the "tension" of the products of combustion at the moment of explosion, and to determine the law according to which the tension varies with the gravimetric density of the powder

3rd To ascertain whether any, and if so what, well defined variation in the nature or proportions of the products accompanies a change in the density or size of grains of the powder

4th. To determine whether any, and if so what, influence is exerted on the nature of the metamorphosis by the pressure under which the gunpowder is fired

5th To determine the volume of permanent gases liberated by the explosion

6th. To compare the explosion of gunpowder fired in a close vessel with that of similar gunpowder when fired in the bore of a gun.

7th. To determine the heat generated by the combustion of gunpowder, and thence to deduce the temperature at the instant of explosion

8th. To determine the work which the gunpowder is capable of performing on a shot in the bore of a gun, and thence to ascertain the total theoretical work if the bore be supposed of indefinite length

This list is sufficient to show the ambitious nature of the programme to be followed

The work was executed with complete thoroughness. Experimental apparatus had to be devised at various stages, and as the field of operations was very largely a new one, this had to be done in many cases with little or no assistance from the work of previous experimenters The results obtained afford the first reliable data on the subject of fired explosives. They were not simply tabulated as those obtained by observations of the recording instruments, but were all carefully weighed, one against another, to ensure the rejection of any that proved unable to withstand crucial investigation and test For instance, the records of crusher gauges were brought under repeated and searching investigation, and the gauges were continually improved till their indications were consistent with the observed muzzle velocity of the projectile

Some earlier experimenters had recorded by crusher gauges pressures so great (in one instance 50 tons per square inch) as to be quite inconsistent with the actual muzzle velocity of the projectile Noble, by his wider knowledge and acute critical faculty, kept free from such errors. His delicate and accurate apparatus, by means of which he ascertained the time taken by a shot in traversing specified distances along the bore of a gun, gave him the means of calculating the effective mean pressure on the base of the projectile and of coupling this up with the indications of the crusher gauges This was the first work of the kind, and was carried out so completely as to leave the results firmly established beyond possibility of disproof He was, indeed, the first to ascertain what took place as regards internal dynamics when a gun was fired

He found that, in a space entirely confined, theoretically the temperature of explosion of ordinary gunpowder is about 2200° C. Further, that the total theoretic work of fired gunpowder when indefinitely expanded is about 486 foot tons per lb of gunpowder

The work done in these researches was so great that it cannot be effectively summarised; the paper itself must be studied by those who wish to follow the subject.

In his next important paper, "Heat Action of Explosives," a lecture delivered at the Institution of Civil Engineers, 1884, he brings under review the properties and behaviour of explosives, such as the fulminates of silver and of mercury, potassium picrate, guncotton, and nitroglycerine, in comparison with those of gunpowder.

The work is largely devoted to the effect of explosives on the interior surfaces of enclosing vessels, on guns as a whole, and particularly on the walls of the bores of guns when explosives are used as propellants. It contains also an elucidation of the considerations which should lead to a choice of propellant, so as to secure as far as possible the best combination of the following objects, in association with a given muzzle velocity of projectile, viz least maximum pressure in the bore, least maximum temperature in the bore, and least rate of erosion of the bore. Other explosives are dealt with, such as gaseous mixtures of hydrogen and oxygen, etc, and of flour dust and of coal dust suspended in air

He points out incidentally that weight for weight the potential energy stored up in a mixture of proper proportions of hydrogen and oxygen is higher than that of any other known mixture Gunpowder is simply an intimate mixture of saltpetre, charcoal, and sulphur, the parts of which do not undergo any chemical change during manufacture It is not a definite chemical composition such as guncotton, nitroglycerine, and similar explosives In the strict sense of the term, gunpowder does not explode, it cannot be detonated as can guncotton and nitroglycerine, it simply burns, although with great rapidity, and at a rate depending largely on the pressure to which it is subject when being fired

It was found that if in the open air a pebble of powder required two seconds for its combustion, a similar pebble in the bore of a gun was consumed in about the 1/200 part of a second, showing that the rapidity of combustion depended on the pressure under which the explosive was burnt.

He also found that the temperature of explosion of guncotton is at least double that of gunpowder Platinum wire and sheet placed in the explosion vessel prior to the explosion either disappear altogether or are found in minute globules welded on the surface of the apparatus

At the date of the lecture he says gunpowder has, as yet, despite some disadvantage, no competitor which can be compared with it as a propelling agent for artillery purposes, at all events in cases where large charges are requisite At this time 830 lb had been fired in a single charge from a 100-ton gun. The potential energy of 1 lb of gunpowder is as nearly as possible 1/10 of that of 1 lb. of coal and 1/40 of that of 1 lb of hydrogen.

From these data he shows that gunpowder or similar explosive can never be economical as a motive power in the ordinary engineering sense of the term He makes the interesting remark that, regarding the earth as a huge projectile, and supposing that we could utilise the whole energy stored up in gunpowder to effect its propulsion, we should require a charge of 150 times its own weight to communicate to the earth its motion in its orbit.

In his address to the Mechanical Science Section of the British Association at Leeds in 1890, he reviews the advances made in mechanical science in relation to the naval and military services, and traces the development of the cast-iron smooth-bore guns, used in the Siege of Sebastopol, and not admitting of much accuracy of aim, to the rifled cannon scientifically built up of the steel in use at the time of his lecture.

He gives details showing "that the changes which were made within the period of ten years following 1854 were far more important and wide spreading in their character than were all the improvements made during the whole of the great wars of the eighteenth and the early part of the nineteenth century" He states, "Indeed it has always struck me as most remarkable that during the long period of the Napoleonic and earlier wars, when the mind of this country must have been to so large an extent fixed on everything connected with our naval and military services, so little real progress was made."

The various steps are given of the progress made in passing from the types of guns and mountings used by Nelson and his predecessors to the 110-ton gun of the "Victoria" Nelson's "Victory" carried an armament of 102 guns consisting of 68, 42, 32, 24, and 12 pounders, the "Victoria," the first British warship, built by the Armstrong Company at their Elswick Yard with armament largely designed by Noble himself, had two 16¼-in guns each of 110 tons, one 10-in 30-ton gun, twelve 6-in 5-ton guns, and several smaller guns The largest charge of powder used on the "Victory" was 10 lb, in the "Victoria" the charge was nearly 1000 lb One broadside from the "Victory" consumed 355 lb of powder, one from the "Victoria" 3120 lb

The paper refers to Noble's growing connection with naval architecture, and is the first of his papers to mention the possible use of cordite, "originated by the Committee of Explosives, of which Sir F Abel is President," as a propellant in ship guns He says "We are as yet hardly able to say that cordite in very large charges is free from this tendency to detonation but I think I may say that up to the 6-in gun we are tolerably safe; at least so far I have been unable, even with charges of fulminate of mercury, to produce detonation"

He traces the gradual change from wood truck carriages, worked entirely by man power, with hand tackles for training, wood "quoins" for giving elevation and depression, and hemp breeching for taking the recoil, up to the methods then in use In modern practice every motion of the guns, training, elevation, depression, running in and out, and also the service of ammunition and loading, was performed by hydraulic power, "worked out by my friend and late partner Mr George Rendel, and up to the end of 1881 all details connected therewith were made under his management."

In a "Note on the Energy absorbed by Friction in the Bores of Rifled Guns," in the 'Proceedings of the Royal Society,' 1891, Noble returns to the question of the effect of rifling on the muzzle velocity of projectiles Many experiments had been made and various kinds of propellants used, including "cordite, a propelling agent which promises to be of great value, and for which we are indebted to the labours and experiments of Sir F Abel and Prof. Dewar."

Three forms of rifling were dealt with, viz.—

(1) Rifling of uniform pitch all cut parallel to the axis of the bore (*i.e.*, it was of infinite pitch).

(2) A uniform pitch of about 1 turn in 35 calibres

(3) A uniformly increasing pitch of from 1 turn in 100 calibres at the breech to 1 turn in 35 calibres at the muzzle Different forms of driving band were used

The general result showed that both with powder and with cordite there was somewhat more loss of energy with the increasing pitch or parabolic system of rifling, than with a rifling of uniform pitch identical with that at the muzzle in the parabolic system The loss of muzzle energy, *i.e.*, energy calculated from the translatory motion alone, was in the case of cordite about 1 43 per cent in the uniform system of rifling of 1 turn in 35 calibres, as compared with the similar form of rifling grooves, but having infinite pitch, and was similarly about 2 3 per cent in the case of the parabolic or increasing twist system of rifling

Noble delivered the James Watt Lecture in his native town of Greenock on February 12, 1892, taking "Internal Ballistics" as his subject

The lecture brings together many matters previously dealt with in these abstracts, but the following may be specially mentioned He says, "although I do not deny that crusher gauges placed in the chase of a gun may give valuable indications, I still consider that unless confirmed by independent means, the accuracy of their results is not to be relied on" He also says —

(*a*) That with service powders about 57 per cent. by weight of the products of explosion are non-gaseous

(*b*) With the same powder about 43 per cent of the products of explosion are in the form of permanent gases, and that these gases at a temperature of 0° C and at a barometric pressure of 760 mm occupy about 280 times the volume of the unexploded powder

(*c*) That, at the moment of explosion, the non-gaseous products are in a liquid state

(*d*) That, at the moment of explosion, the temperature of the products is nearly 2200° C, and that the mean specific heat of the products of explosion at the temperature of explosion is about 0 31

The tension of fired guncotton is very high, and "provisionally I have placed it about 120 tons per square inch, but all efforts actually to measure with any degree of accuracy these enormous pressures have so far proved futile"

In his "Preliminary Note on the Pressure Developed by some New Explosives," contributed to the Royal Society in 1892, Noble says: "Artillerists of all nations are pretty well agreed that, save under exceptional circumstances, the maximum working pressure should not exceed 17 tons per square inch. The reasons for this limitation are weighty" (At the present time, owing to metallurgical improvements, the pressure per square inch has been increased to about 20 tons.)

In a paper contributed to the Royal Society in 1894, he gives an account of some "Researches in Explosives," made in conjunction with Sir F. Abel and Prof Dewar In this he deals with the erosion of guns and the behaviour of guncotton and cordite when fired With regard to guncotton, he says: "The ease with which guncotton can be detonated renders it unsuitable for use as a propulsive agent, unless this property be in some way neutralised"

As regards cordite (still the British Service propellant), he says "Cordite does not detonate, at least, although I have made far more experiments on detonation with this explosive than with any other, I have never succeeded in detonating it"

In his British Association paper at Oxford in 1894, Noble deals with "Methods that have been Adopted for Measuring Processes in the Bores of Guns" He refers to results obtained by earlier experimenters on this subject, and quotes Count Rumford, who estimated the pressure at over 101,000 atmospheres, or 662 tons per square inch He remarks that, "were the pressure anything approaching that which he gives, no gun that ever was made would have a chance of standing it" He then details his own experiments on crusher gauges, and points out how necessary it is to check all such indications by his chronoscopic apparatus for ascertaining the times taken by a projectile to pass along from point to point of the bore

At the summer meeting of the Institution of Naval Architects at Newcastle-on-Tyne, July, 1899, Noble read a paper "On the Rise and Progress of Rifled Naval Artillery" In it he describes the advance made since 1850 from the guns which then formed the principal armament and the means of working them in the most powerful frigates and line-of-battle ships, to the guns and means of working them in the first-class cruisers and battleships at the time of his paper, which constituted the chief work he had been engaged upon during his life. The paper was in every respect worthy of its subject and appropriate to the locality and the occasion It showed that Noble's work in connection with the design of guns, their methods of installation and their means of working, had had a great and lasting influence on the designs of ships, and the methods of armour protection of their more important parts Many famous warships embodying his gunnery arrangements had been built by the Elswick Company for the British Royal Navy and for foreign navies, these included the Chilean "Esmeralda" of 1883, of 2974 tons displacement, which had a speed of 18¼ knots and an armament of two 10-inch B.L. guns and six 6-inch B L. and smaller guns She was an especially famous vessel, and may be called the pioneer of the fast protected cruiser type

In the discussion following the paper, Sir Edward Reed described it "as the very fullest and best record of work in connection with naval gunnery that we have ever had" Sir William White, in following Sir Edward Reed, was equally emphatic on the value of the paper

The last of his papers which will be noticed was read at the Royal Institution in 1900, and deals with "Some Modern Explosives" It discusses the growing use of cordite as a propellant in guns, and the utility and the limitations of crusher gauges. Noble remarks that the kinetic theory (of gases) has for artillerists a special charm, because it indicates that the velocity communicated to a projectile in the bore of a gun is due to the bombardment of that projectile by myriads of small projectiles moving at enormous speeds, and parting with the energy they possess, by impact, to the projectile

The difference between the explosion of gunpowder fired in a close vessel and that of guncotton or lyddite when detonated is very striking. The former explosion is noiseless, or nearly so The latter, even when placed in a bag, gives rise to an exceedingly sharp metallic ring, as if the vessel were struck a sharp blow with a steel hammer

In this paper he says, "I may here mention that with a 100-calibre 6-inch gun, and with a projectile of the dimensions of the ordinary 6-inch projectile, but of aluminium, I have obtained a muzzle velocity of close upon 5000 feet per second" This is probably the high-water mark of muzzle velocity

To conclude with a few more personal notes, it may be said that Noble presented an unusual combination of qualities, for while he was a great man of affairs, immersed in practical questions of administration, he never for a moment lost his interest in science, but remained an eager student almost to the end of his days. He would return from a long spell of Elswick duties to spend his evenings in calculations or investigations in his own laboratory, working late into the night, and sometimes, when he was a young man, all through the night

He always endeavoured to secure the best and most profitable union of science and industrial economics. This combination, on account of the present (1916) war circumstances, and the circumstances which we believe will prevail at the end of the war, has now come to be appraised at high value and diligently cultivated

The President of the Royal Society, Sir J J Thomson, speaking at a recent meeting at Teddington of the General Board of the National Physical Laboratory of the "loss we have sustained by the death of Sir Andrew Noble, who was one of the founders of the Laboratory and always one of its most generous supporters," said: "I think, too, that the principle with which the Laboratory is associated—the application of science to industry—owes a very great deal to the example of Sir Andrew Noble, because he was conspicuous amongst engineers and manufacturers for the zeal with which he applied science to his industry, and nobody could say that he had been ruined by it. An example of that kind speaks to the public much more confidently than any arguments scientific men could put forward."

In his own sphere he came to be recognised as the leading authority of the day, and he served on important committees dealing with questions of guns and gunpowder. Outside these subjects he took little part in public life, and his appearances on the platform were rare. He was a member of the Court of Assistants of the Worshipful Company of Shipwrights. He associated himself with Mr. Chamberlain's proposals for reforming the tariff, and was a member of the Tariff Commission He also served as High Sheriff of Northumberland in 1896 In 1910, on completion of his 50 years' service at Elswick, the city of Newcastle-on-Tyne made him a freeman, and gave a banquet in his honour

With all his overwhelming activity and industry, he was the most human of characters, and enjoyed his brief periods of leisure with the thoroughness of a boy He was extremely fond of all games · real tennis, lawn tennis, racquets, billiards, bridge, all commanded his enthusiasm and gave him genuine pleasure. He was also very fond of shooting, and was a fine shot himself, his great wealth enabled him to cultivate these tastes For many years he rented the splendid estate of Chillingham, and a first-rate shooting in Norfolk He built a real tennis court at Jesmond, and about ten years ago bought the estate of Ardkinglas on Loch Fyne, where he built himself a house

His house at Jesmond, near Newcastle, was for many years the scene of visits from interesting and well-known men. Not only were his own leading fellow-countrymen, statesmen, diplomats, scientific celebrities, soldiers, and sailors entertained there, but the connection of Elswick led to many pleasant visits from foreigners. It was also Sir Andrew's custom to give annual dinners at one or other of his London clubs to his scientific, military and naval friends He was often present, as a member, at the dinners of the Royal Society and Philosophical Club, and the Smeatonian Club, and enjoyed these functions thoroughly He was always proud of his early connection with the Royal Artillery, and took great interest in anything that concerned that famous regiment, especially in any artillery officer who seemed likely to advance those lines of scientific enquiry in which Noble himself was so distinguished

Above everything he was the soul of hospitality, and liked nothing better than to gather about himself his family and his friends He was exceptionally happy in the circumstances of his home life. Lady Noble, herself possessed of great character and enjoying a wide popularity, made a perfect hostess to the many guests who came to her houses in Northumberland and Scotland. There were four sons and two daughters of the marriage The present baronet, Sir George Noble, was in the 13th Hussars, and served in the South African War, two of the sons are directors of Elswick, and the youngest is a director of Lloyd's Bank Sir Andrew centred in himself, to a remarkable degree, the affection of his children and grandchildren, he looked forward eagerly to family meetings at holiday times, at Christmas at Jesmond, or at Ardkinglas in August. No visitor could fail to be impressed with the patriarchal position he filled, and the keen pleasure which he took in seeing others enjoying themselves.

At the beginning of 1912 he resigned his active management of the works at Elswick, though he remained the Chairman of the Board until his death.

He was elected a Fellow of the Royal Society in 1870. He served on the Council 1884–5, 1889–90, 1898–9, 1899–1900, 1909–10, in all five years.

He was Vice-President in 1899–1900 and in 1909–10, and was awarded the Royal Medal in 1880 He was made a C B in 1881, a K C B in 1893, and a Baronet in 1902 He was the recipient of several honorary degrees. He was D Sc of Oxford, Sc D of Cambridge, and D C L of Durham

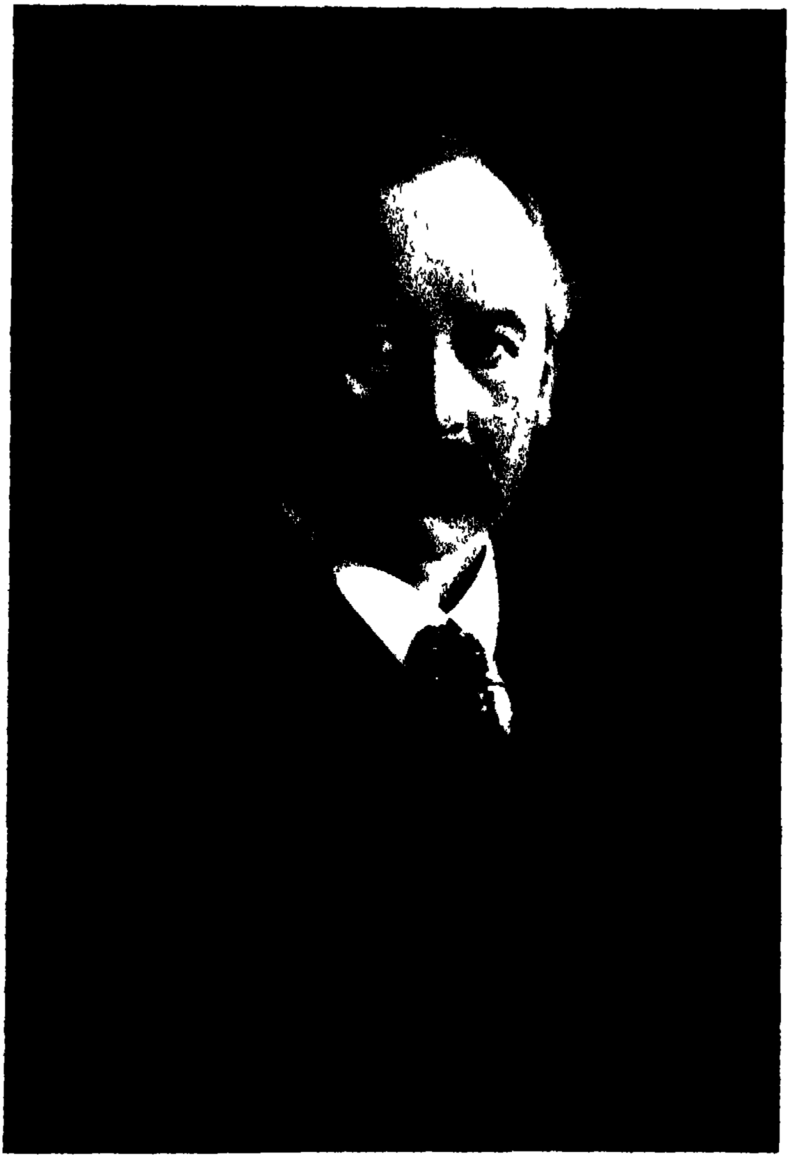
He also received many foreign orders and knighthoods He valued greatly the recognition by Japan of the services which his own work and that of his firm had been able to render to the development of that country His connection with Italy was also very close and of long standing He visited Rome and Naples many times His orders were as follows —First Class Sacred Treasure of Japan, First Class Rising Sun, Grand Cordon of Orders of Osmanie and Medjidie and Rose of Brazil, Grand Cross of Crown of Italy, Dragon of China, Commander of Jesus Christ of Portugal, Knight of Order of Charles III of Spain, Foreign Member of Accademia dei Lincei, Rome

P W

SILVANUS PHILLIPS THOMPSON, 1851–1916

SILVANUS PHILLIPS THOMPSON was born in York in 1851, and died in London, June, 1916 He attended Bootham School, York, in which his father (a botanist) was one of the masters, then the Flounders Institute for training teachers, at Ackworth After becoming a B A of London University in 1869, he was Science Master at Bootham As a scholar he attended the Royal School of Mines in London, and in 1875 graduated B Sc in London University, being first in Science Honours that year Next year he spent at Heidelberg He was Lecturer in Physics at Bristol, and became Professor of Physics in 1878 In 1878 he gained the D.Sc of London He became Principal and Professor of Applied Physics in the City and Guilds Finsbury College in 1885, and held those appointments till he died

He became a Fellow of the Royal Society in 1891, and served on the Council 1906–7 and 1911–13 He was President of the Institution of Electrical Engineers in 1899, and President of the Physical Society of London in 1902 He was the President of other scientific societies The Junior Engineers, the Rontgen Society, the Society of Illuminating Engineers. He held the honorary degree of Doctor of Medicine and Surgery of the University of Königsberg, the honorary LL.D. of Birmingham, and the honorary D.Sc of Bristol. He was a member of many British and foreign scientific societies, of which the Royal Academy of Sciences of Stockholm may be specially mentioned as giving an important vote.



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Silv. P. Thompson

Since 1876 his papers on scientific subjects are so numerous that readers must be referred to the R.S. Catalogue. These papers, of which 166 may be regarded as important, were mainly on experiments in light and electricity, but sometimes on other physics subjects. His own original work was mainly in the discovery of new experimental illustrations of physical laws He had a fine voice, in good training, and it carried far without effort, he was eloquent and very clear in statement, and studied his audience, so that he was a most effective speaker, and might be said to be a perfect lecturer Perhaps it was in consequence of this that he devoted so much time to the elaboration of lecture experiments and illustrations, many of which are described in his published papers He spoke readily in French, German, and Italian, and was of great service in international scientific conferences He was successful as a professional scientific adviser He was of untiring industry and diligently attended scientific meetings, his part in discussions being of great importance He not only described his own work, but he was probably, in his time, the very best expositor of the work of others, and this is what gives value to his books, some of which are still standard text-books The most successful of these is his 'Elementary Lessons on Magnetism and Electricity,' published first in 1881, it ran through more than forty editions and reprints; the last edition in 1915 is likely to maintain its high reputation Like his other books it has been translated into many foreign languages. Other books are · 'Dynamo-electric Machinery,' first edition 1884, seventh edition, in two volumes, 1905, 'The Electromagnet,' 1891, 'Polyphase Electric Currents,' 1895, second enlarged edition 1900, 'Light, Visible and Invisible,' 1896, 'Notes on the Translation of Gilbert's De Magnete,' 1900, 'Design of Dynamos,' 1903, 'The Manufacture of Light,' 1906; 'Calculus Made Easy (by F.R S),' 1910, 'The Quest for Truth,' 1915, 'Life of Philip Reis,' 1883, 'Life of Faraday,' 1898 His Life of Lord Kelvin, 1910, has taken its place as one of the few really great English biographies.

The following books were privately printed.—'William Sturgeon the Electrician,' 1891, 'Gilbert of Colchester, an Elizabethan Magnetiser,' 1891, 'The Magic Mirror of Old Japan' (Sette of Odd Volumes, 1893), 'William Gilbert and Terrestrial Magnetism,' 1903, 'Gilbert of Colchester,' 1903, 'Pied Piper of Hamelin' (Sette of Odd Volumes, 1905), Preface to reprint of two tracts on 'Electricity and Magnetism,' by the Hon Robert Boyle (Sette of Odd Volumes, 1898) His miscellaneous and religious publications were.— "The Sixth Sense," Bachelor's Papers 3, 1870–71, 'On the Progress of the Theory of Natural Selection,' 1871; 'Our National System of Weights and Measures,' 1871, 'The Poems of Morris,' 1871; 'Religion and Science,' 1871; "Two Pictures" and "The Mystery of Nature," in the 'Friends' Quarterly Examiner,' 1876, 'Can a Scientific Man be a Sincere Friend,' 1895, 'Intuitional Religion: A Study of the Divine,' 1906; 'Illumination,' a lecture, 1914; 'The Quest for Truth,' the Swarthmore Lecture of the Society of Friends, 1915.

He collected curious books, and sometimes wrote about them. His collection of manuscripts, old pamphlets, and books relating to electricity is of great value; along with the Ronalds Collection it is for the future to be in the care of the Institution of Electrical Engineers. He had a fondness for old books and old things of all kinds which illustrated the history of his studies. He was of great authority in the bibliography of his studies, and indeed of general subjects. He was familiar with all sorts of byways in literature. He was most catholic in his sympathies, being a keen amateur in music, and almost more than an amateur as a painter of water colours, more especially of Alpine scenery.

I think it well to give the following extract from a paper by Sir George Newman in the 'Friends' Quarterly Examiner,' July, 1916:—

"The most interesting part of his home in Hampstead was the library. It
"was a downstairs room, furnished and fitted to be a compendious receptacle
"of pamphlets and books, severely business-like and scientific in method.
"Here he stored his wonderful collection of works on electricity and
"magnetism, a collection to which additions were continually made, and
"to which precious 'finds' were carefully brought. It consisted of about
"13,000 items, including a few ancient manuscripts relating to magnetism;
"about 900 scarce magnetic and electrical books and works relating to the
"early history of the science, printed before 1825; more than 2500 modern
"works of science and text-books of electricity and magnetism, printed after
"1825; 8000 pamphlets, 1200 volumes of periodicals and proceedings of
"societies, 200 autograph papers of eminent scientific men, and 34 precious
"Faraday MSS. It is hard to say which group of 'items' is the most
"interesting, though it is certain that their learned possessor was able to
"make any of them of absorbing and transcendent interest whenever he
"chose. He would gloat over his collection of autographs, and do a swap
"with the keenness of a schoolboy; or he would handle some old manuscript
"with reverent and even loving care; or he would argue and protest, and
"explain and declare, that before us and in our very hands was a document
"which conclusively demonstrated that such and such a much belauded
"theory of thermodynamics, believed of all men, was unthinkable, incredible,
"impossible!

"It was this unique collection which oftentimes furnished Prof. Thompson
"with the material for those delightful papers and brochures, biographical or
"other, with which he regaled first this learned society and then that. There
"was one on 'Peter Short, Printer, and His Master'; the Boyle Lecture on
"'Magnetism in Growth'; there were several on Dr. William Gilbert,
"physician to Queen Elizabeth, who in 1600 wrote 'De Magnete'; there
"were two on Volta, one on Sturgeon, and a biographical dissertation on
"Petrus Peregrinus de Maricourt, a thirteenth-century authority on the
"magnet and the lodestone; and there was the ingenious paper on the
"origin, history, and development of the compass card entitled 'The Rose of

"the Winds.' Prof. Thompson was a brilliant expositor of the mysterious "and the occult, and, to hear him discourse on the magnet, the lodestone, or "the mariner's compass, was to be taken into a world of magic.

"Many of the books in his library were valued not only for their great age, "authority, or rarity, but also because of some special interest attached to the "particular copy. Here is one with the Aldine Anchor Mark in one of its "rarest forms, some belonged to Ampère, Faraday, William Gilbert, Claude "Delaunay, S T Coleridge, Volta, Archdeacon Barlow, Ridley, and others; "some were peculiar or unique in form or substance, with something quaint "or rare about them which their owner had discovered or understood, and yet "others were like dear and tried friends, who had led him by the hand on his "own upward path of invention or discovery."

To that article I would refer readers who may be interested to know how a man might believe in the latest discoveries of natural science and yet remain a sincere Quaker Newman's description of Thompson's attitude to spiritual things is most impressive.

Thompson's friendships were faithful and sincere His conversation and manners were on the level of the high standards of the Society of Friends He loved work, but he was overworked at Finsbury, as any teacher must be who has charge of both day and night lecture and laboratory classes He died without suffering. To the end he looked happy. He attended his college duties on a Saturday, he had a stroke later in the day, and died on the following Monday

His family history is an interesting one, going back for some centuries Limiting myself to people interested in natural science and not far removed · his grandfather, Thomas Thompson, of Liverpool, was a chemist, and his grandmother's brothers were William Phillips, F R.S, a geologist, and Richard Phillips, F.R.S., a physicist and a correspondent of Faraday Another grandfather, John Tatham, of Settle, chemist, was also a botanist, and his mother was a good field botanist

He married in 1881 Jane, the eldest daughter of James Henderson, of Pollokshields. His four daughters interested themselves in natural science when at school. Sylvia and Irene have, however, taken to music or painting. Helen was second in the Cambridge Natural Science Tripos; she, like her sister, Dorothea (who is a sanitary inspector), is a B.Sc. of London.

J P.

JONS OSKAR BACKLUND (1846–1916).

Oskar Backlund was born on April 28, 1846, in the parish of Långhem, in Wärmland, Sweden His parents were very poor, and after an elementary school education the boy was put into his uncle's business. But by diligently preparing himself he succeeded in entering the University of Upsala in 1866 He there studied mathematics and astronomy under Wackerbarth, to what good purpose became clear in later life. After completing his university career, he was for some time a master in a technical school at Lulendome, in Sweden, but later we find him at the Stockholm Observatory, studying the movement of the comet of 1849, and the minor planet (111) Iphigenia In 1876 he was appointed one of the observers at Dorpat (now Jurjev) under the directorate of Schwarz, and he took a share in the meridian observations for zone +70° to +75° of the 'Astronomische Gesellschaft' Catalogue. In 1879 he was nominated assistant astronomer at Pulkovo, under Otto Struve, and used the Repsold heliometer for measures of the satellites of Jupiter—work which has unfortunately never been published But on the death of von Asten in 1878, at the early age of 36, Backlund took over his researches on Encke's Comet, and for the remainder of Backlund's own life this investigation occupied a large share of his attention. His last memoir on the comet was presented to the Petrograd Academy only in 1916, and has not yet been published

From the first, his attack on this problem attracted attention, especially that of the Imperial Academy of Sciences of Petrograd, who elected him titular member in 1883, on his being first naturalised a Russian subject. In 1886 he left the Pulkovo Observatory to take up his residence at the Academy, where he organised a body of computers to deal with the enormous mass of calculations required, for by this time Backlund had determined that all the perturbations from 1819 onwards must be recomputed The cost was defrayed by the generosity of M. E. Nobel. The results are published in the 'Memoirs' of the Academy. He received the Gold Medal of the Royal Astronomical Society for his successful treatment of this important problem in 1909, and it is needless to repeat here any elaborate account of the work when Prof. Newall's address is so readily accessible ('Mon Not.,' vol 69, p. 324). The main result is that the mean motion of the comet is accelerated, but that there have been several changes in the amount of acceleration. It was constant at the value $\mu' = 0''\cdot126$ in 1819-1858, during which period there was, however, a curious periodic term $+4'' \cos\psi$, the period of ψ being ten years. There were changes in μ' in 1858, when it fell to about $0''\cdot08$, and again in 1868 when it fell to $0''\,066$, remaining at that value until 1891. These assumptions not only give a satisfactory account of the motion of the comet, but a consistent mass for Mercury at 1/9700000, the latter result removing a long standing difficulty.

In 'M.N.R.A.S.,' vol 70, p 429, Backlund gave a supplementary account of his researches on the comet's motion 1895–1908 The value of μ' again fell in January, 1895, to 0″ 050, and probably also again in January, 1905. Backlund inferred that these sudden changes are in some way due to the crossing of a meteoric ring, near the perihelion passage of the comet. But this very considerable work was not sufficient to satisfy him. He went on with his researches on minor planets—it will presently be shown how he regarded them as related to that on Encke's Comet—paying special attention to Gyldén's work. He lectured at the University, and several of his pupils were inspired by him to calculate orbits by Gyldén's methods. He also obtained an instrument for measuring photographs, and interested several ladies who had graduated at the ladies' University in Petrograd in the measurement of stellar clusters

But in 1895, when Bredichin was compelled by ill-health to resign the Directorship of the Pulkovo Observatory, the Academy called upon Backlund to leave his peaceful life among them, and to undertake the administrative duties which he discharged so well and so faithfully until his death His great energy became manifest from the first The time given to calculation was doubled, and the number of computers was increased also In 1916 the total vote for computation was 13,200 roubles as against 1500 or 2000 roubles per year in 1895. His experience led him to engage several ladies in this work. The staff was augmented in other directions, and several first class instruments were obtained But it had long been realised that the high northern latitude of Pulkovo was unfavourable for observation, and Backlund at once set about experiments at other stations In 1896 a transit instrument and a vertical circle were set up at Odessa, and in 1898 regular observations were commenced, which soon showed the advantages of a more southerly site In 1900 the benefaction of two considerable pieces of land made it possible to contemplate a complete observatory on an adequate scale, if only the money could be found. Backlund's influence and energy were thenceforward directed to attack this difficulty, but it was not until June, 1912, that the money was ultimately voted—310,000 roubles for installation and 35,700 roubles a year for maintenance.

The physical portion of the Observatory was to be erected at Simeis (a reflector one metre in diameter), and the astrometrical part at Nicolaiev (refractor of 32 inches). All the instruments were ordered in Great Britain, but, unfortunately, the war arrested their construction, and Backlund was not to live to see his work completed. He superintended the preliminary work at both observatories, which necessitated many long journeys to and from Pulkovo. But these were only a small part of the travelling he did after his appointment as Director. He had already attended the Committee of the Astrographic Chart which met in Paris in 1889, as *savant invité* · and in 1896 he returned in the same capacity; for his

Observatory of Pulkovo was taking no direct share in the work, and he only spoke once (in favour of the greatest possible precision in measurement). But in 1909 he took a leading part in the discussions, presiding over one of the Committees, with satisfaction to all concerned. In 1904 he was invited to the Congress of Arts and Science at St Louis, and took the meeting of the British Association at Cambridge on his way to the United States

In the following year he travelled with the British Association to South Africa He attended the meetings of the International Union for Co-operation in Solar Research at Mount Wilson (1910) and Bonn (1913), various geodetic and other conferences, and the meetings of the International Association of Academies, over the last of which (in 1913 at Petrograd) he presided, entertaining the delegates after the meeting was over at Pulkovo in the most hospitable manner His great friend, Prince Galitzin, was Secretary at this meeting, and the co-operation of these two men, now both lost to the world, secured a wonderful success Both were always welcome in England, and Backlund especially came frequently—in 1909 he was here in February to receive the R A S Gold Medal, and later for the Geodetic Conference in London and Cambridge Altogether he made at least 30 journeys outside Russia in 20 years

Backlund's address at the St. Louis Congress in 1904 was an able review of the situation in Celestial Mechanics, in which he contrasted the successful negotiation of the problems for the larger planets, under Newcomb and Hill, with the uncertainties of the attack on comets and small planets The connection between these two latter problems was brought before the attention of mathematical astronomers by Encke's Comet at the beginning of the 19th century :—

"The aphelion of the comet lies within the orbit of Jupiter, the eccentricity is far greater than that of any of the hitherto known planetary orbits, and the inclination amounts to 12° If the formulæ could be found which represent the motion of this comet, the question in reference to the small planets would also be solved. It was, however, not merely from this point of view that Hansen set himself the problem of obtaining such formulæ. He doubted, in fact, the correctness of the comet's acceleration found by Encke, and hoped by means of general formulæ to be able to settle that question"

Backlund's own work, by establishing not only the existence of an acceleration for Encke's comet, but changes in that acceleration from some cause as yet unidentified, makes it necessary to consider the problem of the small planets separately, at any rate He himself had faith in Gyldén's methods for this problem :—

"The circumstance that a large portion of the small planets occur in the neighbourhood of the so-called gaps, thus causing such an increase in the perturbations that after a relatively short time these can no longer be considered as small quantities, led Gyldén to state the question in the following manner :—'Will it be possible to determine the elements as absolute constants, and so to determine the terms of long periods (thus avoiding completely, the introduction of the time explicitly) that the intermediate orbit thus obtained shall remain included within definite limits, and only differ from the real orbit by quantities of the order of the masses of the planets?' The question includes the question of stability, the principal problem thus consists in proving the convergence of the long-period series. Gyldén believed that he could establish the convergence by means of what he called the horistic method Poincaré, however, disputes the correctness of this method On this assumption, Gyldén's theory would be merely an hypothesis Even if the method is correct, it is applicable only with reference to a limited number of small planets, as it is based upon the development in powers of the eccentricities and inclinations of the disturbing and disturbed planets. Here then we stand, so far as this question is concerned, at the end of the 19th century Upon the problem presented at the beginning of the century much skill and labour has been spent, a satisfactory solution has not, however, been reached

"If now we turn to the larger planets, a more gratifying picture presents itself, etc"

An unsigned appreciation of Backlund in 'The Observatory' for March, 1917, has given us a sequel to this pronouncement from a private letter —

"The main cause why I have been so lazy [in not writing] is my occupation with a theoretical question of great difficulty. You know, perhaps, that Poincaré has criticised the theory of Gyldén in 'Acta Mathematica.' He is in many respects right, but in the most wrong Now, I regarded it as my duty to try to restore the reputation of Gyldén, which such an authority as Poincaré can easily ruin But the question has proved very hard, because it touches the boundaries of mathematical knowledge At last I have arrived at a certain result, so that I can now lay this hard work aside."

If Gyldén's work does ultimately take its stand on firm ground, the event will owe much to Backlund's devoted appreciation and elucidation. He began his study of these methods during his residence at the Petrograd Academy, as already noted, and it remained with him throughout the rest of his life. He supervised the second volume of Gyldén's 'Traité Analytique des Orbites Absolues des Huit Planètes Principales' (1908), of which little more than half was in the press when Gyldén died

Backlund often spoke of his days of residence at the Academy as a specially happy time in his life, which was high appreciation, for though the remainder of his life was perhaps subject to greater and more frequent distractions, it was undoubtedly very happy and prosperous. If he exchanged his peaceful residence in the Academy for a life of constant travel, he was a traveller who was more than welcome everywhere, owing to his wonderful geniality and sound common sense. He had an apparently inexhaustible fund of good stories, and he had a delicate touch with difficult situations. As an instance of the latter quality (though to give instances is a course of doubtful expediency), one may quote the following remark of his, which in itself explains the situation giving rise to it. "When he said that, I saw at once that he was mad, and so I spoke no more astronomy with him." People of slower perceptions might have blundered into an awkward discussion or even a quarrel in this particular case, which was very puzzling; but Backlund's reading was sufficiently prompt

He could speak many languages fluently, though with certain quaintnesses in his diction which gave it a special attractiveness In the charm of his presence it was easy to forget how considerable an achievement one was witnessing—as, for instance, when after receiving the R A S. Medal in 1909, he made, in English, a postprandial speech which few Englishmen could have equalled for grace and quiet humour One forgot to reflect that he could probably have done the same in Russian, French or German, besides, of course, his native language, Swedish And these were not the accomplishments of a professional diplomat, but of a man occupied with work of the first rank in both astronomy and mathematics, who (as we now learn but never suspected) made his own path to the university from unpromising beginnings He had, moreover, a wonderful and infectious capacity for enjoying life, combined with an ability to work under very varied conditions Thus he rejoiced in the long train journeys which in his later days took him from one of the observatories he superintended to another, because they were opportunities for mathematical work, for which he otherwise could not obtain sufficient leisure. In fine, if his life at the Academy was happier than that which brought him, to our great delight, so often to England and elsewhere, it must indeed have been supremely happy.

Many honours naturally fell to him Besides the R.A S Medal he received the Lalande Prize of the Paris Académie des Sciences, of which he was a correspondent since 1895 He was elected a Foreign Member of this Society in 1911

He died suddenly, on August 29, 1916, in his home at Pulkovo. He had just returned from Finland, where he and his wife had spent some weeks in order that (what was hoped to be) a slight ailment might be cured. The day before his death he spent actively in Petrograd, retiring early to bed on his return to Pulkovo, but he slept badly and in the morning felt so weak that he remained in bed. About 2 o'clock in the afternoon he was thirsty, and Madame Backlund was on the point of leaving the room to get a drink for him when he drew three deep breaths and expired He was buried in the cemetery of the Observatory on September 3.

[For many of the details the writer is indebted to M. Belopolski.]

H. H. T

PRINCE BORIS GALITZIN (1862–1916)

AMONG the oldest families of Russia, the Princes Galitzin can trace their pedigree back to the Lithuanian Prince Gedimine in the fourteenth century. During the reign of Ivan the Terrible one of their ancestors settled in the Muscovitish Kingdom, and his son adopted the name of Galitzin A successful General and Field Marshal, Prince Michail Mikailovitch Galitzin served with distinction under Peter the Great, and, descending in a direct line from him, we find Prince Boris Galitzin devoting his life to the peaceful pursuit of science, until the outbreak of the War, when his entire skill and experience were placed with characteristic energy at the disposal of his country

Prince Boris Borissovitch Galitzin was born on February 18, 1862, at Petrograd. His early education was received at home, where—for a time—he was under the influence of a Cambridge graduate, Mr Rogers His early family surroundings do not seem to have been happy ones, when he was 6 years old, his grandmother, the Countess Koushelev, who resided at Athens, took charge of him, and three years later his mother divorced her husband and married the Marquis of Incontri, then Secretary of the Italian Embassy at Petrograd, spending the remainder of her life in Italy With both his mother and stepfather, Prince Galitzin preserved the most friendly relations

The position and influence of the Galitzin family might have led one to expect that the path of success would have been rendered easy to any of its members possessing sufficient intelligence and energy. But this was far from being so. During the critical period of his life, Prince Galitzin had to earn his own living, and, though his fate in this respect was no worse than that of most men of science, he suffered from the additional disadvantage that the academic world of Russia did not—at first—readily receive him as one of their own. He had to fight his way up against much opposition, and his ultimate success was due to sheer merit and force of character At the same time, his early struggles gave him experience of the world, supplied outside interests which added to his enjoyment of life, and engendered a feeling of sympathy and kindness towards young and struggling men of science. This remained one of the most conspicuous features of his character during the period of his success

After the death of his grandmother, Galitzin, desirous of entering the Naval School at Petrograd, went to live with one of the teachers of that institution, who prepared him for the entrance examination. He remained a pupil of the school during five years (1875–1880), participating during the summer months in the sea voyages arranged by the school. In passing out of the school, he obtained the first place in the final examination, and was appointed "garde-marine" on the warship "Prince of Edinburgh." During 1881 he visited a number of ports in the Mediterranean, and, in company with the Grand Dukes Sergius and Paul, made an excursion to Jerusalem and Jericho He also spent some time in Rome, where he became interested in art and archæology.

In the autumn of 1881 Galitzin was promoted midshipman, but immediately resigned his commission, because the "Prince of Edinburgh" was ordered to leave for the Far East, and he saw no hope of advancing his scientific knowledge while he remained in the service. On his return to Petrograd, he was, however, foiled in his wish of entering the University, the authorities putting pressure upon him to join the Naval Academy. A severe illness forced him temporarily to interrupt his studies, and he spent the next two years with his mother in Italy, mostly at Florence. There he devoted some time to the study of social science, history, and civil law, but did not abandon science, attending lectures on chemistry, working at practical physics, and receiving private instruction in the higher mathematics

In the autumn of 1884, Prince Galitzin returned to Petrograd, and entered the Naval Academy, where he stayed two years, and ultimately obtained the second place on the list of graduates. On technical grounds—being one month short of the time required to have been spent at sea—he was unable to obtain a commission, and, unwilling to serve again as midshipman, Galitzin definitely left the Navy He met with a further disappointment when his admission to the University was made conditional on his passing another entrance examination in school subjects, including Arithmetic, Scripture, etc Galitzin was by nature endowed with an even temper, but the one thing that—more than anything else—roused his indignation was to be compelled himself, or to see others compelled, to waste time on useless objects. Those who knew Galitzin only in later life can yet easily picture how he would resent—after several years of serious study—a demand to pass an elementary examination, even though he might be fully prepared for it.

Galitzin left Petrograd, returned to Florence to learn German, and a few months later entered the University at Strassburg in the spring of 1887 His literary activity began almost at the same time It took the form of a compilation entitled 'Material for the Study of Meteorology,' based on the lectures of General Rykatchew at the Naval Academy. The work was published by the Russian Ministry of Marine.

His scientific reading and experimental work soon began to bear fruit. A paper on the relation between the curvature of liquid surfaces and the vapour pressure of the liquid in contact with the surface appeared in 'Wiedemann's Annalen' in 1888, and in the following year the same publication contained his doctor-dissertation. This dealt both from the experimental and theoretical point of view with Dalton's law of the independence of pressure due to each component of a mixture of gases; it was followed by a further paper "On the Range of Action of Molecular Forces." Having obtained his degree at Strassburg, Galitzin returned to Petrograd, where he was admitted to the University without further formalities, and during the winter of 1890 passed the various examinations leading to the doctor's degree. Soon afterwards he was appointed Lecturer in Physics at the University of Moscow, but his troubles were not yet over A dissertation which he presented to the Faculty was adversely criticised by his Professor, and he was asked to revise and amend it There appears to have been something of a split in the University, sides being taken for and against the merits of Galitzin's work, and the discussion even spread to other Universities He once more left Russia to spend another year at Strassburg

It cannot be asserted that the earlier scientific productions of Prince Galitzin showed signs of great originality They are the work of a conscientious student who pursues a systematic study of the various branches of Physics, and in his reading naturally comes across parts which are or seem obscure He set to work to treat the subject in an independent manner, not so much for the sake of arriving at a new result, as to satisfy himself, to clear up doubts, or to supply a more rigid investigation

It was probably during his second visit to Strassburg that Galitzin first became interested in seismology, for Dr E V Rebeur-Paschwitz, one of the pioneers of the subject, was then at work in that University, and corresponding with Dr Milne on an organized system of observatories Galitzin's stay in Germany did not last long, as his countrymen were beginning to recognize his merit During the autumn of 1893 he took over the Professorship of Physics at the University of Juriev, and soon afterwards was appointed to the Directorship of the Physical Laboratory of the Imperial Academy of Science at Petrograd This position he held until shortly before his death In his laboratory he had great opportunities for experimental work, but he was fond of teaching, and also accepted the Professorship of Physics at the Naval Academy. He was interested in the education of women, and lectured on Experimental Physics at the Women's Medical Institute.

Galitzin's various duties forced him to keep himself informed on all branches of Physics, but there are indications that the theory of vibrations attracted him most This showed itself in his seismological and optical work As regards the latter we must note his investigations both theoretical and experimental on the échelon spectroscope, which helped him in furnishing a satisfactory experimental demonstration of the apparent change of wave-length due to the motion of a radiator A few years previously, Belopolsky had constructed an apparatus in which a change of wave-length was produced by the reflection of sunlight from a moving system of mirrors, and what is generally called Döppler's principle was verified in a general manner Replacing sunlight by the radiation from a mercury lamp, and Belopolsky's prism spectroscope by an échelon of high resolving powers, Galitzin in conjunction with J. Wilip was able to obtain accurate numerical results proving that the calculated velocity of the mirror agreed with that directly measured to within less than 1 per cent Careful and detailed determinations of various absorption and emission spectra were also obtained by the collaboration of the two physicists

We must now pass on to his contributions to modern seismometry, which form Galitzin's chief claim to scientific eminence By these researches he not only succeeded in placing a young and struggling art on a sure foundation, but the scientific accuracy and thoroughness of his work may well serve as a model for all similar work

About 1900, or probably before, he appears to have selected seismometry as his ultimate vocation, and his first published paper in this subject emphasizes the ideal that, as the earth movement at any point involves six quantities, viz, three components of linear displacement and three components of rotation, an observing station ought to have six instruments to measure these He soon realised the difficulties of attaining this ideal, and so concentrated his efforts on the major problem of measuring the three linear components, while the minor problem of the rotations was set aside, although not forgotten

Three main defects affected seismographs when he commenced his work These were (1) lack of sensitiveness, (2) incomplete and imperfect damping, (3) reliance on mechanical registration Galitzin appears to have introduced *in succession* the principles of aperiodic electromagnetic damping (which is almost ideal), the principle of electromagnetic magnification (by converting the pendulum movement to movement of a ballistic galvanometer by electric currents generated by the pendulum movement), and photographic registration on a very open time scale (which had been already used by Dr John Milne on a small scale)

Five or six years of pioneer work went on at Poulkovo, during which time the apparatus was perfected, the method of standardisation elaborated, the analysis and deduction of the earth movement from the seismograms systematised, and various special features elucidated. About 1907, Galitzin was able to announce the important result that, in accordance with the accepted view, the first observed impulse represents a longitudinal disturbance, measurement of the amplitudes to N or S, and to E or W, recorded by the horizontal seismographs, gives the azimuth of the epicentre of an earthquake to ½° in favourable cases Moreover, since the distance could be determined from the observed time interval between the first and second phases by aid of Wiechert's empirical time curves, it followed that an epicentre could be determined from "observations at a single station"

This result was not only of practical utility, but it was also of profound importance in the theory of the internal constitution of the earth. This was soon recognized by eminent geophysicists throughout Europe, and in 1910 Galitzin visited Paris to set up one of his horizontal pendulums in the observatory there, and this country to install two of his pendulums at Eskdalemuir Observatory

During this visit, he became personally acquainted with Dr John Milne, and the acquaintance soon ripened into the sincere friendship of these two great pioneers of instrumental seismometry

We may take this opportunity to correct the erroneous impression that Milne was opposed to the damping of seismographs What he objected to was mechanically imperfect damping, but when he became aware of the excellent results obtained by Galitzin, he at once appreciated them, and we have good reasons for knowing that before his death in 1913 Milne had decided to introduce electromagnetic damping into his apparatus

Galitzin continued to work at the various problems of geophysics which are elucidated by earthquake measurement, such are, the significance of the angle of emergence of the rays, the speed and damping coefficient of Rayleigh waves, the determination of the depth of focus, and the classification of microseisms. On the experimental side, he perfected his seismograph for the vertical component of the earth movement The apparatus was exhibited at the meeting of the International Seismological Association at Manchester in 1911, when Galitzin was elected President for the ensuing three years

Meanwhile, the Observatory of Poulkovo had been thoroughly equipped as a continuous recording seismological station, and the first weekly Bulletin was issued in January, 1912. It is the most comprehensive weekly bulletin that has ever been attempted, and it is issued within a few days of the completion of the week to which it refers

The extensive Russian seismological service organized by Prince Galitzin and financed by the Russian Government was completed during the same year It had involved the establishment of five first order stations, and of a number of second order stations

In 1912 Galitzin attended the celebration of the 250th anniversary of the foundation of the Royal Society In the same year he returned to this country to attend the meeting of the Mathematical Congress at Cambridge, and he delivered an important address.

After the meeting he paid a private visit to his friend, Mr Walker, at Eskdalemuir He arrived at 8 A M one morning and left on the following day at 10 P M. to catch the night mail for Paris Seismology was the main topic of conversation, and the idea occurred that from the azimuth determinations of epicentres at two stations, the epicentre could be uniquely determined. Nothing would satisfy Galitzin but to work out an example forthwith He had the requisite data from Poulkovo in his handbag The result was quite satisfactory

In the evening he took up a violin, and played with masterly touch and exquisite feeling some Scotch airs, and afterwards joined in some pianoforte duets. He complained of some bronchial trouble that affected his heart, but was full of enthusiasm about the new work he was to take up in Russia, and which had not yet been publicly announced This was the Directorship of the Russian Meteorological Service, with which he had been entrusted He set about the re-organisation of the department with characteristic zeal, organised extensive observations in atmospheric electricity and a magnetic survey of the Russian Empire His interest in seismology did not, however, diminish, and papers continued to appear We may cite an examination of the value of the method of determining epicentres from azimuths at two stations, and his last paper was read at the Paris Academy of Sciences in June, 1916 (after his death), in which he showed that of all the earthquakes recorded at Poulkovo, the epicentres of 18 per cent had been determined from observations at Poulkovo alone

Before we pass to the close of his career, we must note (what is not generally known) that Prince Galitzin applied his special knowledge of seismometry to the problem of the effects of local mechanical vibration on buildings, and devised special instruments for measuring the accelerating forces experienced

The triennial meeting of the International Association of Seismology was to take place at Petrograd in September, 1914, under the Presidency of Prince Galitzin. All arrangements had been completed, and many were looking forward to the renewal of international friendships. But in August Europe was plunged into a war that struck at the roots of international scientific unity The meeting had to be abandoned, and Galitzin, like others, turned to help his country in a time of need In January, 1915 he wrote with much sadness, but with no bitterness, of the effect of the war on science · "I wonder when our seismological meeting will take place, and under what conditions. It will be a very difficult and painful thing to manage, but, at all events, we Allies must stick together, and try and smooth everything, and try to get the whole business again into the swing" Again, in August, 1915, he wrote, among other matters of importance in the war · "I have constructed lately an instrument which measures directly the instantaneous values of the acceleration of the ground for a quite arbitrary law of motion" "I have been elected Halley Lecturer at Oxford for next year, and, if the war is over then, I hope to be able to come over to England and see all my friends It would be such a great pleasure for me Notwithstanding the hard times we are passing through now, everyone here is sure of final victory"

The war not being over, he had to resign the Halley Lectureship.

In March, 1916, he wrote again "I am awfully busy at present with some special work concerned with the war, and I have very little time to attend pure scientific problems. I am organising also special workshops for making all kinds of meteorological and other instruments, so as to be, as far as possible, quite independent of all foreign instrument makers. It is a hard task, but I am working steadily at it" In his various letters there is no indication of any weakness of health, but it is to be feared that he overtaxed his wonderful powers of endurance

Galitzin was elected a Foreign Member of the Royal Society on March 23, 1916, and died on May 17 of the same year Thus, at the early age of 54, ended a life of great achievement and of great promise for the future.

G Darboux

Somewhat brusque in manner, Prince Galitzin endeared himself to all true scientists in this country by his intense enthusiasm, his frank and fearless expression of opinion, his great kindliness, his appreciation of the work of others, and, perhaps not least, by his intense dislike of humbug

A S
G W W

J G DARBOUX, 1842-1917

JEAN GASTON DARBOUX, Permanent Secretary of the Paris Academy of Sciences, who died on February 23, 1917, was born at Nîmes on August 13, 1842, in a house which had once been a chapel of the cathedral His father having died in 1849, he and his younger brother were brought up under the care of their mother, and their earliest education they received at the local Lycée, which they attended as day-boarders In 1859 young Darboux entered the special class for mathematics at the Lycée of Montpellier, and in 1861 he headed the lists for admission to the École Normale Supérieure and the École Polytechnique Of these two schools he chose the former, somewhat to the surprise of his friends, his decision being based upon a strong inclination towards the teaching profession, to which the École Normale was the recognised avenue At that time Pasteur was the director of the scientific studies of the school, and he became interested in Darboux, whom he recognised as a promising recruit Owing to the influence of Pasteur, after the usual three years' course, Darboux was enabled to remain two years longer as the holder of a teaching post created for him During this period he gave decided evidence of his capacity for advancing mathematical science in the work he sent up to the professors of the school, and he made a profound study of the works of such writers as Monge, Gauss, Poncelet, Dupin, Lamé, and Jacobi

In 1864 he published a note on orthogonal surfaces in the 'Comptes Rendus,' and in 1866 there was presented to the Sorbonne, as a thesis for the doctorate, his memoir "Sur les Surfaces Orthogonales" On the work contained in this thesis he received the congratulations of Chasles, Serret, and Bouquet.

In 1866-67 he assisted Bertrand in the work of the Chair of Mathematical Physics at the Collège de France, and during the years 1868-1872 he was Professor at the Lycée Louis le Grand, where Bouquet was his colleague During this period his pupils were remarkably successful in attaining their practical aims, although the original form of his teaching was such that its reference to the exigencies of examinations was only indirect In 1872 he became Maître de Conférences at the École Normale, and in 1873 he became assistant to Liouville at the Sorbonne in the Chair of Rational Mechanics, among his pupils at this time were Appell and Picard His already established reputation gave him considerable influence amongst the mathematicians of the École Normale, and his clear and elegant exposition of general mechanics at the Sorbonne did much to renovate the teaching of that subject in France On the death of Chasles in 1880, Darboux was appointed as his successor in the Chair of Higher Geometry at the Sorbonne His success not only as professor, but also as an organiser, led to his appointment in 1889, by the Minister of Public Instruction, on the nomination of his professional colleagues, as Doyen de la Faculté des Sciences de Paris, a post which gave full scope to his powers in the work of organisation of the University of Paris

In 1884 he received the great honour of being elected member of the Academy of Sciences in the section of Geometry, the necessary report on his scientific work being composed by Jordan In 1900 he succeeded Joseph Bertrand in his office of Perpetual Secretary of the Academy of Sciences. His remarkable powers as an administrator gave him a position of great authority in this position, in which he exhibited the most untiring zeal for the advance of science and for the prestige of the Academy He excelled in the public orations which it was his duty to pronounce on the death of such members as Bertrand, Hermite, and H Poincaré, He had the great satisfaction of seeing carried out the project, in which he had long been interested, of printing the MS. *procès-verbaux* of the sittings of the Academy since the foundation of the Institute in the year IX up to 1835, when the 'Comptes Rendus' were commenced. Up to the present, seven volumes, up to 1825, have been published, these are of special interest, as containing the work of Lagrange, Laplace, Monge, Cuvier, Lamarck, and other distinguished men

For 17 years Darboux presided over the Society of the Amis des Sciences, a Society founded in 1857 by Baron Thénard for the purpose of aiding men of science in need of pecuniary help.

In his mathematical investigations, Darboux both originated new ideas, and also carried out detailed investigations on previously established lines His expository style was artistic in form, and, while he knew how to utilise the detailed investigation of particular examples, in order to rise by observation and induction to generality of conception, his keen sense of proportion prevented any undue prolixity in his writings, and the detailed consequences of the ideas developed in some of his memoirs were left to be drawn by others He possessed in a high degree the faculty of discerning relations between superficially diverse questions and methods, and thus of exhibiting the fundamental identity which often exists of theories originally developed independently of one another His early work on orthogonal surfaces, in the course of which he discovered an orthogonal system of surfaces of the fourth degree, was contained in the thesis already referred to, this subject, the importance of which, in relation to mathematical physics, had been traced out by Lamé, is one to which he frequently returned in later years.

In 1873 he published his researches on analytical geometry in a work "Sur une Classe Remarquable des Courbes et des Surfaces Algébriques." The principal aim of this work was the study of a class of surfaces, termed cyclides, which have as double line the circle at infinity. The work also deals with plane and spherical cyclics, and, in a note at the end, the differential equation of surfaces applicable to a given surface is formed. The theory of cyclides has been later shown to be of much importance in the theory of the solutions of Laplace's equation applicable to various classes of problems in the theory of the potential. A very remarkable and original memoir was published in 1876 on the approximation to functions of very large numbers. In this memoir, which is of importance in relation to applications in various directions, Darboux established methods for determining the order of magnitude of the coefficients of power-series from a study of the singularities on the circle of convergence, and he studied developments in series of polynomials arising from hypergeometric series. In 1870 there appeared the memoir containing Darboux's new method of integration of partial differential equations of the second order. This work, the most important on the subject since that of Ampère in 1818, has been of much influence in more recent investigations on the subject.

The memoir on discontinuous functions, published in 1875, with a supplement in 1878, contained a critical study of the Riemann definition of an integral, and established for the first time the existence of the upper and the lower integral of any bounded function; the memoir also contains various examples of continuous functions without derivatives. Darboux's monumental work on Differential Geometry, published in four volumes, between 1887 and 1896, under the title, 'Leçons sur la Théorie Générale des Surfaces et les Applications Géométriques du Calcul Infinitésimal,' contains, in a fascinating form, an account of his own researches in Differential Geometry, together with those of his predecessors. In 1898 he commenced the publication of his work, 'Leçons sur les Systèmes Orthogonaux et les Coordonnées Curvilignes,' which supplements and completes the earlier work.

In his geometrical work Darboux exhibits an unsurpassed power of combining geometry and analysis, so that the different points of view support and supplement one another. The theory of geodesic lines led him to consider various questions in Analytical Dynamics connected with the principle of Least Action. He published a memoir on the herpolhode and the theory of Poinsot, and also wrote on the postulates of the statical proofs of the parallelogram of forces, and on the percussion and collision of bodies. He appended a series of elegant notes to an edition of Despeyrous' 'Cours de Mécanique.'

In 1904 he gave an historical lecture at the Exhibition at St Louis, in which he sketched the progress of geometry in the Nineteenth Century. In

1908, at the Rome Congress of Mathematicians, he gave a remarkable discourse on the origin, methods, and problems of Infinitesimal Geometry In the last year of his life he gave a course of lectures at the Sorbonne on the principles of Analytical Geometry, with special reference to the place of the Imaginary and the Infinite in Geometry He intended that these lectures should be incorporated in a book to be edited by himself

As Secretary of the Académie des Sciences, Darboux was brought into contact with other branches of science, and took an active part in international scientific organisations At international meetings, which he frequently attended, his warm interest in the progress of science, the quiet dignity of his sympathetic personality, and his unfailing wisdom and tact helped to surmount difficulties and to ensure success

Darboux was a member of a very large number of Academies and Scientific Societies, and was an Honorary Doctor of the Universities of Cambridge, Christiania, and Heidelberg In 1900 he was elected a Foreign Member of the Royal Society, and in December, 1916, he was awarded the Sylvester Medal This last recognition gave him genuine pleasure The acknowledgment of its receipt was dictated a few hours before his death

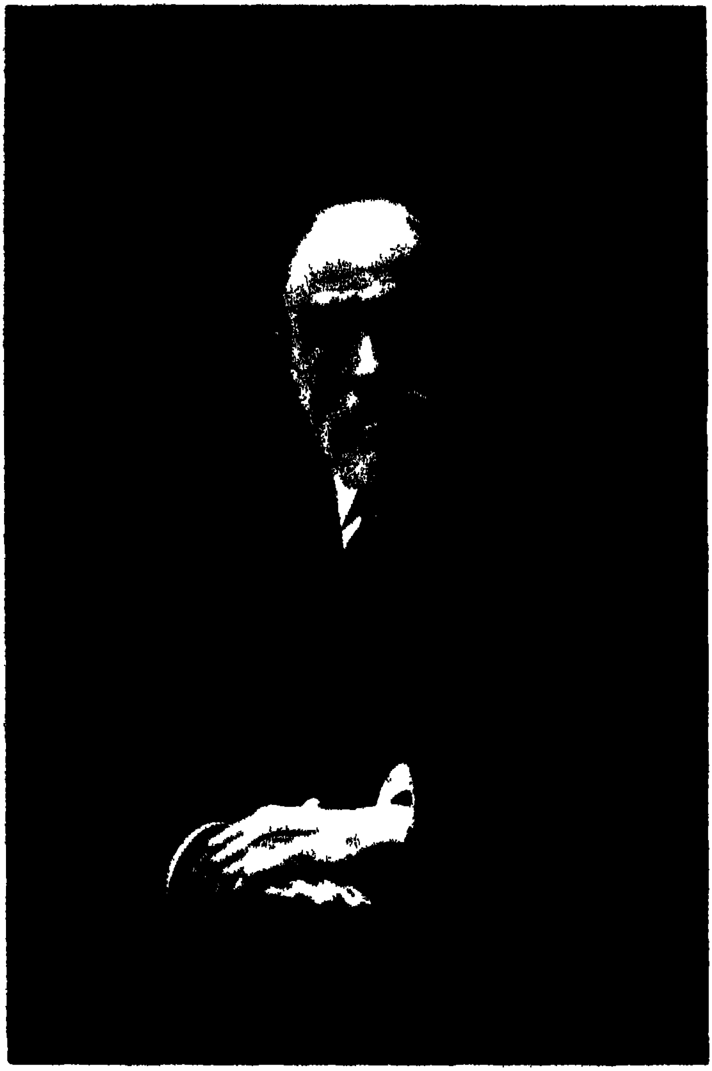
E W H

WILLIAM DU BOIS DUDDELL, 1872-1917.

Born in 1872, and educated in England and France, Duddell served his apprenticeship as engineer to Messrs. Davey, Paxman, and Co., of Colchester He then went to the City and Guilds Institute at the age of 21. He stayed there for some years, as he found the facilities for experimental work were very good, and of exceptional value to him. Joubert had devised a method of tracing alternating pressure or current variations by balancing a series of phases of a period against a standard cell, by a potentiometer bridge. Duddell produced a galvanometer which is quick enough to follow the variations, and to show the curve by a light spot, or to photograph it. This brought him into prominence as a first-rate designer of special instruments.

The next important work was an investigation, with Prof. Marchant, of the ratio of the pressure to the current in the arc, generally called "the resistance." This led to the discovery of the singing arc The singing arc is the basis of a system of generating continuous waves for wireless telegraphy. The arc is made to sing a note so high that it cannot be heard, but not high enough to be seen. The Poulsen generator is a development of this principle, and so is the valve type of transmitter.

William DuBois Duddell.



J. Wolfe Barry

Duddell was a master of the design and making of delicate special instruments. He had a thorough grasp of first principles, the inventiveness of the trained engineer, and had, in addition, the highest skill as a workman.

He was an authority on many branches of electrical engineering, but especially on wireless telegraphy Here his mathematical knowledge, combined with his clear grasp of the physical aspect of electrical phenomena, put him in the first rank Like most clear-headed people, he was a master of exposition As a witness in the law-courts he was very valuable He had the judicial type of mind, so that he gave his evidence with perfect fairness, and that, coupled with his simple lucidity of explanation, made him an ideal witness in technical cases.

In estimating our loss, we must think of what he did, also as an earnest of what he would have done He was only 45 when he died on November 4, 1917, and the last three years of his life were largely spent on secret Government work. To place him, we must compare his work with that done by others before they were 42

Duddell was President of the Institution of Electrical Engineers when only 40, being the youngest President, and he was re-elected to serve a second year. He was elected Fellow of the Royal Society in 1912

J S

SIR JOHN WOLFE BARRY (1836–1918)

THE death of Sir John Wolfe Barry on January 22, in his eighty-second year, removes from the engineering profession one of its most distinguished members, who by his ability, energy, and varied experience, and by his high character, had acquired the position of a leader amongst the civil engineers of his time.

The youngest son of Sir Charles Barry, R.A., the architect of the Houses of Parliament, he was educated at Trinity College, Glenalmond, and at King's College, London. He was a pupil of Sir John Hawkshaw, and afterwards assistant resident engineer under him on the Charing Cross and Cannon Street Railway, and for the bridges over the Thames connected with it

In 1867, he started in private practice, devoting himself largely to the construction of railways, bridges, and docks. It is not possible to enumerate all the important enterprises in which he was concerned in an executive or consultative capacity. Amongst them may be mentioned the Earl's Court Station of the Metropolitan District Railway and the extension of that railway to Ealing and Fulham, the Lewes and East Grinstead Railway, the

Inner Circle extension between the Mansion House, Aldgate, and Whitechapel, a work of great difficulty, the Blackfriars arched railway bridge; the Tower Bridge, with its very successful bascule opening span, in association with the late Sir Horace Jones, the King Edward VII Bridge at Kew, the Barry docks and railways, the Grangemouth dock, the entrance lock and graving dock at Immingham, at Newport extensive works for the Alexandra Dock and Railway Company, and the Avonmouth Docks at Bristol, also the Natal harbour works and dock and railway works in China, Buenos Ayres, and India.

He took, at an early date, an interest in electric traction on railways, and in 1899 persuaded the Metropolitan Companies to permit him to make an experimental installation to work the trains between Earl's Court and High Street, Kensington

Sir John was a Member of the Institution of Civil Engineers for 50 years, on its Council for 34 years, and its President in 1896–7, his authority in its concerns and the value of his services can scarcely be over-rated He became F.R.S in 1895, C B in 1894 on the completion of the Tower Bridge, and K C B in 1897 in recognition of his many public services. He was Chairman of Council of the Society of Arts in 1898–9. He was a Member of the Institution of Mechanical Engineers from 1871, and took much interest in its activities

Sir John gave ungrudging assistance in all public undertakings and enquiries involving engineering considerations, and had great influence in promoting the commercial and industrial prosperity of the country He was Chairman of the Lower Thames Navigation Commission appointed in 1894, member of the Port of London Commission 1900–2, of the Commission on Irish Public Works 1886, of the Western Highlands and Islands Commission 1889, and the Traffic of London Commission 1903–5 He was a member of the Court of Arbitration for the purchase of the waterworks undertakings in London, when they were vested in the Metropolitan Water Board He and Sir Charles Hartley were the representatives of the British Government on the International Commission of Works of the Suez Canal, 1892–1906 He was a colonel in the Engineer and Railway Volunteer Staff Corps. He was Deputy-Lieutenant for the county of London, and Chairman of the Eastern and Western Telegraph Companies.

One of the greatest services rendered by Sir John Wolfe Barry to engineering industries was the part he took in initiating and directing the activities of the Engineering Standards Committee. It was mainly due to his insight and wide influence that representatives of Government departments, registration societies, engineers, manufacturers, shipbuilders, and others were brought together, and have freely given their time and experience in dealing with the complex problems of standardisation, a work of the greatest national importance Sir John, in 1917, in a lecture to the Institution of Civil Engineers, gave an account of the work of the Standards Committee during 16 years and its influence on the prosperity of the country Started in a tentative way, with the object of reducing the wholly unnecessary number of rolled sections of steel bars and rails produced by manufacturers to meet the wishes or whims of different engineers and architects, it was soon found necessary to draw up complete specifications of quality and of the tests of acceptance to which materials should be subjected The work of the Committee soon extended in other directions, especially to the standardisation of machines, fittings, and tests of efficiency in the electrical industry Types of locomotives were standardised for railways in India at the request of the Indian Government The main committee now consists of 22 members, representative of the great professional societies, and there are 64 sectional and sub-committees having in the aggregate more than 500 members Practically all persons interested are represented, and have a consultative voice in proposals for any particular standardisation The specifications are published, and, when necessary, revised annually

Sir John Wolfe Barry pointed out in his lecture that it was difficult to estimate exactly the beneficial results of standardisation, but that he was justified in saying that they have been immense in facilitating production and cheapening output, while securing excellence in the scientific composition of materials and accuracy of workmanship Thus, in the case of Portland cement, whereas formerly very many different specifications were imposed on manufacturers by different users, involving modifications in processes of production, practically now the whole output is made to one standard specification In the case of rolled sections for construction, shipbuilding, and railway and tramway rails, the annual output before the war was 3,700,000 tons, valued at £25,000,000. Of this, now more than 85 per cent is rolled to standard sections and specifications

The war has raised serious questions as to the security in the future of our foreign trade Under Sir John Wolfe Barry's guidance, the Standards Committee has undertaken the task of translating the Standard Specifications into French, Spanish, and Portuguese, and converting British into metric measures. It also contemplates the establishment in twelve important foreign trading centres, of local committees in touch with the London organisation, and concerned with the promotion of British interests

Sir John Wolfe Barry took a great interest in efforts to raise the scientific qualifications of civil engineers It was at his instance that the Institution of Civil Engineers adopted an examination scheme, so that candidates for admission to the Society must now pass an educational test as well as a scrutiny of their experience in constructional work. He was a Governor of the Imperial College of Science and Technology and Chairman of the Executive Committee of the City and Guilds College For a time he was a Member of the Senate of London University

Sir John took an active part in the foundation of the National Physical Laboratory, was a member of its Executive Committee, and greatly assisted it in obtaining the funds required for carrying on its work, and in making it the expert authority in scientific questions arising in connection with standardisation

It would not be just to close this notice without a reference to Sir John Wolfe Barry's unfailing urbanity and his readiness to assist other engineers, and to place his unrivalled stores of information at their disposal He was jealous of the honour and proud of the progress of his profession, and it would be difficult to overestimate the influence he wielded, or the extent to which he raised the work of engineers in the estimation of Government departments and the public.

In 1874 Sir John married Rosalind Grace, daughter of the Rev E E Rowsell, Rector of Hambledon, Surrey He had four sons and three daughters, all of whom survive

W C U

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